On the satisfiability problem for SPARQL patterns

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Abstract

The satisfiability problem for SPARQL patterns is undecidable in general, since SPARQL 1.0 can express the relational algebra. The goal of this paper is to delineate the boundary of decidability of satisfiability in terms of the constraints allowed in filter conditions. The classes of constraints considered are bound-constraints, negated bound-constraints, equalities, nonequalities, constant-equalities, and constant-nonequalities. The main result of the paper can be summarized by saying that, as soon as inconsistent filter conditions can be formed, satisfiability is undecidable. The key insight in each case is to find a way to emulate the set difference operation. Undecidability can then be obtained from a known undecidability result for the algebra of binary relations with union, composition, and set difference. When no inconsistent filter conditions can be formed, satisfiability is decidable by syntactic checks on bound variables and on the use of literals. Although the problem is shown to be NP-complete, it is experimentally shown that the checks can be implemented efficiently in practice. The paper also points out that satisfiability for the so-called ‘well-designed’ patterns can be decided by a check on bound variables and a check for inconsistent filter conditions.

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1 Introduction

The Resource Description Framework [RDF04] is a popular data model for information in the Web. RDF represents information in the form of directed, labeled graphs. The standard query language for RDF data is SPARQL [SPA13]. The current version 1.1 of SPARQL extends SPARQL 1.0 [SPA08] with important features such as aggregation and regular path expressions [ACP12]. Other features, such as negation and subqueries, have also been added, but mainly for efficiency reasons, as they were already expressible, in a more involved manner, in version 1.0. Hence, it is still relevant to study the fundamental properties of SPARQL 1.0. In this paper, we follow the elegant formalization of SPARQL 1.0 by Arenas, Gutierrez and Pérèz [PAG09, APG09] which is eminently suited for theoretical investigations.

The fundamental problem that we investigate is that of satisfiability of SPARQL patterns. A pattern is called satisfiable if there exists an RDF graph under which the pattern evaluates to a nonempty set of mappings. For any query language, satisfiability is clearly one of the essential properties one needs to understand if one wants to do automated reasoning. Since SPARQL patterns can emulate relational algebra expressions [AG08, Pol07, AP11], and satisfiability for relational algebra is undecidable [AHV95], the general satisfiability problem for SPARQL is undecidable as well.

Whether or not a pattern is satisfiable depends mainly on the filter operations appearing in the pattern. The goal of this paper is to precisely delineate the decidability of SPARQL fragments that are defined in terms of the constraints that can be used as filter conditions. The six basic classes of constraints we consider are bound-constraints; equalities; constant-equalities; and their negations. In this way, fragments of SPARQL can be constructed by specifying which kinds of constraints are allowed as filter conditions. For example, in the fragment SPARQL(bound, ≠, ≠), filter conditions can only be bound constraints, nonequalities, and constant-nonequalities.

Our main result states that the only fragments for which satisfiability is decidable are the two fragments SPARQL(bound, =, ≠, ≠) and SPARQL(bound, ≠, ≠, ≠) and their subfragments. Consequently, as soon as either negated bound-constraints, or constant-equalities, or combinations of equalities and nonequalities are allowed, the satisfiability problem becomes undecidable. Each undecidable case is established by showing how the set difference operation can be emulated. This was already known using negated bound-constraints [AG08, AP11]; so we show it is also possible using constant-equalities, and using combinations of equalities and nonequalities, but in no other way. Undecidability can then be obtained from a known undecidability result for the algebra of binary relations with union, composition, and set difference [TVdBZ14].

In the decidable cases, satisfiability can be decided by syntactic checks on bound variables and the use of literals. Although the problem is shown to be NP-complete, it is experimentally shown that the checks can be implemented efficiently in practice.

At the end of the paper we look at a well-behaved class of patterns known
as the ‘well-designed’ patterns [PAG09]. We observe that satisfiability of well-designed patterns can be decided by combining the check on bound variables with a check for inconsistent filter conditions.

This paper is further organized as follows. In the next section, we introduce syntax and semantics of SPARQL patterns and introduce the different fragments under consideration. Section 3 introduces the satisfiability problem and shows satisfiability checking for the fragments SPARQL(bound, $=, \neq, \land$) and SPARQL(bound, $\neq, \neq, \land$). Section 4 shows undecidability for the fragments SPARQL($\neg$bound), SPARQL($=, \land$), and SPARQL($=, \land, \neq$). Section 5 considers well-designed patterns.

Section 6 reports on experiments that test our decision methods in practice. In Section 7 we briefly discuss how our results extend to the new operators that have been added to SPARQL 1.1. We conclude in Section 8.

2 SPARQL and fragments

In this section we recall the syntax and semantics of SPARQL patterns, closely following the core SPARQL formalization given by Arenas, Gutierrez and Pérez [PAG09, APG09, AP11]. The semantics we use is set-based, whereas the semantics of real SPARQL is bag-based. However, for satisfiability (the main topic of this paper), it makes no difference whether we use a set or bag semantics [SML10, Lemma 1].

In this section we will also define the language fragments defined in terms of allowed filter conditions, which will form the object of this paper.

2.1 RDF graphs

Let $I$, $B$, and $L$ be infinite sets of IRIs, blank nodes and literals, respectively. These three sets are pairwise disjoint. We denote the union $I \cup B \cup L$ by $U$, and elements of $I \cup L$ will be referred to as constants. Note that blank nodes are not constants.

A triple $(s, p, o) \in (I \cup B) \times I \times U$ is called an RDF triple. An RDF graph is a finite set of RDF triples.

2.2 Syntax of SPARQL patterns

Assume furthermore an infinite set $V$ of variables, disjoint from $U$. The convention in SPARQL is that variables are written beginning with a question mark, to distinguish them from constants. We will follow this convention in this paper.

SPARQL patterns are inductively defined as follows.

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1The cited works are seminal works on the semantics and complexity of SPARQL patterns, but they do not investigate the satisfiability of SPARQL patterns which is the main topic of the present paper. The cited works also extensively discuss minor deviations between the formalization and real SPARQL, and why these differences are inessential for the purpose of formal investigation.
• Any triple from \((I \cup L \cup V) \times (I \cup V) \times (I \cup L \cup V)\) is a pattern (called a triple pattern).

• If \(P_1\) and \(P_2\) are patterns, then so are the following:
  
  \(- P_1 \text{ UNION } P_2\);
  
  \(- P_1 \text{ AND } P_2\);
  
  \(- P_1 \text{ OPT } P_2\).

• If \(P\) is a pattern and \(C\) is a constraint (defined next), then \(P \text{ FILTER } C\) is a pattern; we call \(C\) the filter condition.

Here, a constraint can have one of the six following forms:

1. bound-constraint: \(\text{bound}(?x)\)
2. negated bound-constraint: \(\neg \text{bound}(?x)\)
3. equality: \(?x = ?y\)
4. nonequality: \(?x \neq ?y\) with \(?x\) and \(?y\) distinct variables
5. constant-equality: \(?x = c\) with \(c\) a constant
6. constant-nonequality: \(?x \neq c\)

We do not need to consider conjunctions and disjunctions in filter conditions, since conjunctions can be expressed by repeated application of filter, and disjunctions can be expressed using UNION. Hence, by going to disjunctive normal form, any predicate built using negation, conjunction, and disjunction is indirectly supported by our language.

Moreover, real SPARQL also allows blank nodes in triple patterns. This feature has been omitted from the formalization [PAG09, APG09, AP11], because blank nodes in triple patterns can be equivalently replaced by variables.

2.3 Semantics of SPARQL patterns

The semantics of patterns is defined in terms of sets of so-called solution mappings, hereinafter simply called mappings. A solution mapping is a total function \(\mu : S \rightarrow U\) on some finite set \(S\) of variables. We denote the domain \(S\) of \(\mu\) by \(\text{dom}(\mu)\).

We make use of the following convention.

**Convention.** For any mapping \(\mu\) and any constant \(c \in I \cup L\), we agree that \(\mu(c)\) equals \(c\) itself.

In other words, mappings are by default extended to constants according to the identity mapping.

Now given a graph \(G\) and a pattern \(P\), we define the semantics of \(P\) on \(G\), denoted by \(\llbracket P \rrbracket_G\), as a set of mappings, in the following manner.
- If $P$ is a triple pattern $(u, v, w)$, then
  \[ [P]_G := \{ \mu : \{u, v, w\} \cap V \rightarrow U \mid (\mu(u), \mu(v), \mu(w)) \in G \}. \]
  This definition relies on Convention 2.3 formulated above.

- If $P$ is of the form $P_1$ UNION $P_2$, then
  \[ [P]_G := [P_1]_G \cup [P_2]_G. \]

- If $P$ is of the form $P_1$ AND $P_2$, then
  \[ [P]_G := [P_1]_G \otimes [P_2]_G, \]
  where, for any two sets of mappings $\Omega_1$ and $\Omega_2$, we define
  \[ \Omega_1 \otimes \Omega_2 = \{ \mu_1 \cup \mu_2 \mid \mu_1 \in \Omega_1 \text{ and } \mu_2 \in \Omega_2 \} \text{ and } \mu_1 \sim \mu_2. \]
  Here, two mappings $\mu_1$ and $\mu_2$ are called compatible, denoted by $\mu_1 \sim \mu_2$, if they agree on the intersection of their domains, i.e., if for every variable $?x \in \text{dom}(\mu_1) \cap \text{dom}(\mu_2)$, we have $\mu_1(?x) = \mu_2(?x)$.
  Note that when $\mu_1$ and $\mu_2$ are compatible, their union $\mu_1 \cup \mu_2$ is a well-defined mapping; this property is used in the formal definition above.

- If $P$ is of the form $P_1$ OPT $P_2$, then
  \[ [P]_G := ([P_1]_G \otimes [P_2]_G) \cup ([P_1]_G \setminus [P_2]_G), \]
  where, for any two sets of mappings $\Omega_1$ and $\Omega_2$, we define
  \[ \Omega_1 \setminus \Omega_2 = \{ \mu_1 \in \Omega_1 \mid \exists \mu_2 \in \Omega_2 : \mu_1 \sim \mu_2 \}. \]

- Finally, if $P$ is of the form $P_1$ FILTER $C$, then
  \[ [P]_G := \{ \mu \in [P_1]_G \mid \mu \models C \} \]
  where the satisfaction of a constraint $C$ by a mapping $\mu$, denoted by $\mu \models C$, is defined as follows:
  1. $\mu \models \text{bound}(?x)$ if $?x \in \text{dom}(\mu)$;
  2. $\mu \models \neg\text{bound}(?x)$ if $?x \notin \text{dom}(\mu)$;
  3. $\mu \models ?x = ?y$ if $?x, ?y \in \text{dom}(\mu)$ and $\mu(?x) = \mu(?y)$;
  4. $\mu \models ?x \neq ?y$ if $?x, ?y \in \text{dom}(\mu)$ and $\mu(?x) \neq \mu(?y)$;
  5. $\mu \models ?x = c$ if $?x \in \text{dom}(\mu)$ and $\mu(?x) = c$;
  6. $\mu \models ?x \neq c$ if $?x \in \text{dom}(\mu)$ and $\mu(?x) \neq c$.

  Note that $\mu \models ?x \neq ?y$ is not the same as $\mu \not\models ?x = ?y$, and similarly for $\mu \models ?x \neq c$. This is in line with the three-valued logic semantics for filter conditions used in the official semantics [APG09]. For example, if $?x \notin \text{dom}(\mu)$, then in three-valued logic $?x = c$ evaluates to error under $\mu$; consequently, also $\neg ?x = c$ evaluates to error under $\mu$. Accordingly, in the semantics above, we have both $\mu \not\models ?x = c$ and $\mu \not\models ?x \neq c$. 
2.4 SPARQL fragments

We can form fragments of SPARQL by specifying which of the six classes of constraints are allowed as filter conditions. We denote the class of bound-constraints by ‘bound’, negated bound-constraints by ‘¬bound’, equalities by ‘=’, nonequalities by ‘̸=’, constant-equalities by ‘=c’, and constant-nonequalities by ‘̸=c’. Then for any subset $F$ of \{bound, ¬bound, =, ̸=, =c, ̸=c\} we can form the fragment $\text{SPARQL}(F)$. For example, in the fragment $\text{SPARQL}(\text{bound, } =, ̸=c)$, filter conditions can only be bound constraints, equalities, and constant-nonequalities.

3 Satisfiability: decidable fragments

A pattern $P$ is called satisfiable if there exists a graph $G$ such that $[P]_G$ is nonempty. In general, checking satisfiability is a very complicated, indeed undecidable, problem. But for the two fragments $\text{SPARQL}(\text{bound, } =, ̸=c)$ and $\text{SPARQL}(\text{bound, } ̸=, ̸=c)$, it will turn out that there are essentially only two possible reasons for unsatisfiability.

The first possible reason is that the pattern specifies a literal value in the first position of some RDF triple, whereas RDF triples can only have literals in the third position. For example, using the literal 42, the triple pattern $(42, ?x, ?y)$ is unsatisfiable. Note that literals in the middle position of a triple pattern are already disallowed by the definition of triple pattern, so we only need to worry about the first position.

This discrepancy between triple patterns and RDF triples is easy to sidestep, however. In the Appendix we show how, without loss of generality, we may assume from now on that patterns do not contain any triple pattern $(u, v, w)$ where $u$ is a literal.

The second and main possible reason for unsatisfiability is that filter conditions require variables to be bound together in a way that cannot be satisfied by the subpattern to which the filter applies. For example, the pattern

$((?x, a, ?y) \text{ UNION } (?x, b, ?z)) \text{ FILTER } (\text{bound}(?y) \land \text{bound}(?z))$

is unsatisfiable. Note that bound constraints are not strictly necessary to illustrate this phenomenon: if in the above example we replace the filter condition by $?y = ?z$ the resulting pattern is still unsatisfiable.

We next prove formally that satisfiability for patterns in $\text{SPARQL}(\text{bound, } =, ̸=c)$ and $\text{SPARQL}(\text{bound, } ̸=, ̸=c)$ is effectively decidable, by catching the reason for unsatisfiability described above. Note also that the two fragments can not be combined, since satisfiability for $\text{SPARQL}(=, ̸=)$ is undecidable as we will see in the next Section.

3.1 Checking bound variables

To perform bound checks on variables, we associate to every pattern $P$ a set $\Gamma(P)$ of schemes, where a scheme is simply a set of variables, in the following
way.\footnote{We define $\Gamma(P)$ for general patterns, not only for those belonging to the fragments considered in this Section, because we will make another use of $\Gamma(P)$ in Section 5.}

- If $P$ is a triple pattern $(u, v, w)$, then $\Gamma(P) := \{\{u, v, w\} \cap V\}$.
- $\Gamma(P_1 \text{ UNION } P_2) := \Gamma(P_1) \cup \Gamma(P_2)$.
- $\Gamma(P_1 \text{ AND } P_2) := \{S_1 \cup S_2 \mid S_1 \in \Gamma(P_1) \text{ and } S_2 \in \Gamma(P_2)\}$.
- $\Gamma(P_1 \text{ OPT } P_2) := \Gamma(P_1 \text{ AND } P_2) \cup \Gamma(P_1)$.
- $\Gamma(P_1 \text{ FILTER } C) := \{S \in \Gamma(P_1) \mid S \vdash C\}$, where $S \vdash C$ is defined as follows:
  - If $C$ is of the form $\text{bound}(?x)$ or $?x = c$ or $?x \neq c$, then $S \vdash C$ if $?x \in S$;
  - If $C$ is of the form $?x = ?y$ or $?x \neq ?y$, then $S \vdash C$ if $?x, ?y \in S$;
  - $S \vdash \neg \text{bound}(?x)$ if $?x \notin S$.

\textbf{Example 1.} Consider the pattern

$$P = (?x, p, ?y) \text{ OPT } ((?x, q, ?z) \text{ UNION } (?x, r, ?u)).$$

For the subpattern $P_1 = (?x, q, ?z) \text{ UNION } (?x, r, ?u)$ we have $\Gamma(P_1) = \{\{?x, ?z\}, \{?x, ?u\}\}$. Hence, $\Gamma((?x, p, ?y) \text{ AND } P_1) = \{\{?x, ?y, ?z\}, \{?x, ?y, ?u\}\}$. We conclude that $\Gamma(P) = \{\{?x, ?y\}, \{?x, ?y, ?z\}, \{?x, ?y, ?u\}\}$.

\textbf{Example 2.} For another example, consider the pattern

$$P = ((?x, p, ?y) \text{ OPT } ((?x, q, ?z) \text{ FILTER } ?y = ?z)) \text{ FILTER } ?x \neq c.$$ We have $\Gamma(?x, q, ?z) = \{\{?x, ?z\}\}$. Note that $\{?x, ?z\} \not\vdash ?y = ?z$, because $?y \notin \{?x, ?z\}$. Hence, for the subpattern $P_1 = (?x, q, ?z) \text{ FILTER } ?y = ?z$ we have $\Gamma(P_1) = \emptyset$. For the subpattern $P_2 = (?x, p, ?y) \text{ OPT } P_1$ we then have $\Gamma(P_2) = \Gamma(?x, p, ?y) = \{\{?x, ?y\}\}$. Since $\{?x, ?y\} \not\vdash ?x \neq c$, we conclude that $\Gamma(P) = \{\{?x, ?y\}\}$.

We now establish the main result of this Section.

\textbf{Theorem 3.} Let $P$ be a SPARQL(bound, $=, \neq$) or SPARQL(bound, $\neq, \neq$) pattern. Then $P$ is satisfiable if and only if $\Gamma(P)$ is nonempty.

The only-if direction of Theorem 3 is the easy direction and is given by the following Lemma 4. Note that this lemma holds for general patterns; it can be straightforwardly proven by induction on the structure of $P$.

\textbf{Lemma 4.} Let $P$ be a pattern. If $\mu \in \mathbb{P}_G$ then there exists $S \in \Gamma(P)$ such that $\text{dom}(\mu) = S$. 

\begin{align*}
\text{dom}(\mu) &= \{x \mid \text{there exists } S \in \Gamma(P) \text{ such that } \mu_x = S\} \\
\text{dom}(\mu) &= \{x \mid \text{there exists } S \in \Gamma(P) \text{ such that } \mu_x = S\}.
\end{align*}
The if direction of Theorem 3 for SPARQL(\text{bound},=,\neq,e) is given by the following Lemma 5.

In the following we use \text{var}(P) to denote the set of all variables occurring in a pattern $P$.\footnote{We also use the following standard notion of restriction of a mapping. If $f:X\to Y$ is a total function and $Z\subseteq X$, then the restriction $f|_Z$ of $f$ to $Z$ is the total function from $Z$ to $Y$ defined by $f|_Z(z) = f(z)$ for every $z \in Z$. That is, $f|_Z$ is the same as $f$ but is only defined on the subdomain $Z$.}

**Lemma 5.** Let $P$ be a pattern in SPARQL(\text{bound},=,\neq,e). Let $c \in I$ be a constant that does not appear in any constant-nonequality filter condition in $P$.

With the constant mapping $\mu:\text{var}(P) \to \{c\}$, let $G$ be the RDF graph consisting of all possible triples $(\mu(u),\mu(v),\mu(w))$ where $(u,v,w)$ is a triple pattern in $P$.

Then for every $S \in \Gamma(P)$ there exists $S' \supseteq S$ such that $\mu|_{S'}$ belongs to $[P]_G$.

**Proof.** By induction on the structure of $P$. If $P$ is a triple pattern $(u,v,w)$ then $S = \{u,v,w\} \cap V$. Since $(\mu|_S(u),\mu|_S(v),\mu|_S(w)) = (\mu(u),\mu(v),\mu(w)) \in G$, we have $\mu|_S \in [P]_G$ and we can take $S' = S$.

If $P$ is of the form $P_1 \text{UNION} P_2$, then the claim follows readily by induction.

If $P$ is of the form $P_1 \text{AND} P_2$, then we have $S = S_1 \cup S_2$ with $S_i \in \Gamma(P_i)$ for $i = 1,2$. By induction, there exists $S'_i \supseteq S_i$ such that $\mu|_{S'_i} \in [P_i]_G$. Clearly $\mu|_{S_1} \sim \mu|_{S_2}$ since they are restrictions of the same mapping. Hence $\mu|_{S'_1} \cup \mu|_{S'_2} = \mu|_{S'_1 \cup S'_2} \in [P]_G$ and we can take $S' = S'_1 \cup S'_2$.

If $P$ is of the form $P_1 \text{OPT} P_2$, then there are two possibilities.

- If $S \in \Gamma(P_1 \text{AND} P_2)$ then we can reason as in the previous case.
- If $S \in \Gamma(P_1)$ then by induction there exists $S'_1 \supseteq S$ so that $\mu|_{S'_1} \in [P_1]_G$.

Now there are two further possibilities:

- If $\Gamma(P_2)$ is nonempty then by induction there exists some $S'_2$ so that $\mu|_{S'_2} \in [P_2]_G$. We can now reason again as in the case $P_1 \text{AND} P_2$.
- Otherwise, by Lemma 4 we know that $[P_2]_G$ is empty. But then $[P]_G = [P_1]_G$ and we can take $S' = S'_1$.

Finally, if $P$ is of the form $P_1 \text{FILTER} C$, then we know that $S \in \Gamma(P_1)$ and $S \models C$. By induction, there exists $S' \supseteq S$ such that $\mu|_{S'} \in [P_1]_G$. We show that $\mu|_{S'} \models C$. There are three possibilities for $C$.

- If $C$ is of the form $\text{bound}(?x)$, then we know by $S \models C$ that $?x \in S'$. Hence $\mu|_{S'} \models C$.
- If $C$ is of the form $?x = ?y$, then we again know $?x,?y \in S'$, and certainly $\mu|_{S'} \models C$ since $\mu$ maps everything to $c$.
- If $C$ is of the form $?x \neq d$, then we have $d \neq c$ by the choice of $c$, so $\mu|_{S'} \models C$ since $\mu(?x) = c$. \qed
Example 6. To illustrate the above Lemma, consider the pattern

\[ P = (\langle ?x, p, ?y \rangle \text{ FILTER } ?x \neq a) \cup \text{ OPT } (\langle ?x, q, ?z \rangle \text{ UNION } \langle ?x, r, ?u \rangle) \]

which is a variant of the pattern from Example 1. As in that example, we have \( \Gamma(P) = \{ \{?x, ?y\}, \{?x, ?y, ?z\}, \{?x, ?y, ?u\} \} \). In this case, the mapping \( \mu \) from the Lemma maps \(?x, ?y, ?z\) and \(?u\) to \(c\). The graph \( G \) from the Lemma equals \( \{(c, p, c), (c, q, c), (c, r, c)\} \), and \( [P]_G = \{\mu_1, \mu_2\} \) where \( \mu_1 = \mu_1(?x, ?y, ?z) \) and \( \mu_2 = \mu_1(?x, ?y, ?u) \). Now consider \( S = \{?x, ?y\} \in \Gamma(P) \). Then for \( S' = \{?x, ?y, ?z\} \) we indeed have \( S' \supseteq S \) and \( \mu|_{S'} = \mu_1 \in [P]_G \). Note that in this example we could also have chosen \( \{?x, ?y, ?u\} \) for \( S' \).

The counterpart to Lemma 5 for the fragment SPARQL(bound, \(\neq\), \(=\)) is given by the following Lemma, thus settling Theorem 3 for that fragment.

Lemma 7. Let \( P \) be a pattern in SPARQL(bound, \(\neq\), \(=\)). Let \( W \) be the set of all constants appearing in a constant-nonequality filter condition in \( P \). Let \( Z \subseteq I \) be a finite set of constants of the same cardinality as \( \text{var}(P) \), and disjoint from \( W \). With \( \mu : \text{var}(P) \rightarrow Z \) an arbitrary but fixed injective mapping, let \( G \) be the RDF graph consisting of all possible triples \((\mu(u), \mu(v), \mu(w))\) where \((u, v, w)\) is a triple pattern in \( P \).

Then for every \( S \in \Gamma(P) \) there exists \( S' \supseteq S \) such that \( \mu|_{S'} \) belongs to \([P]_G\).

Proof. We prove for every subpattern \( Q \) of \( P \) that for every \( S \in \Gamma(Q) \) there exists \( S' \supseteq S \) such that \( \mu|_{S'} \in [Q]_G \). The proof is by induction on the height of \( Q \). The reasoning is largely the same as in the proof of Lemma 5. The only difference is in the case where \( Q \) is of the form \( Q_1 \text{ FILTER } C \). In showing that \( \mu|_{S'} \models C \), we now argue as follows for the last two cases:

- If \( C \) is of the form \(?x \neq ?y\), then \( \mu|_{S'} \models C \) since \( \mu \) is injective.
- If \( C \) is of the form \(?x \neq c\), then \( \mu|_{S'} \models C \) since \( Z \) and \( W \) are disjoint. \(\Box\)

3.2 Computational complexity

In this section we show that satisfiability for the decidable fragments is NP-complete. Note that this does not immediately follow from the NP-completeness of SAT, since boolean formulas are not part of the syntax of the decidable fragments.

Theorem 3 implies the following complexity upper bound:

Corollary 8. The satisfiability problem for SPARQL(bound, =, \(\neq\)) patterns, as well as for SPARQL(bound, \(\neq\), \(\neq\)) patterns, belongs to the complexity class NP.

Proof. By Theorem 3, a SPARQL(bound, =, \(\neq\)) or SPARQL(bound, \(\neq\), \(\neq\)) pattern \( P \) is satisfiable if and only if there exists a scheme in \( \Gamma(P) \). Following the definition of \( \Gamma(P) \), it is clear that there is a polynomial-time nondeterministic algorithm such that, on input \( P \), each accepting possible run computes
a scheme in $\Gamma(P)$, and such that every scheme in $\Gamma(P)$ is computed by some accepting possible run.

Specifically, the algorithm works bottom-up on the syntax tree of $P$ and computes a scheme for every subpattern. At every leaf $Q$, corresponding to a triple pattern in $P$, we compute the unique scheme in $\Gamma(Q)$. At every UNION operator we nondeterministically choose between continuing with the scheme from the left or from right child. At every AND operator we continue with the union of the left and right child schemes. At every OPT operator, we nondeterministically choose between treating it as an AND, or simply continuing with the scheme from the left. At every FILTER operation with constraint $C$ we check for the child scheme $S$ whether $S \vdash C$. If the check succeeds, we continue with $S$; if the check fails, the run is rejected. When the computation has reached the root of the syntax tree and we can compute a scheme for the root, the run is accepting and the computed scheme is the output.

We next show that satisfiability is actually NP-hard, even for patterns not using any OPT operators and using only bound constraints in filter conditions.

**Proposition 9.** The satisfiability problem for OPT-free patterns in the fragment SPARQL(bound) is NP-hard.

**Proof.** We define the problem Nested Set Cover as follows:

**Input:** A finite set $T$ and a finite set $E$ of sets of subsets of $T$. (So, every element of $E$ is a set of subsets of $T$.)

**Decide:** Whether for each element $e$ of $E$ we can choose a subset $S_e$ in $e$, so that $\bigcup_{e \in E} S_e = T$.

Let us first describe how the above problem can be reduced in polynomial time to the satisfiability problem at hand. Consider an input $(T, E)$ for Nested Set Cover. Without loss of generality we may assume that $T$ is a set of variables $\{?x_1, ?x_2, \ldots, ?x_n\}$. Fix some constant $c$. For any subset $S$ of $T$, we can make a pattern $P_S$ by taking the AND of all $(x, c, c)$ for $x \in S$. Now for a set $e$ of subsets of $T$, we can form the pattern $P_e$ by taking the UNION of all $P_S$ for $S \in e$. Finally, we form the pattern $P_E$ by taking the AND of all $P_e$ for $e \in E$.

Now consider the following pattern which we denote by $P_{(T, E)}$:

$$P_E \text{ FILTER } \text{bound}(?x_1) \text{ FILTER } \text{bound}(?x_2) \ldots \text{FILTER } \text{bound}(?x_n)$$

We claim that $P_{(T, E)}$ is satisfiable if and only if $(T, E)$ is a yes-instance for Nested Set Cover. To see the only-if direction, let $G$ be a graph such that $[P_{(T, E)}]_G$ is nonempty, i.e., has as an element some solution mapping $\mu$. Then in particular $\mu \in [P_E]_G$. Hence, for every $e \in E$ there exists $\mu_e \in [P_e]_G$ such that $\mu = \bigcup_{e \in E} \mu_e$. Since $P_e$ is the UNION of all $P_S$ for $S \in e$, for each $e \in E$ there exists $S_e \in e$ such that $\mu_e \in [P_{S_e}]_G$. Since $P_{S_e}$ is the AND of all $(x, c, c)$ for $x \in S_e$, it follows that $\text{dom}(\mu_e) = S_e$. Hence, since $\text{dom}(\mu) = \bigcup_{e \in E} \text{dom}(\mu_e)$, we have $\text{dom}(\mu) = \bigcup_{e \in E} S_e$. However, by the bound constraints in the filters
applied in \( P(T,E) \), we also have \( \text{dom}(\mu) = \{?x_1, \ldots, ?x_n\} = T \). We conclude that \( T = \bigcup_{e \in E} S_e \) as desired.

For the if-direction, assume that for each \( e \in E \) there exists \( S_e \in e \) such that \( T = \bigcup_{e \in E} S_e \). Consider the singleton graph \( G = \{(c,c,c)\} \). For any subset \( S \) of \( T \), let \( \mu_S : S \rightarrow \{c\} \) be the constant solution mapping with domain \( S \). Clearly, \( \mu_S \in [P_S]_G \), so \( \mu_S \in \bigcup_{e \in E} [P_e]_G \) for every \( e \in E \). All the \( \mu_S \) map to the same constant, so they are all compatible. Hence, for \( \mu = \bigcup_{e \in E} \mu_S \), we have \( \mu \in [P]_G \). Since \( \text{dom}(\mu) = \bigcup_{e \in E} \text{dom}(\mu_S) = \bigcup_{e \in E} S_e = T = \{?x_1, \ldots, ?x_n\} \), the mapping \( \mu \) satisfies every constraint \( \text{bound}(?x_i) \) for \( i = 1, \ldots, n \). We conclude that \( \mu \in \bigcup_{(E,T)} P \) as desired.

It remains to show that Nested Set Cover is NP-hard. Thereto we reduce the classical CNF-SAT problem. Assume given a boolean formula \( \phi \) in CNF, so \( \phi \) is a conjunction of clauses, where each clauses is a disjunction of literals (variables or negated variables). We construct an input \((T,E)\) for Nested Set Cover as follows. Denote the set of variables used in \( \phi \) by \( W \).

For \( T \) we take the set of clauses of \( \phi \). For any variable \( x \in W \), consider the set \( \text{Pos}_x \) consisting of all clauses that contain a positive occurrence of \( x \), and the set \( \text{Neg}_x \) consisting of all clauses that contain a negative occurrence of \( x \). Then we define \( e_x \) as the pair \( \{\text{Pos}_x, \text{Neg}_x\} \).

Now \( E \) is defined as the set \( \{e_x \mid x \in W\} \). It is clear that \( \phi \) is satisfiable if and only if the constructed input is a yes-instance for Nested Set Cover. Indeed, truth assignments to the variables correspond to selecting either \( \text{Pos}_x \) or \( \text{Neg}_x \) from \( e_x \) for each \( x \in W \).

\[ \square \]

4 Undecidable fragments

In this Section we show that the two decidable fragments \( \text{SPARQL}(\text{bound}, =, \neq_c) \) and \( \text{SPARQL}(\text{bound}, \neq, \neq_c) \) are, in a sense, maximal. Specifically, the three minimal fragments not subsumed by one of these two fragments are \( \text{SPARQL}(\neg \text{bound}), \text{SPARQL}(=, \neq), \) and \( \text{SPARQL}(=, c) \). The main result of this Section is:

**Theorem 10.** Satisfiability is undecidable for \( \text{SPARQL}(\neg \text{bound}) \) patterns, for \( \text{SPARQL}(=, \neq) \) patterns, and for \( \text{SPARQL}(=, c) \) patterns.

We will first present the proof for \( \text{SPARQL}(\neg \text{bound}) \); after that we explain how the proof can be adapted for the other two fragments.

4.1 \( \text{SPARQL}(\neg \text{bound}) \)

Our approach is to reduce from the satisfiability problem for the algebra of finite binary relations with union, difference, and composition [TVdBZ14]. This algebra is also called the Downward Algebra and denoted by \( DA \). The expressions of \( DA \) are defined as follows. Let \( R \) be an arbitrary fixed binary relation symbol.

- The symbol \( R \) is a \( DA \)-expression.
If $e_1$ and $e_2$ are DA-expressions, then so are $e_1 \cup e_2$, $e_1 - e_2$, and $e_1 \circ e_2$.

Semantically, DA-expressions represent binary queries on binary relations, i.e., mappings from binary relations to binary relations. Let $J$ be a binary relation. For DA-expression $e$, we define the binary relation $e(J)$ inductively as follows:

- $R(J) = J$;
- $(e_1 \cup e_2)(J) = e_1(J) \cup e_2(J)$;
- $(e_1 - e_2)(J) = e_1(J) - e_2(J)$ (set difference);
- $(e_1 \circ e_2)(J) = \{(x,z) : \exists y : (x,y) \in e_1(J) \text{ and } (y,z) \in e_2(J)\}$.

A DA-expression is called **satisfiable** if there exists a finite binary relation $J$ such that $e(J)$ is nonempty.

**Example 11.** An example of a DA-expression is $e = (R \circ R) - R$. If $J$ is the binary relation $\{(a,b),(b,c),(a,c),(c,d)\}$ then $e(J) = \{(b,d),(a,d)\}$. An example of an unsatisfiable DA expression is $(R \circ R - R) \circ R - R \circ R \circ R$.

We recall the following result. It is actually well known [AGN97] that relational composition together with union and complementation leads to an undecidable algebra; the following result simplifies matters by showing that undecidability already holds for expressions over a single relation symbol and using set difference instead of complementation. The following result has been proven by reduction from the universality problem for context-free grammars.

**Theorem 12 ([TVdBZ14]).** The satisfiability problem for DA-expressions is undecidable.

We are now ready to formulate the reduction from the satisfiability problem for DA to the satisfiability problem for SPARQL(¬bound).

**Lemma 13.** Let $r \in I$ be an arbitrary fixed constant. For any binary relation $J$, let $G_J$ be the RDF graph $\{(c,r,d) \mid (c,d) \in J\}$. Then for every DA-expression $e$ there exists a SPARQL(¬bound) pattern $P_e$ with the following properties:

1. there exist two distinct fixed variables $?x$ and $?y$ such that for every RDF graph $G$ and every $\mu \in [P_e]_G$, $?x$ and $?y$ belong to dom($\mu$);
2. for every binary relation $J$, we have $e(J) = \{(\mu(?x),\mu(?y)) : \mu \in [P_e]_{G,J}\}$;
3. for every RDF graph $G$, we have $[P_e]_G = [P_e]_{G'}$, where $G' := \{(u,v,w) \in G \mid v = r\}$.
Proof. By induction on the structure of \( e \). If \( e = R \) then \( P_e \) is the triple pattern 
\( (?x, r, ?y) \).

If \( e \) is of the form \( e_1 \cup e_2 \), then \( P_e \) is \( P_{e_1} \) UNION \( P_{e_2} \).

If \( e \) is of the form \( e_1 \circ e_2 \), then \( P_e \) is \( P'_{e_1} \) AND \( P'_{e_2} \) where \( P'_{e_1} \) and \( P'_{e_2} \) are obtained as follows. First, by renaming variables, we may assume without loss of generality that \( P_{e_1} \) and \( P_{e_2} \) have no variables in common other than \(?x\) and \(?y\). Let \(?z\) be a fresh variable. Now in \( P_{e_1} \), rename \(?y\) to \(?z\), yielding \( P'_{e_1} \), and in \( P_{e_2} \), rename \(?x\) to \(?z\), yielding \( P'_{e_2} \).

Finally, if \( e \) is of the form \( e_1 - e_2 \), then we use a known idea [AP11]. As before we may assume without loss of generality that \( P_{e_1} \) and \( P_{e_2} \) have no variables in common other than \(?x\) and \(?y\). Let \(?u\) and \(?w\) be two fresh variables. Then \( P_e \) is equal to

\[
(P_{e_1} \text{ OPT } (P_{e_2} \text{ AND } (?u, r, ?w))) \text{ FILTER } \neg \text{ bound(?u)}.
\]

The above lemma provides us with a reduction from satisfiability for DA to satisfiability for SPARQL(\( \neg \text{ bound} \)), thus showing undecidability of the latter problem. Indeed, if \( e \) is satisfiable, then clearly \( P_e \) is satisfiable as well, by property 2 of the lemma. Conversely, if \( P_e \) is satisfiable by some RDF graph \( G \), then, by property 3 of the lemma, \( [P_e]_{\neg \text{ bound}} \) is nonempty. Now define the binary relation \( J = \{(c, d) \mid (c, r, d) \in G\} \). Then \( G_J = G' \), so by property 2 of the lemma we obtain the nonemptiness of \( e(J) \) as desired.

4.2 SPARQL(\( =, \neq \))

We now consider a minor variant of satisfiability for DA-expressions where we restrict attention to binary relations over at least two elements. Formally, the active domain of a binary relation \( J \) is the set of all entries in pairs belonging to \( J \), so \( \text{adom}(J) := \{x \mid \exists y : (x, y) \in J \text{ or } (y, x) \in J\} \). Then a DA-expression \( e \) is called two-satisfiable if \( e(J) \) is nonempty for some \( J \) such that \( \text{adom}(J) \) has at least two distinct elements.

Clearly, two-satisfiability is undecidable as well, for if it were decidable, then satisfiability would be decidable too. Indeed, \( e \) is satisfiable if and only if it is two-satisfiable, or satisfiable by a binary relation \( J \) over a single element. Up to isomorphism there is only one such \( J \) (the singleton \( \{(x, x)\} \)), and DA-expressions commute with isomorphisms.

Lemma 13 can now be adapted as follows. Property 2 of the lemma is only claimed for every binary relations \( J \) over at least two distinct elements. In the proof for the case where \( e \) is \( e_1 - e_2 \), we use six fresh variables \(?u, ?u', ?v, ?v', ?w, \) and \(?w'\). We use the abbreviation \( \text{adom}_{?u} \) for \((?u, r, ?w) \cup \text{UNION } (?v, r, ?u) \) and similarly for \( \text{adom}_{?u'} \). We now use the following pattern for \( P_e \):

\[
\left( (P_{e_1} \text{ OPT } (P_{e_2} \text{ AND } \text{adom}_{?u} \text{ AND } \text{adom}_{?u'})) \text{ FILTER } ?u \neq ?u' \right) \\
\text{ AND } \text{adom}_{?u} \text{ AND } \text{adom}_{?u'} \text{ FILTER } ?u = ?u'.
\]
Let us verify that \( P_e \) satisfies the three properties of Lemma 13.

Proof. 1. By induction, \( P_{e_1} \) has the property that every returned solution mapping has \( \exists x \) and \( \exists y \) in its domain. Since \( P_e \) is of the form

\[
(P_{e_1} \text{ OPT} \ldots) \text{ FILTER} \ldots
\]

the same property holds for \( P_e \).

2. Let \( J \) be a binary relation on at least two distinct elements. To prove the equality

\[
e(J) = \{(\mu(?x), \mu(?y)) \mid \mu \in [P_e]_J\}
\]

we are going to consider both inclusions. For easy reference we name some subpatterns of \( P_e \) as follows.

- \( P_2 \) denotes \((P_{e_2} \text{ AND adom}_u \text{ AND adom}_v) \text{ FILTER } \exists u \neq \exists v'\);
- \( P_3 \) denotes \( P_{e_1} \text{ OPT } P_2 \).
- Thus, \( P \) is \((P_3 \text{ AND adom}_u \text{ AND adom}_v) \text{ FILTER } \exists u = \exists v'\).

To prove the inclusion from right to left, let \( \mu \in [P_e]_J \). Then \( \mu = \mu_3 \cup \varepsilon \), where \( \mu_3 \in [P_3]_J \) and \( \varepsilon \) is a mapping defined on \( \exists u \) and \( \exists v' \) such that \( \varepsilon(?u) = \varepsilon(?v') \). In particular, \( \mu_3 \sim \varepsilon \). Since \( P_3 = P_{e_1} \text{ OPT } P_2 \), there are two possibilities for \( \mu_3 \):

- \( \mu_3 \in [P_{e_1}]_J \), and there is no \( \mu_2 \in [P_2]_J \), such that \( \mu_3 \sim \mu_2 \).
  - By induction, both \( \exists x \) and \( \exists y \) belong to \( \text{dom}(\mu_3) \), so \( (\mu(?x), \mu(?y)) \) equals \( (\mu_3(?x), \mu_3(?y)) \), which belongs to \( e_2(J) \) again by induction.
  - So it remains to show that \( (\mu(?x), \mu(?y)) \notin e_2(J) \). Assume the contrary. Then there exists \( \mu_2' \in [P_{e_2}]_J \), such that \( (\mu_3(?x), \mu_3(?y)) = (\mu_2'(?x), \mu_2'(?y)) \). Since \( \text{adom}(J) \) has at least two distinct elements, \( \mu_2' \) can be extended to a mapping \( \mu_2 \in [P_2]_J \). Since \( \exists x \) and \( \exists y \) are the only variables common to \( \text{var}(P_{e_1}) \) and \( \text{var}(P_2) \), we conclude \( \mu_3 \sim \mu_2 \) which is a contradiction.

- \( \mu_3 = \mu_1 \cup \mu_2 \) with \( \mu_1 \in [P_{e_1}]_J \), and \( \mu_2 \in [P_2]_J \). In particular, \( \mu_3 \) is defined on \( ?u \) and \( ?v' \), and \( \mu_3(?u) \neq \mu_3(?v') \). On the other hand, since \( \mu_3 \sim \varepsilon \), and \( \varepsilon(?u) = \varepsilon(?v') \), also \( \mu_3(?u) = \mu_3(?v') \). This is a contradiction, so the possibility under consideration cannot happen.

To prove the inclusion from left to right, let \( (c, d) \in e_1(J) \). Since \( (c, d) \in e_1(J) \), there exists \( \mu_1 \in [P_{e_1}]_J \), such that \( (c, d) = (\mu_1(?x), \mu_1(?y)) \). Assume, for the sake of argument, that there would exist \( \mu_2 \in [P_2]_J \), such that \( \mu_1 \sim \mu_2 \). Mapping \( \mu_2 \) contains a mapping \( \mu_2' \in [P_{e_2}]_J \), by definition of \( P_2 \). Since \( (\mu_2'?(?x), \mu_2'?(?y)) \in e_2(J) \) and \( \mu_1 \sim \mu_2 \), it follows that \( (c, d) \in e_2(J) \) which is a contradiction.

So, we now know that there does not exist \( \mu_2 \in [P_2]_J \), such that \( \mu_1 \sim \mu_2 \). Hence, \( \mu_1 \in [P_3]_J \). Note that the six variables \( ?u, ?u', ?v, ?v', ?w, \) and
\( ?u' \) do not belong to \( \text{dom}(\mu_1) \). Since \( J \) is nonempty, \( \mu_1 \) can thus be extended to a mapping \( \mu \in [P]_{G^r} \). We conclude \((c,d) = (\mu_1(?x),\mu_1(?y)) = (\mu(?x),\mu(?y))\) as desired.

3. The third property of Lemma 13 holds because \([\text{adom}_u]_G = [\text{adom}_u]_{G^r}\) (and similarly for \( \text{adom}_{u'} \)).

\[
\square
\]

Using the adapted lemma, we can now reduce two-satisfiability for DA to satisfiability for SPARQL\( (=, \neq) \). Indeed, a DA-expression \( e \) is two-satisfiable if and only if the pattern

\[
P_e \land ((\text{adom}_u \land \text{adom}_{u'}) \land ?u \neq ?u')
\]

is satisfiable, where all variables used in \( \text{adom}_u \) and \( \text{adom}_{u'} \) are distinct and disjoint from those used in \( P_e \).

### 4.3 SPARQL\( (=, c) \)

We consider a further variant of two-satisfiability, called \( ab \)-satisfiability, for two arbitrary fixed constants \( a,b \in I \) that are distinct from the constant \( r \) already used for Lemma 13. A DA-expression is called \( ab \)-satisfiable if \( e(J) \) is nonempty for some binary relation \( J \) where \( a,b \in \text{adom}(J) \).

Since DA-expressions do not distinguish between isomorphic binary relations, \( ab \)-satisfiability is equivalent to two-satisfiability, and thus still undecidable.

We now again adapt Lemma 13, as follows. Property 2 is only claimed for every binary relation \( J \) such that \( a,b \in \text{adom}(J) \). In the proof for the case \( e = e_1 - e_2 \), we now use the following pattern for \( P_e \):

\[
\left( (P_{e_1} \land ((P_{e_2} \land \text{adom}_{u}) \land \text{FILTER} ?u = a)) \land \text{adom}_{u} \right) \land \text{FILTER} ?u = b.
\]

The proof correctness of this construction is analogous to the proof given in the previous Section 4.2; instead of exploiting the inconsistency between \( ?u \neq ?u' \) and \( ?u = ?u' \) as done in that proof, we now exploit the inconsistency between \( ?u = a \) and \( ?u = b \).

We then obtain that \( e \) is \( ab \)-satisfiable if and only if

\[
P_e \land (\text{adom}_u \land \text{adom}_{u'}) \land ?u = a \land ?u' = b
\]

is satisfiable, establishing a reduction from \( ab \)-satisfiability for DA to satisfiability for SPARQL\( (=, c) \).

Remark 14. Recall that literals cannot appear in first or second position in an RDF triple. Patterns using constant-equality predicates can be unsatisfiable because of that reason. For example, using the literal 42, the pattern \((?x,?y,?z) \land \text{FILTER} ?y = 42\) is unsatisfiable. However, we have seen here that the use of constant-equality predicates leads to undecidability of satisfiability for a much more fundamental reason, that has nothing to do with literals, namely, the ability to emulate set difference.
5 Satisfiability of well-designed patterns

The well-designed patterns [PAG09] have been identified as a well-behaved class of SPARQL patterns, with properties similar to the conjunctive queries for relational databases [AHV95]. Standard conjunctive queries are always satisfiable, and conjunctive queries extended with equality and nonequality constraints, possibly involving constants, can only be unsatisfiable if the constraints are inconsistent. An analogous behavior is present in what we call AF-patterns: patterns that only use the AND and FILTER operators. We will formalize this in Proposition 15. We will then show in Theorem 18 that a well-designed pattern is satisfiable if and only if its reduction to an AF-pattern is satisfiable. In other words, as far as satisfiability is concerned, well-designed patterns can be treated like AF-patterns.

5.1 Satisfiability of AF-patterns

In Section 3.1 we have associated a set of schemes \( \Gamma(P) \) to every pattern \( P \). When \( \Gamma(P) \) is empty, \( P \) is unsatisfiable (Lemma 4).

Now when \( P \) is an AF-pattern and \( \Gamma(P) \) is nonempty, the satisfiability of \( P \) will turn out to depend solely on the equalities, nonequalities, constant-equalities, and constant-nonequalities occurring as filter conditions in \( P \). We will denote the set of these constraints by \( C(P) \).

Any set \( \Sigma \) of constraints is called consistent if there exists a mapping that satisfies every constraint in \( \Sigma \).

We establish:

**Proposition 15.** An AF-pattern \( P \) is satisfiable if and only if \( \Gamma(P) \) is nonempty and \( C(P) \) is consistent.

**Proof.** The only-if direction of this proposition is given by Lemma 4 together with the observation that if \( \mu \in [P]_G \), then \( \mu \) satisfies every constraint in \( C(P) \). Since \( P \) is satisfiable, such \( G \) and \( \mu \) exist, so \( C(P) \) is consistent.

For the if direction, since \( P \) does not have the UNION and OPT operators, \( \Gamma(P) \) is a singleton \( \{S\} \). Since \( C(P) \) is consistent, there exists a mapping \( \mu : S \rightarrow U \) satisfying every constraint in \( C(P) \). Let \( G \) be the graph consisting of all triples \( (\mu(u), \mu(v), \mu(w)) \) where \( (u, v, w) \) is a triple pattern in \( P \). It is straightforward to show by induction on the height of \( Q \) that for every subpattern \( Q \) of \( P \), we have \( \mu \mid_{S'} \in [Q]_G \), where \( \Gamma(Q) = \{S'\} \). Hence \( \mu \in [P]_G \) and \( P \) is satisfiable.

Note that \( \Gamma(P) \) can “blow up” only because of possible UNION and OPT operators, which are missing in an AF-pattern. Hence, for an AF-pattern \( P \), we can efficiently compute \( \Gamma(P) \) by a single bottom-up pass over \( P \). Moreover, \( C(P) \) is a conjunction of possibly negated equalities and constant equalities. It is well known that consistency of such conjunctions can be decided in polynomial time [KS08]. Hence, we conclude:

**Corollary 16.** Satisfiability for AF-patterns can be checked in polynomial time.
5.2 AF-reduction of well-designed patterns

A well-designed pattern is defined as a union of union-free well-designed patterns. Since a union is satisfiable if and only if one of its terms is, we will focus on union-free patterns in what follows. Formally, a union-free pattern \( P \) is called well-designed [PAG09] if

1. for every subpattern of \( P \) of the form \( Q \ \text{FILTER} \ C \), all variables mentioned in \( C \) also occur in \( Q \); and

2. for every subpattern \( Q \) of \( P \) of the form \( Q_1 \ \text{OPT} \ Q_2 \), and every \( ?x \in \text{var}(Q_2) \), if \( ?x \) also occurs in \( P \) outside of \( Q \), then \( ?x \in \text{var}(Q_1) \).

We associate to every union-free pattern \( P \) an AF-pattern \( \rho(P) \) obtained by removing all applications of OPT and their right operands; the left operand remains in place. Formally, we define the following:

- If \( P \) is a triple pattern, then \( \rho(P) \) equals \( P \).
- If \( P \) is of the form \( P_1 \ \text{AND} \ P_2 \), then \( \rho(P) = \rho(P_1) \ \text{AND} \ \rho(P_2) \).
- If \( P \) is of the form \( P_1 \ \text{FILTER} \ C \), then \( \rho(P) = \rho(P_1) \ \text{FILTER} \ C \).
- If \( P \) is of the form \( P_1 \ \text{OPT} \ P_2 \), then \( \rho(P) = \rho(P_1) \).

For further use we note that \( \Gamma(P) \) and \( \Gamma(\rho(P)) \) are related in the following way. The proof by induction is straightforward.

**Lemma 17.** Let \( S \in \Gamma(P) \) and let \( S' \in \Gamma(\rho(P)) \). Then \( S' \subseteq S \).

The announced result is now given by the following theorem. The if direction of this theorem is already known from a result by Pérez et al. [PAG09, Lemma 4.3].

**Theorem 18.** Let \( P \) be a union-free well-designed pattern. Then \( P \) is satisfiable if and only if \( \rho(P) \) is.

Since \( \rho(P) \) can be efficiently computed from \( P \), the above Theorem and Corollary 16 imply:

**Corollary 19.** Satisfiability of union-free well-designed patterns can be tested in polynomial time.

5.3 Proof

We prove the only-if direction of Theorem 18. We begin by introducing two auxiliary notations.

1. For any pattern \( P \) and subpattern \( Q \) of \( P \), we denote by \( \text{var}'(Q) \) the set of variables from \( \text{var}(Q) \) that also occur in \( P \) outside of \( Q \).
2. When $P$ is an AF-pattern with nonempty $\Gamma(P)$, it is readily seen that $\Gamma(P)$ in that case consists of a single scheme. We denote the unique scheme in $\Gamma(P)$ by $\mathcal{S}(P)$.

The following lemma connects the above two notations:

**Lemma 20.** Let $P$ be a union-free well-designed pattern, and let $Q$ be a sub-pattern of $P$ such that $\Gamma(Q)$ is nonempty. Then $\Gamma(\rho(Q))$ is nonempty as well, and $\text{var}^p(Q) \subseteq \mathcal{S}(\rho(Q))$.

**Proof.** By induction on the height of $Q$. If $Q$ is a triple pattern $(u, v, w)$, then we have $Q = \rho(Q)$ and $\text{var}^p(Q) \subseteq \text{var}(Q) = \{u, v, w\} \cap V = \mathcal{S}(Q) = \mathcal{S}(\rho(Q))$ as desired.

If $Q$ is of the form $Q_1 \text{ AND } Q_2$, then the definition of $\Gamma(Q)$ immediately implies that $\Gamma(Q_1)$ and $\Gamma(Q_2)$ must both be nonempty. Since $\rho(Q) = \rho(Q_1) \text{ AND } \rho(Q_2)$ we then obtain $\mathcal{S}(\rho(Q)) = \mathcal{S}(\rho(Q_1)) \cup \mathcal{S}(\rho(Q_2))$. Any $?x \in \text{var}^p(Q)$ belongs to $\text{var}^p(Q_1)$ or $\text{var}^p(Q_2)$; we assume the former case as the latter case is analogous. By induction, we then have $?x \in \mathcal{S}(\rho(Q_1)) \subseteq \mathcal{S}(\rho(Q))$ as desired.

If $Q$ is of the form $Q_1 \text{ OPT } Q_2$, then $\rho(Q) = \rho(Q_1)$. Recall that $\Gamma(Q) = \Gamma(Q_1) \cup \Gamma(Q_1 \text{ AND } Q_2)$. If $\Gamma(Q_1)$ is nonempty we obtain by induction that $\Gamma(\rho(Q_1)) = \Gamma(\rho(Q))$ is nonempty; if $\Gamma(Q_1 \text{ AND } Q_2)$ is nonempty we obtain $\Gamma(\rho(Q_1))$ nonempty as in the case for AND. So, $\mathcal{S}(\rho(Q))$ exists and is equal to $\mathcal{S}(\rho(Q_1))$. Now let $?x \in \text{var}^p(Q)$. If $?x \in \text{var}^p(Q_1)$ then $?x \in \mathcal{S}(\rho(Q_1))$ by induction. But if $?x \in \text{var}^p(Q_2)$, then also $?x \in \text{var}^p(Q_1)$ since $P$ is well-designed. Hence we are done with this case.

Finally, let $Q$ be of the form $Q_1 \text{ FILTER } C$. Since $\Gamma(Q)$ is nonempty, $\Gamma(Q_1)$ is nonempty as well. To show that $\Gamma(\rho(Q))$ is nonempty we must show that $\mathcal{S}(\rho(Q_1)) \models C$. Thereto, consider a variable $?x$ mentioned in $C$. Since $P$ is well-designed, $?x \in \text{var}(Q_1)$ and thus $?x \in \text{var}^p(Q_1)$. By induction we obtain $?x \in \mathcal{S}(\rho(Q_1))$. By Lemma 17, then also $?x \in \mathcal{S}$ for every $\mathcal{S} \in \Gamma(Q_1)$. In other words, $\mathcal{S} \not\models \text{bound}(?x)$ for every $\mathcal{S} \in \Gamma(Q_1)$. This rules out the possibility that $C$ is a negated bound-constraint, since we are given that $\Gamma(Q)$ is nonempty. On the other hand, this argument also shows that $\mathcal{S}(\rho(Q_1)) \models C$ in the other cases, where $C$ is a bound-constraint or an (constant) (non)equality, as desired.

It remains to show that $\text{var}^p(Q) \subseteq \mathcal{S}(\rho(Q)) = \mathcal{S}(\rho(Q_1))$. Let $?x \in \text{var}^p(Q)$. If $?x \in \text{var}(Q_1)$ the result follows by induction. If $?x$ occurs in $C$ then, because $P$ is well-designed, also $?x \in \text{var}(Q_1)$ and thus we are done. 

We mention in passing an interesting corollary of the reasoning in the above proof, to the effect that well-designedness rules out any nontrivial use of negated bound-constraints:

**Corollary 21.** If $P$ is a union-free well-designed pattern and $Q$ is a subpattern of $P$ of the form $Q_1 \text{ FILTER } \neg\text{bound}(?x)$, then $\Gamma(Q)$ is empty, in particular, $Q$ is unsatisfiable.

We are now ready to make the final step in the proof of Theorem 18:
Lemma 22. Let $P$ be a union-free well-designed pattern. If $\mu \in [P]_G$ and $\Gamma(\rho(P))$ is nonempty, then $\mu|_{S(\rho(P))} \in [\rho(P)]_G$.

Proof. By induction on the structure of $P$. If $P$ is a triple pattern, then the claim is trivial.

So let $P$ be of the form $P_1 \text{ AND } P_2$. Since $\Gamma(\rho(P))$ is nonempty and $\rho(P) = \rho(P_1) \text{ AND } \rho(P_2)$, also $\Gamma(\rho(P_i))$ is nonempty for $i = 1, 2$. Then by induction, $\mu|_{S(\rho(P_i))} \in [\rho(P_i)]_G$. Since they are restrictions of the same mapping $\mu$, we also have $\mu|_{S(\rho(P_i))} \sim \mu|_{S(\rho(P_2))}$, so the mapping $\mu|_{S(\rho(P))} \cup \mu|_{S(\rho(P_2))}$ belongs to $[\rho(P)]_G$. Since $S(\rho(P)) = S(\rho(P_1)) \cup S(\rho(P_2))$, we obtain $\mu|_{S(\rho(P))} \in [\rho(P)]_G$ as desired.

If $P$ is of the form $P_1 \text{ OPT } P_2$, then we have $\rho(P) = \rho(P_1)$, so we are given that $\Gamma(\rho(P_1))$ is nonempty. By induction, $\mu|_{S(\rho(P_1))} \in [\rho(P_1)]_G = [\rho(P)]_G$ as desired.

Finally, if $P$ is of the form $P_1 \text{ FILTER } C$ then by the nonemptiness of $\Gamma(\rho(P))$ we know that $S(\rho(P_1)) \models C$ and $S(\rho(P)) = S(\rho(P_1))$. Hence, by induction, $\mu|_{S(\rho(P_1))} \in [\rho(P_1)]_G$. It remains to show that $\mu|_{S(\rho(P_1))} \models C$, but this follows immediately because $\mu \models C$ and $S(\rho(P_1)) \models C$.

With the above lemmas in hand, the only-if direction of Theorem 18 can now be argued as follows. Since $P$ is satisfiable, $\Gamma(P)$ is nonempty by Lemma 4. By Lemma 20 applied to $Q = P$, also $\Gamma(\rho(P))$ is nonempty. Since $P$ is satisfiable, there exist $G$ and $\mu$ such that $\mu \in [P]_G$. Now applying Lemma 22 yields that $[\rho(P)]_G$ is nonempty. We conclude that $\rho(P)$ is satisfiable.

6 Experimental evaluation

We want to evaluate experimentally the positive results presented so far:

1. Wrong literal reduction (Proposition 24);

2. Satisfiability checking for the two fragments SPARQL(bound, $=$, $\neq$) and SPARQL(bound, $\neq$, $\neq$) by computing $\Gamma(P)$ (Theorem 3);

3. Satisfiability checking for well-designed patterns, by reduction to AF-patterns (Proposition 15 and Theorem 18).

Our experiments follow up on those reported earlier by the third author and Vansummeren [PV11]. As test datasets of real-life SPARQL queries, we use logs of the SPARQL endpoint for DBpedia, available at ftp://download.openlinksw.com/support/dbpedia/. This data source contains the “query dumps” from the year 2012, divided into 14 logfiles. Out of these we chose the three logs 20120913, 20120929 and 20121031 to obtain a span of roughly three months; we then took a sample of 100,000 queries from each of them. A typical query in the log has size between 75 and 125 (size measured as number of nodes in the syntax tree). About 10% of the queries in each log is not usable because they have syntax errors or because they use features not covered by our analysis.
Table 1: Timings of experiments (averaged over five repeats). Times are in ms. Baseline is time to read and parse 1000000 queries; WL stands for baseline plus time for wrong-literal reduction. $\Gamma(P)$ stands for WL plus time for computing $\Gamma(P)$. AF stands for baseline, plus testing well-designedness, plus doing AF-reduction and testing satisfiability (Proposition 15). The percentages show the increases relative to the baseline.

<table>
<thead>
<tr>
<th>logfile</th>
<th>baseline</th>
<th>WL</th>
<th>$\Gamma(P)$</th>
<th>AF</th>
</tr>
</thead>
<tbody>
<tr>
<td>20120913</td>
<td>39220</td>
<td>41254</td>
<td>44395</td>
<td>48329</td>
</tr>
<tr>
<td>20120929</td>
<td>34234</td>
<td>35868</td>
<td>38102</td>
<td>41087</td>
</tr>
<tr>
<td>20121031</td>
<td>32286</td>
<td>33186</td>
<td>34419</td>
<td>36993</td>
</tr>
</tbody>
</table>

The implementation of the tests was done in Java 7 under Windows 7, on an Intel Core 2 Duo SU9400 processor (1.40GHz, 800MHz, 3MB) with 3GB of memory (SDRAM DDR3 at 1067MHz).

Our tests measure the time needed to perform the analyses of SPARQL queries presented above. The timings are averaged over all queries in a log, and each experiment is repeated five times to smooth out accidental quirks of the operating system. Although we give absolute timings, the main emphasis is on the percentage of the time needed to analyse a query, with respect to the time needed simply to read and parse that query. If this percentage is small this demonstrates efficient, linear time complexity in practice. It will turn out that this is indeed achieved by our experiments, as shown in Table 1.

In the following subsections we discuss the results in more detail.

6.1 Wrong literal reduction

Testing for and removing triple patterns with wrong literals in a pattern $P$ is performed by the reduction $\lambda(P)$ defined in the Appendix. From the definition of $\lambda(P)$ it is clear that it can be computed by a single bottom-up traversal of $P$ and this is indeed borne out by our experiments. Table 1 shows that on average, wrong-literal reduction takes between 3 and 5% of the time needed to read and parse the input.

Interestingly, some real-life queries with literals in the wrong position were indeed found; one example is the following:

```
SELECT DISTINCT *
WHERE { 49 dbpedia-owl:wikiPageRedirects ?redirectLink .}
```

6.2 Computing $\Gamma(P)$

In Section 3 we have seen that satisfiability for the decidable fragments can be tested by computing $\Gamma(P)$, but that the problem is NP-complete. Intuitively, the problem is intractable because $\Gamma(P)$ may be of size exponential in the size
of $P$. This actually occurs in real life; a common SPARQL query pattern is to use many nested OPTIONAL operators to gather additional information that is not strictly required by the query but may or may not be present. We found in our experiments queries with up to 50 nested OPT operators, which naively would lead to a $\Gamma(P)$ of size $2^{50}$. A shortened example of such a query is shown in Figure 1.

In practice, however, the blowup of $\Gamma(P)$ can be avoided as follows. Recall that Theorem 3 states that $P$ is satisfiable if and only if $\Gamma(P)$ is nonempty. The elements of $\Gamma(P)$ are sets of variables. Looking at the definition of $\Gamma(P)$, a set may be removed from $\Gamma(P)$ only by the application of a FILTER. Hence, only variables that are mentioned in FILTER conditions can influence the emptiness of $\Gamma(P)$; other variables can be ignored. For example, in the query in Figure 1, only two variables appear in a filter, namely ?ontology_abstract and ?ontology_motto, so that the maximal size of $\Gamma(P)$ is reduced to $2^2$.

In our experiments, it turns out that typically few variables are involved in filter conditions. Hence, the above strategy works well in practice.

Another practical issue is that, in this paper, we have only considered filter conditions that are bound checks, equalities, and constant-equalities, possibly negated. In practice, filter conditions typically apply built-in SPARQL predicates such as the predicate langMatches in Figure 1. For the experimental purpose of testing the practicality of computing $\Gamma(P)$, however, such predicates can simply be treated as bound checks. In this way we can apply our experiments to 70% of the queries in the testfiles.

With the above practical adaptations, our experiments show that computing $\Gamma(P)$ is efficient: Table 1 shows that it requires, on average, between 4 and 8% of the time needed to read and parse the input, and these timings even include the wrong-literal reduction.

### 6.3 Satisfiability testing for well-designed patterns

In Section 5 we have seen that testing satisfiability of a well-designed pattern can be done by testing satisfiability of the AF-reduction (Theorem 18). The latter can be done by testing nonemptiness of $\Gamma(P)$ and testing consistency of the filter conditions (Proposition 15).

Computing the AF-reduction can be done by a simple bottom-up traversal of the pattern. Moreover, for an AF-pattern $P$, computing $\Gamma(P)$ poses no problems since it is either empty or a singleton. As far as testing consistency of filter conditions is concerned, our experiments yield a rather baffling observation: almost all well-designed patterns in the test sets have no filters at all. We cannot explain this phenomenon, but it implies that we have not been able to test the performance of the consistency checks on real-life SPARQL queries.

Anyhow, Table 1 shows that doing the entire analysis of wrong-literal reduction, testing well-designedness, AF-reduction, computing $\Gamma(P)$, and consistency checking (in the few cases where the latter was necessary), incurs at most a 10% increase relative to reading and parsing the input.
Figure 1: A real-life query with many nested OPTIONAL operators, retrieving as much information as possible about universities in Brazil.
Table 2: Scalability experiment (times in ms). Timings clearly scale linearly for increasing input size.

<table>
<thead>
<tr>
<th>input size</th>
<th>200,000</th>
<th>100,000</th>
<th>50,000</th>
<th>10,000</th>
<th>5,000</th>
<th>Pearson coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>baseline</td>
<td>74.168</td>
<td>39.422</td>
<td>21.315</td>
<td>3.596</td>
<td>1.851</td>
<td>0.999924005</td>
</tr>
<tr>
<td>WL</td>
<td>77.800</td>
<td>41.253</td>
<td>21.876</td>
<td>3.762</td>
<td>1.942</td>
<td>0.999989454</td>
</tr>
<tr>
<td>Γ(P)</td>
<td>81.730</td>
<td>44.395</td>
<td>23.552</td>
<td>4.016</td>
<td>2.036</td>
<td>0.999900948</td>
</tr>
<tr>
<td>AF</td>
<td>91.470</td>
<td>48.329</td>
<td>26.023</td>
<td>4.463</td>
<td>2.254</td>
<td>0.999044542</td>
</tr>
</tbody>
</table>

6.4 Scalability

The experiments described above were run on sets of 100,000 queries each. We also did a modest scaling experiment where we varied the number of queries from 5,000 to 200,000. Table 2 shows that the performance scales linearly.

7 Extension to SPARQL 1.1

As already mentioned in the Introduction, SPARQL 1.0 has been extended to SPARQL 1.1 with a number of new operators for building patterns. The main new features are property paths; grouping and aggregates; BIND; VALUES; MINUS; EXISTS and NOT EXISTS-subqueries; and SELECT. A complete analysis of SPARQL 1.1 goes beyond the scope of the present paper. Nevertheless, in this section, we briefly discuss how our results may be extended to this new setting.

Property paths provide a form of regular path querying over graphs. This aspect of graph querying has already been extensively investigated, including questions of satisfiability and other kinds of static analysis such as query containment [KRV14, KRRV15]. Therefore we do not discuss property paths any further here.

The SPARQL 1.1 features that we discuss can be grouped in two categories: those that cause undecidability, and those that are harmless as far as satisfiability is concerned. We begin with the harmless category.

7.1 SELECT operator and EXISTS-subqueries

SPARQL 1.1 allows patterns of the form SELECTₘP, where S is a finite set of variables and P is a pattern. The semantics is that of projection: solution mappings are restricted to the variables listed in S. Formally, we define

\[ [\text{SELECT}_S P]_G = \{ \mu | [S \cap \text{dom}(\mu)] \mid \mu \in [P]_G \}. \]

This feature in itself does not influence the satisfiability of patterns. Indeed, patterns extended with SELECT operators can be reduced to patterns without said operators. The reduction amounts simply to rename the variables that
are projected out by fresh variables that are not used anywhere else in the
pattern; then the SELECT operators themselves can be removed. The resulting,
SELECT-free, pattern is equivalent to the original one if we omit the freshly
introduced variables from the solution mappings in the final result. In particular,
the two patterns are equisatisfiable.

Example 23. Rather than giving the formal definition of SELECT-reduction and
formally stating and proving the equivalence, we give an example. Consider the
pattern $P$:

$$(c, p, ?x) \text{ OPT } ((?x, p, ?y) \text{ AND } \text{SELECT}_y(?y, q, ?z) \text{ AND } \text{SELECT}_y(?y, r, ?z))$$

Renaming projected-out variables by fresh variables and omitting the SELECT
operators yields the following pattern $P'$:

$$(c, p, ?x) \text{ OPT } ((?x, p, ?y) \text{ AND } (?y, q, ?z_1) \text{ AND } (?y, r, ?z_2))$$

Pattern $P'$ is equivalent to $P$ in the sense that for any graph $G$, we have $[P]_G =
\{\hat{\mu} | \mu \in [P']_G\}$, where $\hat{\mu}$ denotes the mapping obtained from
$\mu$ by omitting the values for $?z_1$ and $?z_2$ (if at all present in dom($\mu$)).

Now that we know how to handle SELECT operators, we can also handle
EXISTS-subqueries. Indeed, a pattern $P \text{ FILTER EXISTS}(Q)$ (with the obvious
SQL-like semantics) is equivalent to $\text{SELECT}_{\text{var}(P)}(P \text{ AND } Q)$.

7.2 Features leading to undecidability

In Section 4 we have seen that as soon as one can express the union, composition
and difference of binary relations, the satisfiability problem becomes undecid-
able. Since union and composition are readily expressed in basic SPARQL
(UNION and AND), the key lies in the expressibility of the difference operator.
In this subsection we will see that various new features of SPARQL 1.1 indeed
allow expressing difference.

MINUS operator and NOT EXISTS subqueries  Any of these two fea-
tures can quite obviously be used to express difference, so we do not dwell on
them any further.

Grouping and aggregates  A known trick for expressing difference using
grouping and counting [Cel05] can be emulated in the extension of SPARQL 1.0
with grouping. We illustrate the technique with an example.

Consider the query $(?x, p, ?y) \text{ MINUS } (?x, q, ?y)$ asking for all pairs $(a, b)$
such that $(a, p, b)$ holds but $(a, q, b)$ does not. We can express this query (with
the obvious SQL-like semantics) as follows:

SELECT$_{?x, ?y}$($(?x, p, ?y) \text{ OPT } ((?x, q, ?y) \text{ AND } (?x, p, ?yy))$

GROUP BY $?x, ?y$

HAVING count(?xx) = 0

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Note that this technique of looking for the (?x, ?y) groups with a zero count for ?xx is very similar to the technique used to express difference using a negated bound constraint (seen in the proof of Lemma 13).

**BIND and VALUES** We have seen in Section 4.3 that allowing constant equalities in filter constraints allows us to emulate the difference operator. Two mechanisms introduced in SPARQL 1.1, BIND and VALUES, allow the introduction of constants in solution mappings. Together with equality constraints this allows us to express constant equalities, and hence, difference.

Specifically, using VALUES, we can express $P \text{ FILTER } ?x = c$ as

$$SELECT_{\text{var}(P)}(P \text{ AND VALUES}_x(c)).$$

Using BIND, it can be expressed as

$$SELECT_{\text{var}(P)}((P \text{ BIND}_x ?x' (c)) \text{ FILTER } ?x = ?x')$$

where ?x' is a fresh variable. Note the use of SELECT, which, however, does not influence satisfiability as discussed above. We conclude that SPARQL(=) extended with BIND, or SPARQL(=) extended with VALUES, have an undecidable satisfiability problem.

### 8 Conclusion

The results of this paper may be summarized by saying that, as long as the kinds of constraints allowed in filter conditions cannot be combined to yield inconsistent sets of constraints, satisfiability for SPARQL patterns is decidable; otherwise, the problem is undecidable. Moreover, for well-designed patterns, satisfiability is decidable as well. All our positive results yield straightforward bottom-up syntactic checks that can be implemented efficiently in practice.

We thus have attempted to paint a rather complete picture of the satisfiability problem for SPARQL 1.0. Of course, satisfiability is only the most basic automated reasoning task. One may now move on to more complex tasks such as equivalence, implication, containment, or query answering over ontologies. Indeed, investigations along this line for limited fragments of SPARQL are already happening [LPPS13, WEGL12, KG13, CGMSH12] and we hope that our work may serve to provide some additional grounding to these investigations.

We also note that in query optimization it is standard to check for satisfiability of subexpressions, to avoid executing useless code. Some specific works on SPARQL query optimization [SM13, GGK09] do mention that inconsistent constraints can cause unsatisfiability, but they have not provided sound and complete characterizations of satisfiability, like we have offered in this paper. Thus, our results will be useful in this direction as well.
Acknowledgment

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References


Appendix

Literals in the wrong place in triple patterns are easily dealt with in the following manner. We define the wrong-literal reduction of a pattern \( P \), denoted by \( \lambda(P) \), as a set that is either empty or is a singleton containing a single pattern \( P' \):

- If \( P \) is a triple pattern \((u, v, w)\) and \( u \) is a literal, then \( \lambda(P) := \emptyset \); else \( \lambda(P) := \{P\} \).
- \( \lambda(P_1 \cup P_2) := \lambda(P_1) \cup \lambda(P_2) \) if \( \lambda(P_1) \) or \( \lambda(P_2) \) is empty;
- \( \lambda(P_1 \cup P_2) := \{P'_1 \cup P'_2 \mid P'_1 \in \lambda(P_1) \text{ and } P'_2 \in \lambda(P_2)\} \) otherwise.
- \( \lambda(P_1 \land P_2) := \{P'_1 \land P'_2 \mid P'_1 \in \lambda(P_1) \text{ and } P'_2 \in \lambda(P_2)\} \).
- \( \lambda(P_1 \text{ OPT } P_2) := \emptyset \) if \( \lambda(P_1) \) is empty;
- \( \lambda(P_1 \text{ OPT } P_2) := \lambda(P_1) \) if \( \lambda(P_2) \) is empty but \( \lambda(P_1) \) is nonempty;
- \( \lambda(P_1 \text{ OPT } P_2) := \{P'_1 \text{ OPT } P'_2 \mid P'_1 \in \lambda(P_1) \text{ and } P'_2 \in \lambda(P_2)\} \) otherwise.
- \( \lambda(P_1 \text{ FILTER } C) := \{P'_1 \text{ FILTER } C \mid P'_1 \in \lambda(P_1)\} \).

Note that the wrong-literal reduction never has a literal in the subject position of a triple pattern. The next proposition shows that, as far as satisfiability checking is concerned, we may always perform the wrong-literal reduction.

**Proposition 24.** Let \( P \) be a pattern. If \( \lambda(P) \) is empty then \( P \) is unsatisfiable; if \( \lambda(P) = \{P'\} \) then \( P \) and \( P' \) are equivalent, i.e., \( [P]_G = [P']_G \) for every RDF graph \( G \). Moreover, if \( \lambda(P) = \{P'\} \) then \( P' \) does not contain any triple pattern \((u, v, w)\) where \( u \) is a literal.

**Proof.** Assume \( P \) is a triple pattern \((u, v, w)\) and \( u \) is a literal, so that \( \lambda(P) = \emptyset \). Since \( u \) is a constant, \( \mu(u) \) equals the literal \( u \) for every solution mapping \( \mu \). Since no triple in an RDF graph can have a literal in its first position, \( [P]_G \) is empty for every RDF graph \( G \), i.e., \( P \) is unsatisfiable. If \( u \) is not a literal, \( \lambda(P) = \{P\} \) and the claims of the Proposition are trivial.

If \( P \) is of the form \( P_1 \cup P_2 \), or \( P_1 \land P_2 \), or \( P_1 \text{ OPT } P_2 \), or \( P_1 \text{ FILTER } C \), the claims of the Proposition follow straightforwardly by induction.

If \( P \) is of the form \( P_1 \text{ OPT } P_2 \), there are three cases to consider:

- If \( \lambda(P_1) \) is empty then so is \( \lambda(P) \). In this case, by induction, \( P_1 \) is unsatisfiable, whence so is \( P \).
- If \( \lambda(P_1) = \{P'_1\} \) is nonempty but \( \lambda(P_2) \) is empty, then \( \lambda(P) = \{P'_1\} \). By induction, \( P_2 \) is unsatisfiable. Hence, \( P \) is equivalent to \( P_1 \), which in turn is equivalent to \( P'_1 \) by induction. That \( P'_1 \) does not contain any triple pattern with a literal in first position again follows by induction.
• If \( \lambda(P_1) = \{ P_1' \} \) and \( \lambda(P_2) = \{ P_2' \} \) are both nonempty, then \( \lambda(P) = P_1' \text{OPT} P_2' \). By induction, \( P_1 \) is equivalent to \( P_1' \) and so is \( P_2 \) to \( P_2' \). Hence, \( P \) is equivalent to \( P_1' \text{OPT} P_2' \) as desired. By induction, neither \( P_1' \) nor \( P_2' \) contain any triple pattern with a literal in first position, so neither does \( P_1' \text{OPT} P_2' \).