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Bounds on the Cost of Stabilizing a Cooperative Game

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Abstract

A key issue in cooperative game theory is coalitional stability, usually captured by the notion of the *core*—the set of outcomes that are resistant to group deviations. However, some coalitional games have empty cores, and any outcome in such a game is unstable. We investigate the possibility of stabilizing a coalitional game by using subsidies. We consider scenarios where an external party that is interested in having the players work together offers a supplemental payment to the grand coalition, or, more generally, a particular coalition structure. This payment is conditional on players not deviating from this coalition structure, and may be divided among the players in any way they wish. We define the *cost of stability* as the minimum external payment that stabilizes the game. We provide tight bounds on the cost of stability, both for games where the coalitional values are nonnegative (profit-sharing games) and for games where the coalitional values are nonpositive (cost-sharing games), under natural assumptions on the characteristic function, such as superadditivity, anonymity, or both. We also investigate the relationship between the cost of stability and several variants of the least core. Finally, we study the computational complexity of problems related to the cost of stability, with a focus on weighted voting games.

1. Introduction

There are many settings where self-interested agents find it profitable to cooperate and form teams in order to achieve their individual goals: Workers with diverse skills might want to found a start-up company; software agents can pool their resources to perform large tasks; and firms can join databases they own in order to uncover new patterns. Such settings are modeled using the toolkit of *coalitional game theory*, which studies what teams, or *coalitions*, are most likely to arise and how their members should distribute the gains from cooperation.

An important consideration in identifying acceptable outcomes of a coalitional game is *stability*: The agents should prefer the current outcome to the ones they can feasibly achieve by deviating. The most prominent solution concept that aims at formalizing the idea of stability in coalitional games is the *core*: An outcome is said to be in the core if it distributes the gains or costs so that no subset of agents has an incentive to abandon the existing arrangement and form a coalition of their own. However, a coalitional game may have an empty core. This observation has led to the development of several alternative solution concepts, which are based on relaxing the core constraints (this includes the ε-core, the least core, and the (pre)nucleolus, Shapley & Shubik, 1966; Schmeidler, 1969) or employing alternative notions of stability, such as the kernel and the bargaining set (Davis & Maschler, 1965, 1967).

In this paper, we examine the possibility of stabilizing an outcome of a game by means of subsidies. That is, we investigate settings where an external party, which can be seen as a central authority interested in a stable outcome of the interaction, attempts to incentivize the agents to cooperate. This party implements its agenda by offering the agents a supplemental payment that is conditional on the agents working together as a team. This payment is given to the grand coalition as a whole, and can be divided arbitrarily among its members. We call the minimum payment necessary to stabilize the game the *cost of stability*. It is useful both as a measure of the inherent instability of a given game (similarly to the value of the least core), and as a design metric for an external party that considers subsidizing the agents. It is convenient to distinguish between the minimum subsidy itself, which we refer to as the *additive cost of stability*, and the ratio between the total payment (the subsidy plus earnings) and the maximum amount that the agents can earn on their own, which we call the *multiplicative cost of stability*, though of course the two quantities are closely related.

The following example illustrates the concept of the cost of stability.

Example 1 (Sharing the cost) *Three private hospitals in a large city plan to purchase an X-ray machine. The standard X-ray machine costs \$5 million, and can fulfill the needs of up to two hospitals. There is also a more advanced machine, which is capable of serving all three hospitals, but costs \$9 million. The hospital managers understand that the right thing to do is to buy the more expensive machine, which will serve all three hospitals and cost less than two standard machines, but cannot agree on how to share the cost: There will always be a pair of hospitals that (together) pay at least \$6 million, and would then rather split off and buy the cheaper machine for themselves. The generous mayor solves the problem by subsidizing the advanced X-ray machine: She offers to contribute \$3 million, and asks each hospital to add \$2 million. Pairs of hospitals now have no incentive to buy the cheaper machine, as each pair (together) only pays \$4 million.*

When the supplemental payment is large enough, the resulting outcome is stable: The profit that the deviators can make on their own is dwarfed by the subsidy they could receive by sticking to the prescribed solution. For instance, an easy way to stabilize the game above would be for the mayor to pay for the advanced X-ray machine (a subsidy of \$9 million in total). However, normally the external party would want to minimize its expenditure: In our example, a total subsidy of \$1.5 million will suffice, thus the additive cost of stability of the game in this example is \$1.5 million.

The observation that an external party may be willing to subsidize the grand coalition in the interest of stability has a long history; we provide an overview of this stream of work in Section 7. However, most prior papers focus on analyzing the cost of stability (or its variants) in specific combinatorial optimization games. In contrast, the goal of this paper is to establish general bounds for classes of games that are characterized by simple axiomatic conditions.

In Section 2, we present the required notions from cooperative game theory; for more background, the reader is referred to the books by Peleg and Sudhölter (2003) and Chalkiadakis, Elkind, and Wooldridge (2011) and to the book chapter by Elkind and Rothe (2015). In Section 3, we introduce the notion of cost of stability and list some easy observations. We then present our main contribution, which is twofold. First, we provide tight bounds on the (multiplicative) cost of stability in games where the characteristic function satisfies additional constraints, such as superadditivity, anonymity, or both (Section 4). These results are summarized in Table 1 on page 1014. Second, we explore the relationship between the cost of stability and several variants of the least core (Section 5 and Table 2 on page 1014). Notably, some of our results rely on the tight connection between the cost of stability and the Bondareva–Shapley theorem (see Section 3.2 for details), as well as on a characterization of the core of superadditive games due to Shapley (1967). In Section 6, we study the algorithmic properties of problems related to the cost of stability in the context of a specific compactly representable class of cooperative games, namely weighted voting games. Related work is discussed in Section 7, and we conclude in Section 8 with a brief discussion of future work.

2. Preliminaries

For brevity, we only provide formal definitions for games with positive values (where the players share the profits of cooperation). The symmetric case where players share expenses, or costs, is similar; we comment where special attention is required.

Given two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, we write $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also, given a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ and a set $S \subseteq \{1, \ldots, n\}$, we write $x(S)$ to denote $\sum_{i \in S} x_i$.

A *transferable utility game* (*TU game*; note that we sometimes drop "TU") is a pair $G = \langle N, v \rangle$, where $N = \{1, \ldots, n\}$ is a set of n players (or agents) and $v : 2^N \to \mathbb{R}_+$ is the characteristic function. By convention, $v(\emptyset) = 0$ and $v(N) > 0$. Subsets of N are called *coalitions*; the set N itself is called the *grand coalition*. A *payoff vector* for a TU game $G = \langle N, v \rangle$ is a vector $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$, where p_i is the share of player i. A payoff vector $\boldsymbol{p}$ is *budget-balanced*, or a *pre-imputation*, if $\sum_{i \in N} p_i = v(N)$. In general, we do not impose any constraints on the signs of the entries of $\boldsymbol{p}$. We denote by $\mathbb{I}(G)$ the set of all pre-imputations for G. A pre-imputation $\boldsymbol{p} \in \mathbb{I}(G)$ is said to be an *imputation* if it is *individually rational*, i.e., $p_i \geq v(\{i\})$ for all $i \in N$. We say that a vector $\boldsymbol{p} \in \mathbb{R}^n$ is a *super-imputation* for a game $G = \langle N, v \rangle$ if $p_i \geq 0$ for all $i \in N$ and $p(N) \geq v(N)$. A coalition S *blocks a payoff vector* $\boldsymbol{p}$ if $v(S) > p(S)$. A payoff vector is *stable* if no coalition blocks it. We denote the set of all stable payoff vectors by $\mathbb{S}(G)$. The *core of* G, denoted by $\mathbb{C}(G)$, is the set of all payoff vectors that are both stable and budget-balanced. Thus $\mathbb{C}(G) = \mathbb{I}(G) \cap \mathbb{S}(G)$.

A game $G = \langle N, v\rangle$ is called *monotone* if $v(S) \leq v(T)$ for all coalitions S and T with $S \subseteq T$; it is called *simple* if it is monotone and $v(S) \in \{0, 1\}$ for all $S \subseteq N$. Given a simple game $G = \langle N, v\rangle$, we say that a coalition $S \subseteq N$ *wins* if $v(S) = 1$, and it *loses* if $v(S) = 0$. A player i in a simple game $G = \langle N, v\rangle$ is called a *veto player* if $v(N \setminus \{i\}) = 0$; it is known that a simple game G has an empty core if and only if no player in this game is a veto player (see, e.g., Peleg & Sudhölter, 2003). A game $G = \langle N, v\rangle$ is called *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. (For cost-sharing games, the analogous property is *subadditivity*, where the inequality is flipped.) A game $G = \langle N, v\rangle$ is called *anonymous* if the payoff of a coalition depends on its size only, i.e., $v(S) = v(T)$ whenever $|S| = |T|$. Given an anonymous game $G = \langle N, v\rangle$, for every $k = 1, \dots, n$ we define $v_k = v(\{1, \dots, k\})$; we have $v(S) = v_k$ for every coalition $S \subseteq N$ of size k.

3. The Cost of Stability

In this section, we first define the notion of cost of stability and state some easy observations. We then explore the relationship between the cost of stability and the Bondareva–Shapley theorem.

3.1 Definition and Some Easy Observations

We consider settings where an external authority can provide a subsidy that increases the value of the grand coalition. This subsidy is conditional on agents forming the grand coalition, and can be divided arbitrarily among them. We will refer to the new game that arises as a result of this subsidy as the *adjusted coalitional game*. Technically, this game is derived from the original game by relaxing the budget-balance requirement.

Definition 1 *Given a TU game $G = \langle N, v\rangle$ and a real value $\Delta \geq 0$, the* adjusted coalitional game *$G(\Delta) = \langle N, v'\rangle$ is given by*

$$v'(S) = \begin{cases} v(S) & \textit{if } S \neq N \\ v(S) + \Delta & \textit{if } S = N. \end{cases}$$

We will refer to the quantity Δ as the *subsidy* for the game $G = \langle N, v\rangle$. Note that a super-imputation $\boldsymbol{p}$ with $p(N) = v(N) + \Delta$ distributes the adjusted gains, i.e., it is a pre-imputation for $G(\Delta)$; it is stable if and only if it is in the core of $G(\Delta)$. We say that a subsidy Δ *stabilizes* the game G if the adjusted game $G(\Delta)$ has a nonempty core.

Observe that every TU game can be stabilized by an appropriate choice of Δ: We can set $\Delta = n \max_{S\subseteq N} v(S)$ and distribute the profits so that each player receives at least $\max_{S\subseteq N} v(S)$. However, the central authority typically wants to spend as little money as possible. Hence, we are interested in the *smallest* subsidy that stabilizes the grand coalition.

Definition 2 *Given a TU game $G = \langle N, v\rangle$, its* additive cost of stability *is the quantity*

$$\mathrm{CoS}^{+}(G) = \inf\{\Delta \in \mathbb{R}_{+} \mid \mathbb{C}(G(\Delta)) \neq \emptyset\} = \inf\{p(N) - v(N) \mid \boldsymbol{p} \in \mathbb{S}(G)\}, \tag{1}$$

and its multiplicative cost of stability *is the quantity*

$$\mathrm{CoS}^{\times}(G) \;=\; \frac{\mathrm{CoS}^{+}(G) + v(N)}{v(N)} \;=\; \inf\left\{\frac{p(N)}{v(N)} \;\middle|\; \boldsymbol{p} \in \mathbb{S}(G)\right\}. \tag{2}$$

In what follows, we will alternate between the additive and the multiplicative notation; typically, results for the additive cost of stability can be restated for its multiplicative sibling, and vice versa. Note that $\mathrm{CoS}^{\times}(G) \geq 1$.[1]

We will denote the game $G(\mathrm{CoS}^{+}(G))$ by $\overline{G} = \langle N, \overline{v}\rangle$. As argued above, the constraint $\Delta \in \mathbb{R}_{+}$ in (1) can be replaced with $0 \leq \Delta \leq n \max_{S \subseteq N} v(S)$. Since the inequalities defining $\mathbb{S}(G)$ are not strict, $\mathbb{S}(G)$ is a closed set, so the infimum in (1) is in fact a *minimum*, and thus $\overline{G}$ has a nonempty core.

To illustrate the notions introduced in this section, we provide two examples of analyzing the cost of stability.

Our first example is weighted voting games where all players have the same weight. Recall that a *weighted voting game (WVG)* is a simple game given by a set of agents $N = \{1, \ldots, n\}$, a vector of agents' *weights* $\boldsymbol{w} = (w_1, \ldots, w_n) \in (\mathbb{R}_{+})^n$, and a *quota* $q \in \mathbb{R}_{+}$; we write $G = [\boldsymbol{w}; q]$, or, dropping the vector parentheses, $G = [w_1, \ldots, w_n; q]$. The *weight* of a coalition $S \subseteq N$ is $w(S) = \sum_{i \in S} w_i$; we require $0 < q \leq w(N)$. A coalition S wins if $w(S) \geq q$ and loses otherwise. For weighted voting games with $w_i = w$ for all $i \in N$ we can give an explicit formula for the cost of stability.

Proposition 1 *In a weighted voting game $G = [w, w, \ldots, w; q]$ with n players, we have*

$$\mathrm{CoS}^{\times}(G) = \frac{n}{\lceil q/w \rceil}.$$

Equivalently, $\mathrm{CoS}^{+}(G) = \frac{n}{\lceil q/w \rceil} - 1$.

Proof. By scaling w and q we can assume that $w = 1$. Set $\Delta = n/\lceil q \rceil - 1$ and note that since $q \leq nw$, $\Delta \geq 0$.

Consider the payoff vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ given by $p_i = 1/\lceil q \rceil$ for $i = 1, \ldots, n$. Clearly, we have $p(N) = n/\lceil q \rceil$, so $\boldsymbol{p} \in \mathbb{I}(G(\Delta))$. Moreover, for every winning coalition S we have $|S| \geq \lceil q \rceil$, so $p(S) \geq \lceil q \rceil \cdot 1/\lceil q \rceil = 1$. Therefore, $\boldsymbol{p}$ is in the core of $G(\Delta)$, and hence $\mathrm{CoS}^{+}(G) \leq \Delta$.

Conversely, let $\boldsymbol{p}$ be in the core of $\overline{G}$. Set $s = \lceil q \rceil$. Consider a collection of coalitions $S^1, \ldots, S^n$, where $S^i = \{(i \bmod n) + 1, (i + 1 \bmod n) + 1, \ldots, (i + s - 1 \bmod n) + 1\}$; for example, we have $S^{n-1} = \{n, 1, \ldots, s - 2, s - 1\}$. We have $|S^i| = s$ and hence $p(S^i) \geq 1$ for all $i = 1, \ldots, n$, so $p(S^1) + \cdots + p(S^n) \geq n$. On the other hand, each player i occurs in exactly s of these coalitions, so we have $p(N) \cdot s = p(S^1) + \cdots + p(S^n)$. Hence, $p(N) \geq n/s = n/\lceil q \rceil$, and therefore $\mathrm{CoS}^{+}(G) \geq \Delta$, and our claim follows. ❑

Our second example is given by simple games defined by finite projective planes; subsequently, we will use this example to prove lower bounds on the cost of stability.

[1] For cost-sharing games, the multiplicative cost of stability is also known as the *cost recovery ratio* (Xu & Du, 2006), and we have $0 \leq \mathrm{CoS}^{\times}(G) \leq 1$.

Recall that the finite projective plane of order q, where q is a prime, has q^2+q+1 points and the same number of lines; every line contains $q+1$ points, every two lines intersect at a single point, and every point belongs to exactly $q+1$ lines. Given a finite projective plane P of order q, we construct a simple game $G_q = \langle N, v\rangle$ as follows. We let N be the set of points in P, and for every $S \subseteq N$, we let $v(S) = 1$ if S contains a line, and $v(S) = 0$ otherwise. Observe that this game is superadditive: Since any two lines intersect, there do not exist two disjoint winning coalitions. The following proposition provides bounds on the cost of stability in such games.

Proposition 2 *We have* $\mathrm{CoS}^{\times}(G_q) = q + \frac{1}{q+1} \in (\sqrt{n}-1, \sqrt{n})$.

Proof. Consider a stable payoff vector $\boldsymbol{p}$. For each line R we have $p(R) \geq 1$. Summing over all q^2+q+1 lines, and using the fact that each point belongs to $q+1$ lines, we obtain $(q+1)\sum_{i\in N} p_i \geq q^2+q+1$, i.e.,

$$p(N) \geq \frac{q^2+q+1}{q+1} = q + \frac{1}{q+1} > \sqrt{n}-1,$$

where the last inequality holds since $n = |N| = q^2+q+1$. On the other hand, it is clear that the payoff vector $\boldsymbol{p}$ with $p_i = \frac{1}{q+1}$ for each $i \in N$ is stable and $p(N) = q + \frac{1}{q+1} < \sqrt{n}$. ❑

To conclude this section, we note that the cost of stability is essentially independent of the utility scales, as shown by the following proposition.

Proposition 3 *Given a game* $G = \langle N, v\rangle$ *and a positive real* α*, let* αG *be the game* $\langle N, v'\rangle$*, where* $v'(S) = \alpha v(S)$ *for all* $S \subseteq N$*. Then* $\mathrm{CoS}^{+}(\alpha G) = \alpha \mathrm{CoS}^{+}(G)$, $\mathrm{CoS}^{\times}(\alpha G) = \mathrm{CoS}^{\times}(G)$.

Proof. Let $\Delta = \mathrm{CoS}^{+}(G)$. Consider a payoff vector $\boldsymbol{p} \in \mathbb{C}(G(\Delta))$. The payoff vector $\alpha\boldsymbol{p}$ satisfies $\alpha p(S) \geq \alpha v(S) = v'(S)$ and $\alpha p(N) = v'(N) + \alpha\Delta$, which means that $\mathrm{CoS}^{+}(\alpha G) \leq \alpha \mathrm{CoS}^{+}(G)$. Conversely, by observing that $G = \frac{1}{\alpha}\cdot \alpha G$ and applying the same argument, we obtain $\mathrm{CoS}^{+}(G) \leq \frac{1}{\alpha}\mathrm{CoS}^{+}(\alpha G)$ and hence $\mathrm{CoS}^{+}(\alpha G) = \alpha \mathrm{CoS}^{+}(G)$. It then follows that $\mathrm{CoS}^{\times}(\alpha G) = \mathrm{CoS}^{\times}(G)$. ❑

On the other hand, adding a constant to the value of each coalition may change the cost of stability in a hard-to-predict way. For instance, consider the two-player game $G = \langle N, v\rangle$, where $v(\{1\}) = v(\{2\}) = 0$ and $v(\{1,2\}) = 1$. Every pre-imputation that allocates each agent a nonnegative payoff is stable. Consequently, $\mathrm{CoS}^{+}(G) = 0$. If we increase the value of every coalition by 1, i.e., set $v' = v+1$, the resulting game $G' = \langle N, v'\rangle$ still has a nonempty core (as witnessed by the payoff vector $p_1 = p_2 = 1$) and hence $\mathrm{CoS}^{+}(G') = 0$. However, if we increase the value of every coalition by 2, i.e., set $v'' = v+2$, any stable payoff vector for the resulting game $G'' = \langle N, v''\rangle$ would have to satisfy $p_1 \geq 2$, $p_2 \geq 2$, so $p(N) - v''(N) \geq 1$ and hence $\mathrm{CoS}^{+}(G'') \geq 1$.

3.2 The Cost of Stability and Balanced Collections of Weights

It is well-known that the core of a game $G = \langle N, v\rangle$ is the set of solutions to a linear feasibility program with n variables $p_1, \ldots, p_n$, whose constraints are $p(S) \geq v(S)$ for all

$S \subseteq N$, and $p(N) = v(N)$. Replacing the constraint $p(N) = v(N)$ with the optimization goal $\min p(N) - v(N)$ (while keeping the constraint $p(N) \geq v(N)$), we obtain a linear optimization program, which we call LP_G; it is immediate that the optimal value of this linear program is exactly $\mathrm{CoS}^+(G)$.

The *dual* linear program to LP_G has a variable δ_S for every subset $S \subseteq N$, and n "balance" constraints. We will now explain how to use this dual LP to obtain a closed-form expression for the cost of stability.

Definition 3 *A vector $\mathcal{B} = \{\delta_S\}_{S \in 2^N}$ is said to be a* balanced collection of weights *if $\delta_S \in \mathbb{R}_+$ for every $S \in 2^N$, and for every agent $i \in N$ it holds that $\sum_{S \in 2^N : i \in S} \delta_S = 1$.*

Given a balanced collection of weights $\mathcal{B} = \{\delta_S\}_{S \in 2^N}$, define $\mathcal{D}(\mathcal{B}) = \{S \in 2^N \mid \delta_S > 0\}$; we say that $\mathcal{D}(\mathcal{B})$ is the *balanced collection of subsets associated with* $\mathcal{B}$. The following classic result is a direct application of the LP duality theorem.

Theorem 4 (Bondareva, 1963; Shapley, 1967) *A game $G = \langle N, v \rangle$ has a nonempty core if and only if for every balanced collection of weights $\{\delta_S\}_{S \in 2^N}$ it holds that*

$$\sum_{S \in 2^N} \delta_S v(S) \leq v(N).$$

Now, fix a TU game $G = \langle N, v \rangle$ with an empty core and consider the associated game $\overline{G} = \langle N, \overline{v} \rangle$. Since the core of G is empty, the constraint $p(N) \geq \overline{v}(N)$ in $LP_{\overline{G}}$ is redundant. Removing this constraint corresponds to setting $\delta_N = 0$ in the dual LP; the value of the modified dual LP is the same as that of the original dual LP for $LP_{\overline{G}}$. Hence, $\overline{G}$ admits a balanced collection of weights $\{\delta_S\}_{S \in 2^N}$ with $\delta_N = 0$ for which the inequality in the statement of the Bondareva–Shapley theorem holds with equality, i.e., $\sum_{S \in 2^N} \delta_S \overline{v}(S) = v(N) + \mathrm{CoS}^+(G) = v(N) \cdot \mathrm{CoS}^\times(G)$. As $\overline{v}(S) = v(S)$ for $S \neq N$, we can write the multiplicative cost of stability of the game G as follows:

$$\mathrm{CoS}^\times(G) = \frac{1}{v(N)} \max \left\{ \sum_{S \in 2^N} \delta_S v(S) \;\middle|\; \{\delta_S\}_{S \in 2^N} \text{ is balanced} \right\}. \tag{3}$$

Note that a balanced collection of weights $\{\delta_S\}_{S \in 2^N}$ is a feasible solution of the linear program that is dual to LP_G. This inspires the following definition.

Definition 4 *A* solution *to $\overline{G}$ is a balanced collection of weights $\{\delta_S\}_{S \in 2^N}$ that satisfies $\sum_{S \in 2^N} \delta_S v(S) = \overline{v}(N)$ (note that $\overline{v}(N) = v(N) \cdot \mathrm{CoS}^\times(G)$).*

4. Bounds on the Cost of Stability

Consider an arbitrary game $G = \langle N, v \rangle$ with an empty core. We have observed that G can be stabilized by paying the maximum possible coalitional value to each agent, i.e., $\mathrm{CoS}^+(G) \leq n \cdot \max_{S \subseteq N} v(S) - v(N)$. For monotone games we have $\max_{S \subseteq N} v(S) = v(N)$, so this bound can be simplified to $\mathrm{CoS}^+(G) \leq (n-1)v(N)$, or, equivalently, $\mathrm{CoS}^\times(G) \leq n$. In fact, this bound is tight, as illustrated by the (simple, anonymous) game G' given by

$v'(S) = 1$ for all $S \neq \emptyset$: Clearly, in this game any payoff vector that offers some agent less than 1 will not be stable, whereas setting $p_i = 1$ for all $i \in N$ ensures stability. We summarize these observations as follows.

Observation 5 *Let $G = \langle N, v\rangle$ be a monotone TU game. Then $\mathrm{CoS}^{\times}(G) \leq n$ and this bound is tight, even if G is simple and anonymous.*

We will now show how to refine this upper bound for specific subclasses of profit-sharing games.

4.1 Superadditive Games

In superadditive games the grand coalition has the highest social welfare among all coalition structures formed by disjoint coalitions, so its stability is particularly desirable. Yet, we will see that ensuring stability may turn out to be quite costly even in this restricted setting.

A collection of subsets $\mathcal{D}$ is said to be *proper* if for every $R, T \in \mathcal{D}$ we have $R \cap T \neq \emptyset$. Shapley (1967) proved that when applying the Bondareva–Shapley theorem to superadditive games, it suffices to consider balanced collections of weights whose associated balanced collections of subsets are proper (see Theorem 3 therein). Consequently, there is a solution $\mathcal{B}$ to $\overline{G}$ such that $\mathcal{D}(\mathcal{B})$ is a proper collection of subsets, which can be stated as the following result:[2]

Lemma 6 (Shapley (1967), Theorem 3) *Let $G = \langle N, v\rangle$ be a superadditive game. Then there exists a solution $\mathcal{B}$ to $\overline{G}$ such that $\mathcal{D}(\mathcal{B})$ is a proper collection of subsets.*

Lemma 6 enables us to prove a tight upper bound on the cost of stability in superadditive games.

Theorem 7 *Let $G = \langle N, v\rangle$ be a superadditive game. Then $\mathrm{CoS}^{\times}(G) \leq \sqrt{n}$, and this bound is asymptotically tight even for simple games.*

Proof. Consider a solution $\mathcal{B} = \{\delta_S\}_{S\in 2^N}$ to $\overline{G}$ such that $\mathcal{D}(\mathcal{B})$ is a proper collection of subsets; its existence is ensured by Lemma 6. Since $\mathcal{B}$ is a solution to $\overline{G}$ and G is monotone, we obtain

$$\mathrm{CoS}^{\times}(G) = \frac{1}{v(N)}\left(\sum_{S\subseteq N} \delta_S v(S)\right) \leq \sum_{S\subseteq N} \delta_S.$$

To complete the proof, we will argue that $\sum_{S\subseteq N} \delta_S \leq \sqrt{n}$.

Suppose first that there exists a set $T \in \mathcal{D}(\mathcal{B})$ with $|T| \leq \sqrt{n}$. Every set $S \in \mathcal{D}(\mathcal{B})$ intersects T. Thus we have

$$\sum_{S\subseteq N} \delta_S \leq \sum_{i\in T} \sum_{S\subseteq N: i\in S} \delta_S = \sum_{i\in T} 1 = |T| \leq \sqrt{n}.$$

Now, suppose that $|S| > \sqrt{n}$ for every $S \in \mathcal{D}(\mathcal{B})$. Then we have

$$\sqrt{n}\sum_{S\subseteq N} \delta_S < \sum_{S\subseteq N} |S|\delta_S = \sum_{S\subseteq N}\sum_{i\in S} \delta_S = \sum_{i\in N}\sum_{S\subseteq N: i\in S} \delta_S = \sum_{i\in N} 1 = n,$$

[2] We thank an anonymous reviewer for helping us find this lemma in Shapley's paper.

which implies $\sum_{S\subseteq N}\delta_S \le \sqrt{n}$.

This bound is asymptotically tight: By Proposition 2, for each $n_0 > 0$ there exists an $n \ge n_0$ such that there is a (simple) superadditive game G_q with n players satisfying $\mathrm{CoS}^{\times}(G_q) > \sqrt{n} - 1$. ❑

For small values of n we can get a stronger bound on the cost of stability by considering all balanced collections of weights whose associated balanced collections of subsets are proper. For example, for $n = 3$ the only such collection is $\delta_{12} = \frac{1}{2}$, $\delta_{23} = \frac{1}{2}$, $\delta_{13} = \frac{1}{2}$ and thus $\mathrm{CoS}^{\times}(G) \le \frac{3}{2} < \sqrt{3}$.

4.2 Anonymous Games

Recall that an anonymous game $G = \langle N, v\rangle$ can be specified by a list of n numbers $v_1, \dots, v_n$, where $v_k = v(\{1, \dots, k\})$: We have $v(S) = v_k$ for every coalition $S \subseteq N$ of size k. Using this notation, we can simplify equation (3) for anonymous games.

Theorem 8 *Let $G = \langle N, v\rangle$ be an anonymous game. Then*

$$\mathrm{CoS}^{\times}(G) = \frac{n}{v_n} \cdot \max_{k\le n} \frac{v_k}{k}.$$

Proof. Pick $k^* \in \mathrm{argmax}_{k\le n}\, v_k/k$, and let $\boldsymbol{p}$ be the payoff vector given by $p_i = v_{k^*}/k^*$ for all $i \in N$. Clearly, $\boldsymbol{p}$ is stable: For every $S \subseteq N$, we have $p(S) = |S|v_{k^*}/k^* \ge v(S)$ by our choice of k^*.

Now, suppose that there is a stable payoff vector $\boldsymbol{q}$ with $q(N) < p(N)$. Renumber the players so that $q_1 \le \cdots \le q_n$ and set $S^* = \{1, \dots, k^*\}$. Clearly, we have $q(S^*)/k^* \le q(N)/n$, and hence

$$q(S^*) \le \frac{k^*}{n} q(N) < \frac{k^*}{n} p(N) = v_{k^*},$$

which means that $\boldsymbol{q}$ is not stable. Hence,

$$\mathrm{CoS}^{\times}(G) = \frac{p(N)}{v(N)} = \frac{n}{v_n} \cdot \frac{v_{k^*}}{k^*},$$

which completes the proof. ❑

4.3 Superadditive and Anonymous Games

If we assume both superadditivity and anonymity, we can strengthen Theorem 7 considerably. Note that an anonymous game $\langle N, v\rangle$ is superadditive if and only if for every pair of positive integers k, ℓ such that $k + \ell \le n$ it holds that $v_k + v_\ell \le v_{k+\ell}$.

Theorem 9 *Let $G = \langle N, v\rangle$ be an anonymous superadditive game. Then* $\mathrm{CoS}^{\times}(G) \le 2 - \frac{2}{n+1}$*, and this bound is tight for odd values of n.*

Proof. Fix an anonymous superadditive game $G = \langle N, v\rangle$ with $|N| = n$.

For every positive integer $k \geq (n+1)/2$, we have

$$\frac{n}{v_n} \cdot \frac{v_k}{k} \leq \frac{2nv_k}{(n+1)v_n} \leq \frac{2n}{n+1}.$$

Now, consider a positive integer $k < (n+1)/2$ and let $q = \lfloor n/k \rfloor$. Since k is integer, we have $q \geq 2$. By superadditivity we have $v_n \geq qv_k$. Let $\alpha = n/k - q < 1$. Then if $n \geq 3$, we have

$$\frac{n}{v_n} \cdot \frac{v_k}{k} = (q+\alpha)\frac{v_k}{v_n} \leq (q+\alpha)\frac{1}{q} = 1 + \frac{\alpha}{q} < \frac{3}{2} \leq \frac{2n}{n+1},$$

and if $n = 2$, we obtain $k = 1$ and $(n/v_n) \cdot (v_k/k) = 2v_1/v_2 \leq 1 < \frac{4}{3}$.

Thus, by Theorem 8, we obtain

$$\mathrm{CoS}^{\times}(G) = \max_{k \leq n} \frac{n}{v_n} \cdot \frac{v_k}{k} \leq \frac{2n}{n+1} = 2 - \frac{2}{n+1}.$$

To show that this bound is tight, for every odd $n > 1$ we define a simple game $G_n = \langle N, v \rangle$ with $|N| = n$ by setting $v(S) = 1$ if $|S| > n/2$, and $v(S) = 0$ otherwise. In G_n every two winning coalitions intersect. Thus for any two disjoint sets S, T, either both are losing, in which case $v(S \cup T) \geq 0 = v(S) + v(T)$; or exactly one is winning, in which case $v(S \cup T) \geq 1 = v(S) + v(T)$. Hence G_n is superadditive. Let $k^* = \lceil n/2 \rceil$. Then by Theorem 8 we have

$$\mathrm{CoS}^{\times}(G_n) \geq \frac{n}{v_n} \cdot \frac{v_{k^*}}{k^*} = \frac{n}{\lceil n/2 \rceil} = \frac{2n}{n+1},$$

which completes the proof. ❑

We obtain similar results for cost-sharing games (Appendix A). All these results are summarized in Table 1 on page 1014.

4.4 Weighted Voting Games

We first note that the game used to establish the lower bound of n in Observation 5 is equivalent to the weighted voting game $[1, 1, 1, \ldots, 1; 1]$. So, without further constraints, for weighted voting games we cannot improve over the trivial upper bound on the cost of stability.

However, we can show that the multiplicative cost of stability of *superadditive* weighted voting games is always less than two. This bound is essentially tight, even for anonymous superadditive weighted voting games: The game used to prove the lower bound of $2 - \frac{2}{n+1}$ in Theorem 9 can be written as the anonymous weighted voting game $[1, 1, \ldots, 1; \frac{n+1}{2}]$.

Theorem 10 *Let $G = \langle N, v \rangle$ be a superadditive weighted voting game. Then* $\mathrm{CoS}^{\times}(G) < 2$.

Proof. Our first observation is that since G is a simple game, it is superadditive if and only if every pair of winning coalitions has a nonempty intersection.

Recall that we assume that $w(N) \geq q$. Suppose that there is an agent i^* with weight $w_{i^*} \geq q$. In that case, by superadditivity i^* must be a veto player, so the core of G is nonempty and hence $\mathrm{CoS}^{\times}(G) = 1$.

Otherwise, let S be a minimum-weight winning coalition in G. Pick an agent $j \in S$ such that $w_j \leq w_i$ for all $i \in S$, and set $s = 1 - \frac{w(S \setminus \{j\})}{q}$. Note that $s > 0$ by our choice of S.

We define a payoff vector $\boldsymbol{p}$ by setting $p_j = s$, $p_i = \frac{w_i}{q}$ for $i \in N \setminus \{j\}$. We claim that $\boldsymbol{p}$ is stable. Indeed, consider a winning coalition R. If $j \notin R$, then $p(R) = \frac{w(R)}{q} \geq 1$, so R does not block $\boldsymbol{p}$. Hence, suppose that $j \in R$. Note that $w(R) \geq w(S)$ by our choice of S. Thus

$$p(R) = p(R \setminus \{j\}) + p_j = \frac{w(R \setminus \{j\})}{q} + p_j \geq \frac{w(S \setminus \{j\})}{q} + s = 1.$$

It remains to bound the total payment:

$$\begin{aligned} p(N) = p(S \setminus \{j\}) + p_j + p(N \setminus S) &= \frac{w(S \setminus \{j\})}{q} + s + \frac{w(N \setminus S)}{q} \\ = 1 + \frac{w(N \setminus S)}{q} &< 1 + 1 = 2, \end{aligned}$$

where the inequality holds because $N \setminus S$ is a losing coalition. ❑

We conjecture that the worst-case cost of stability is obtained for weighted voting games where all players have identical weights, i.e., that the lower bound of $2 - \frac{2}{n+1}$ is tight.

5. The Cost of Stability and the Least Core

In this section we explore the relationship between the cost of stability and several variants of the least core. We start by defining the relevant stability concepts.

Strong and weak least core. Consider a TU game $G = \langle N, v \rangle$ and some $\varepsilon \geq 0$. The *strong ε-core of G* (Shapley & Shubik, 1966) is the set $\mathrm{SC}_\varepsilon(G)$ of all pre-imputations for G such that no coalition can gain more than ε by deviating:

$$\mathrm{SC}_\varepsilon(G) = \{\boldsymbol{p} \in \mathbb{I}(G) \mid p(S) \geq v(S) - \varepsilon \text{ for all } S \subseteq N\}.$$

Clearly, if ε is large enough, we have $\mathrm{SC}_\varepsilon(G) \neq \emptyset$. The quantity $\varepsilon_{\mathbf{S}}(G) = \inf\{\varepsilon \geq 0 \mid \mathrm{SC}_\varepsilon(G) \neq \emptyset\}$ is called the *value of the strong least core of G*.[3] The strong $\varepsilon_{\mathbf{S}}$-core of G is referred to as the *strong least core of G*, and is denoted by $\mathbb{SLC}(G)$.

In contrast, the *weak ε-core of G* (Shapley & Shubik, 1966) consists of pre-imputations such that no coalition can deviate in a way that profits each deviator by at least ε:

$$\mathrm{WC}_\varepsilon(G) = \{\boldsymbol{p} \in \mathbb{I}(G) \mid p(S) \geq v(S) - \varepsilon|S| \text{ for all } S \subseteq N\}.$$

Just as for the strong least core, we define the *value of the weak least core of G* as $\varepsilon_{\mathbf{W}}(G) = \inf\{\varepsilon \geq 0 \mid \mathrm{WC}_\varepsilon(G) \neq \emptyset\}$; the *weak least core of G* (denoted by $\mathbb{WLC}(G)$) is its $\varepsilon_{\mathbf{W}}$-core. Clearly both $\mathbb{SLC}(G)$ and $\mathbb{WLC}(G)$ are nonempty.

Note that $\mathrm{SC}_\varepsilon(G) \subseteq \mathrm{WC}_\varepsilon(G)$ for any $\varepsilon > 0$ and, consequently, $\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G)$.

[3] We remark that there are variants of this definition with and without the constraint $\varepsilon \geq 0$; we impose this constraint since we also require the additive cost of stability to be nonnegative.

Positive strong/weak least core. By definition, both weak and strong ε-core may contain payoff vectors with negative entries. However, in many cases negative payoffs are not acceptable. To capture such settings, we define the *positive strong ε-core of G* as

$$\mathrm{PSC}_\varepsilon(G) = \{\boldsymbol{p} \in \mathrm{SC}_\varepsilon(G) \mid p_i \geq 0 \text{ for all } i \in N\}.$$

Similarly, the notions of the *positive weak ε-core* (denoted by $\mathrm{PWC}_\varepsilon(G)$), the *value of the positive strong/weak least core* (denoted by $\varepsilon_{\mathbf{PS}}$ and $\varepsilon_{\mathbf{PW}}$, respectively) and the *positive strong/weak least core* (denoted by $\mathbb{PSLC}(G)$ and $\mathbb{PWLC}(G)$, respectively), are defined by adding the requirement that $p_i \geq 0$ for all $i \in N$ to the corresponding definitions above. Clearly, we have $\mathrm{PSC}_\varepsilon(G) \subseteq \mathrm{SC}_\varepsilon(G)$, $\mathrm{PWC}_\varepsilon(G) \subseteq \mathrm{WC}_\varepsilon(G)$ and $\varepsilon_{\mathbf{PS}} \geq \varepsilon_{\mathbf{S}}$, $\varepsilon_{\mathbf{PW}} \geq \varepsilon_{\mathbf{W}}$.

The impact of the requirement that $p_i \geq 0$ for all $i \in N$ is illustrated by the following example.

Example 2 *Consider a two-player game $G = \langle N, v\rangle$, where $v(\{1\}) = 2$, $v(\{2\}) = 0$, and $v(\{1, 2\}) = 1$. The pre-imputation (p_1, p_2) with $p_1 = 1.5$, $p_2 = -.5$ satisfies $p(S) \geq v(S) - .5$ for each $S \subseteq N$ and hence $\varepsilon_{\mathbf{W}}, \varepsilon_{\mathbf{S}} \leq .5$. In fact, it is not hard to see that $\varepsilon_{\mathbf{W}} = \varepsilon_{\mathbf{S}} = .5$: If $v(\{1\}) - p'_1 < .5$ and $v(\{2\}) - p'_2 < .5$, we get $p'_1 + p'_2 > 1$, so (p'_1, p'_2) cannot be a pre-imputation for G. On the other hand, every pre-imputation (q_1, q_2) with $q_1, q_2 \geq 0$ has $q_1 \leq 1$ and hence $q(\{1\}) \leq v(\{1\}) - 1$, so $\varepsilon_{\mathbf{PW}}, \varepsilon_{\mathbf{PS}} \geq 1$. Note that we have $\mathrm{CoS}^\times(G) = 1$: If (p_1, p_2) is a stable payoff vector for G then $p_1 \geq 2$, and $(2, 0)$ is in the core of $G(1)$.*

Similarly to the additive cost of stability, the values of the (positive) strong least core and the (positive) weak least core can be obtained as optimal values of certain linear programs. We can think of all these notions as different *measures* of (in)stability. For instance, it is clear that conditions $\varepsilon_{\mathbf{S}}(G) > 0$, $\varepsilon_{\mathbf{W}}(G) > 0$, $\varepsilon_{\mathbf{PS}}(G) > 0$, $\varepsilon_{\mathbf{PW}}(G) > 0$, and $\mathrm{CoS}^+(G) > 0$ are all equivalent, as each of them holds if and only if the core of G is empty. The goal of this section is to discuss the relationship between the (positive) weak least core, the (positive) strong least core, and the cost of stability in more detail.

The extended core. Following Bejan and Gómez (2009), we set

$$\mathrm{CoS}^+(\boldsymbol{p}, G) = \inf\{\Delta \geq 0 \mid \ \exists \boldsymbol{p}' \geq \boldsymbol{p} \text{ such that } \boldsymbol{p}' \in \mathbb{C}(G(\Delta)) \ \}$$

and define the *extended core* of a TU game G as

$$\mathbb{EC}(G) = \{\boldsymbol{p} \in \mathbb{I}(G) \mid \mathrm{CoS}^+(\boldsymbol{p}, G) = \mathrm{CoS}^+(G)\}.$$

We say that a payoff vector $\boldsymbol{p}'$ is a *stable extension* of a pre-imputation $\boldsymbol{p} \in \mathbb{I}(G)$ if $\boldsymbol{p}' \in \mathbb{S}(G)$ and $\boldsymbol{p}' \geq \boldsymbol{p}$. Note that the set $\mathbb{EC}(G)$ can be viewed as the set of feasible solutions to a linear program. Intuitively, $\mathbb{EC}(G)$ is the set of "the most stable" pre-imputations, i.e., pre-imputations that require the lowest amount of subsidy to prevent coalitional deviations.

5.1 The Weak Least Core

Bejan and Gómez (2009) observe that the additive cost of stability and the value of the weak least core are closely related and, moreover, every vector in $\mathbb{WLC}(G)$ can be stabilized by adding $\varepsilon_{\mathbf{W}}$ to each agent's payoff.

Proposition 11 (Bejan and Gómez (2009)) *Let $G = \langle N, v \rangle$ be a TU game. Then*

1. $\mathrm{CoS}^+(G) = n\varepsilon_{\mathbf{W}}(G)$.

2. $\mathbb{WLC}(G) \subseteq \mathbb{EC}(G)$.

However, Proposition 11 does not hold for the positive weak least core, as can be seen from Example 2. Our next result relates the value of the positive weak least core and the cost of stability for monotone games.

Proposition 12 *Let $G = \langle N, v \rangle$ be a monotone TU game. Then $2\varepsilon_{\mathbf{PW}}(G) \leq \mathrm{CoS}^+(G) \leq n\varepsilon_{\mathbf{PW}}(G)$, and these bounds are tight.*

Proof. The upper bound is immediate, since $\mathrm{CoS}^+(G) = n\varepsilon_{\mathbf{W}}(G) \leq n\varepsilon_{\mathbf{PW}}(G)$. To see that it is tight, consider the game $G = \langle N, v \rangle$ such that $v(S) = 1$ for every nonempty coalition S. We have $\mathrm{CoS}^+(G) = n - 1$. On the other hand, we have $\varepsilon_{\mathbf{PW}}(G) = \frac{n-1}{n}$: Indeed, it is at most $\frac{n-1}{n}$, because if we distribute the value of the grand coalition by paying each agent $\frac{1}{n}$, then for each coalition of size s, $s \geq 1$, the difference between its value and the payment it receives is at most $\frac{n-s}{n} \leq \frac{n-1}{n}$, and it is at least $\frac{n-1}{n}$, because for every payoff vector $\boldsymbol{p}$ with $p(N) = v(N)$ we have $p_i \leq \frac{1}{n}$ for some $i \in N$, but $v(\{i\}) = 1$.

For the lower bound, let $\Delta = \mathrm{CoS}^+(G)$ and consider a payoff vector $\boldsymbol{q} \in \mathbb{C}(G(\Delta))$; note that $q_i \geq v(\{i\}) \geq 0$ for all $i \in N$. Suppose first that there exist two players i and j, $i \neq j$, such that $q_i \geq \Delta/2$ and $q_j \geq \Delta/2$. Consider the payoff vector $\boldsymbol{q}'$ defined by

$$q'_\ell = \begin{cases} q_\ell & \text{if } \ell \neq i, j, \\ q_\ell - \Delta/2 & \text{if } \ell = i \text{ or } \ell = j. \end{cases}$$

Clearly, $\boldsymbol{q}'$ is a pre-imputation for G, all of its entries are nonnegative, and for every coalition $S \subseteq N$ we have $q'(S) \geq q(S) - (\Delta/2)|S| \geq v(S) - (\Delta/2)|S|$, so $\varepsilon_{\mathbf{PW}}(G) \leq \Delta/2$.

Now, suppose that there is at most one player i such that $q_i \geq \Delta/2$; assume without loss of generality that $q_i < \Delta/2$ for $i = 2, \ldots, n$ and set $S = \{2, \ldots, n\}$.

Observe that $q_1 \leq v(N)$. Indeed, if $q_1 > v(N)$, the payoff vector $\boldsymbol{q}'$ given by $q'_1 = v(N)$, $q'_i = q_i$ for $i \in S$ satisfies $q'(N) < q(N)$. Further, $\boldsymbol{q}'$ is stable: We have $q'(T) = q(T) \geq v(T)$ for every $T \subseteq S$ and, since G is monotone, $q'(T) \geq v(N) \geq v(T)$ for every $T \subseteq N$ such that $1 \in T$. This is a contradiction with our choice of $\boldsymbol{q}$.

Thus we have $q(S) = q(N) - q_1 \geq \Delta$. Consider an arbitrary pre-imputation $\boldsymbol{q}''$ for G such that $q''_1 = q_1$ and $0 \leq q''_i \leq q_i$ for $i \in S$; note that $q''(S) = q(S) - \Delta \geq 0$. Since $q_i < \Delta/2$ for all $i \in S$, we have $q_i - q''_i \leq \Delta/2$ for all $i \in N$. Consequently, for every set $T \subseteq N$ we have $q''(T) \geq q(T) - (\Delta/2)|T|$, which means that $\boldsymbol{q}'' \in \mathrm{PWC}_{\Delta/2}(G)$.

The weighted voting game $G = [1, 1, 0, 0, \ldots, 0; 1]$ shows that this bound is tight: We have $\mathrm{CoS}^+(G) = 1$ and $\varepsilon_{\mathbf{PW}}(G) = \frac{1}{2}$ (since any payoff vector for G allocates at most $\frac{1}{2}$ to one of the first two agents). ❑

5.2 The Strong Least Core

In this section we derive upper and lower bounds on the ratio $\mathrm{CoS}^+(G)/\varepsilon_{\mathbf{S}}(G)$. Since $\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G)$, Proposition 11 immediately implies that $\mathrm{CoS}^+(G) \leq n\varepsilon_{\mathbf{S}}(G)$. For general TU games this bound is tight. To see this, consider the game $G = \langle N, v\rangle$ with $v(S) = 1$ for all $S \neq \emptyset$: We have $\varepsilon_{\mathbf{S}}(G) = \frac{n-1}{n}$ and $\mathrm{CoS}^+(G) = n - 1$. We will now explore whether this bound can be improved if we place additional restrictions on the characteristic function.

In what follows, we use the following construction. Given a game G with an empty core, we set $\varepsilon = \varepsilon_{\mathbf{S}}(G)$ and define a new game $G_\varepsilon = \langle N, v_\varepsilon\rangle$, where $v_\varepsilon(S) = \max\{0, v(S) - \varepsilon\}$ for all $S \subsetneq N$, and $v_\varepsilon(N) = v(N)$. Intuitively, G_ε is obtained by imposing the minimum penalty on deviating coalitions that ensures stability, just as $\overline{G}$ is obtained by providing the minimum subsidy that ensures stability. We have $\mathbb{C}(G_\varepsilon) = \mathbb{SLC}(G)$.

We will now show that for superadditive games we can strengthen the upper bound on the ratio $\mathrm{CoS}^+(G)/\varepsilon_{\mathbf{S}}(G)$ to $\sqrt{n}$.

Theorem 13 *Let $G = \langle N, v\rangle$ be a superadditive game. Then $\mathrm{CoS}^+(G) \leq \sqrt{n} \cdot \varepsilon_{\mathbf{S}}(G)$, and this bound is tight up to a small additive constant.*

PROOF. By Lemma 6 there exists a solution $(\delta_S)_{S\in 2^N}$ to $\overline{G}$ such that every two sets S and T with $\delta_S \neq 0$ and $\delta_T \neq 0$ have a nonempty intersection.

Since $(\delta_S)_{S\in 2^N}$ is a balanced collection of weights, applying the Bondareva–Shapley theorem to the game G_ε (which has a nonempty core), we obtain

$$\sum_{S\subseteq N} \delta_S(v(S) - \varepsilon) \leq \sum_{S\subsetneq N} \delta_S(v(S) - \epsilon) + v(N) \leq \sum_{S\subseteq N} \delta_S v_\varepsilon(S) \leq v_\varepsilon(N) = v(N).$$

Together with the fact that $\sum_{S\subseteq N} \delta_S \leq \sqrt{n}$ (cf. the proof of Theorem 7), this implies

$$\begin{aligned}\mathrm{CoS}^+(G) &= \sum_{S\subseteq N} \delta_S v(S) - v(N) \leq \sum_{S\subseteq N} \delta_S v(S) - \sum_{S\subseteq N} \delta_S(v(S) - \varepsilon) \\ &= \varepsilon \sum_{S\subseteq N} \delta_S \leq \sqrt{n}\varepsilon = \sqrt{n}\varepsilon_{\mathbf{S}}(G),\end{aligned}$$

which completes the proof of the upper bound.

To see that this bound is tight, consider the game G_q (see Proposition 2). Since G_q is a simple game, we have $\varepsilon_{\mathbf{S}}(G_q) \leq 1$. Moreover, consider the payoff vector $\boldsymbol{p}$ given by $p_i = 1/n$ for all $i \in N$. If S is a winning coalition in G_q, then $|S| \geq q + 1$ and thus $p(S) \geq (q+1)/n \geq 1/\sqrt{n}$. Therefore, $\varepsilon_{\mathbf{S}}(G_q) \leq 1 - 1/\sqrt{n}$, and hence $\sqrt{n} \cdot \varepsilon_{\mathbf{S}}(G_q) \leq \sqrt{n} - 1$. On the other hand, we have seen that $\mathrm{CoS}^\times(G_q) > \sqrt{n} - 1$. Since G_q is a simple game, this implies

$$\mathrm{CoS}^+(G_q) = \mathrm{CoS}^\times(G_q) - 1 > (\sqrt{n} - 1) - 1 \geq \sqrt{n} \cdot \varepsilon_{\mathbf{S}}(G) - 1,$$

which completes the proof. ❑

It is instructive to compare Theorem 7 with Theorem 13: For any superadditive TU game $G = \langle N, v\rangle$, the former shows that $\mathrm{CoS}^\times(G) \leq \sqrt{n}$, while the latter can be rewritten as $\mathrm{CoS}^\times(G) \leq \sqrt{n}\, \frac{\varepsilon_{\mathbf{S}}(G)}{v(N)} + 1$. For games where $\varepsilon_{\mathbf{S}}(G)$ is significantly smaller than $v(N)$, Theorem 13 is substantially stronger than Theorem 7.

Our next proposition establishes a lower bound on the ratio $\mathrm{CoS}^+(G)/\varepsilon_{\mathbf{S}}(G)$.

Proposition 14 *Let $G = \langle N, v\rangle$ be a TU game. Then* $\mathrm{CoS}^+(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{S}}(G)$, *and this bound is tight.*

Proof. Observe that $\mathrm{WC}_\varepsilon(G) \subseteq \mathrm{SC}_{(n-1)\varepsilon}(G)$ for any $\varepsilon > 0$. Indeed, let $\varepsilon > 0$, and consider a pre-imputation $\boldsymbol{p} \in \mathrm{WC}_\varepsilon(G)$. For every coalition $S \subsetneq N$ we have $p(S) \geq v(S) - |S|\varepsilon \geq v(S) - (n-1)\varepsilon$, and hence $\boldsymbol{p} \in \mathrm{SC}_{(n-1)\varepsilon}(G)$. This means that $\varepsilon_{\mathbf{S}} \leq (n-1)\varepsilon_{\mathbf{W}}$. Now, by Proposition 11 we obtain $\mathrm{CoS}^+(G) = n\varepsilon_{\mathbf{W}}(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{S}}(G)$.

To see that this bound is tight, consider the game $G = \langle N, v\rangle$, where $v(S) = 1$ if $|S| \geq n-1$, and $v(S) = 0$ otherwise.

We have $\mathrm{CoS}^+(G) = \frac{1}{n-1}$. Indeed, the payoff vector $\boldsymbol{p}$ given by $p_i = \frac{1}{n-1}$ for all $i \in N$ is stable and satisfies $p(N) = 1 + \frac{1}{n-1}$. On the other hand, consider a payoff vector $\boldsymbol{q}$ with $q(N) < \frac{n}{n-1}$ and an agent i such that $q_i \geq q_j$ for all $j \in N$. Either $q_i < \frac{1}{n-1}$, in which case for every coalition S, $|S| = n-1$, we have $q(S) \leq |S|q_i < 1$, but $v(S) = 1$, or $q(N \setminus \{i\}) = q(N) - q_i < 1$, even though $|N \setminus \{i\}| = n-1$, i.e., in either case $\boldsymbol{q}$ admits a blocking coalition.

Further, we have $\varepsilon_{\mathbf{S}}(G) = \frac{1}{n}$: For the imputation $\boldsymbol{p}$ given by $p_i = \frac{1}{n}$ for all $i \in N$ we have $p(S) \geq \frac{n-1}{n}$ for every winning coalition S, and for every imputation $\boldsymbol{q}$ we have $q_i \geq \frac{1}{n}$ and hence $v(N \setminus \{i\}) - q(N \setminus \{i\}) \geq \frac{1}{n}$. ❑

For superadditive games, we can combine Proposition 11, Theorem 13, and Proposition 14 to relate the value of the strong least core and the value of the weak least core.

Corollary 15 *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game. Then*

$$\sqrt{n}\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G) \leq (n-1)\varepsilon_{\mathbf{W}}(G)$$

(superadditivity is not required for the second inequality).

For monotone games, we can strengthen Proposition 14, by showing that it extends to the positive strong least core.

Theorem 16 *Let $G = \langle N, v\rangle$ be a monotone profit-sharing game. Then* $\mathrm{CoS}^+(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{PS}}(G)$, *and this bound is tight.*

Proof. Let $\Delta = \mathrm{CoS}^+(G)$. We will construct a pre-imputation $\boldsymbol{p}^* \in \mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$.

Set $\mathbb{EC}^+(G) = \mathbb{EC}(G) \cap \mathbb{R}^n_+$. First, we will argue that $\mathbb{EC}^+(G) \neq \emptyset$. To see this, pick an arbitrary pre-imputation $\boldsymbol{p} \in \mathbb{EC}(G)$, and let $\overline{\boldsymbol{p}}$ be some stable extension of $\boldsymbol{p}$. Note that $\overline{p}_i \geq \max\{p_i, 0\}$ for all $i \in N$. Without loss of generality, assume that $p_1 \leq \cdots \leq p_n$, and let j be the smallest index such that $\sum_{i=1}^j p_i \geq 0$; note that $p(N) = v(N) > 0$, so j is well-defined. If $j = 1$, we are done, since $\boldsymbol{p} \in \mathbb{R}^n_+$. Otherwise, consider the pre-imputation $\boldsymbol{p}'$ given by $p'_i = 0$ for $i = 1, \ldots, j-1$, $p'_j = \sum_{i=1}^j p_i$, and $p'_i = p_i$ for $i = j+1, \ldots, n$. Note that $p'(N) = p(N)$. We have $0 \leq p'_j \leq p_j \leq \overline{p}_j$ by our choice of j. Further, for $i < j$ we have $p'_i = 0 \leq \overline{p}_i$ and for $i > j$ we obtain $0 \leq p'_i = p_i \leq \overline{p}_i$. Hence, $\boldsymbol{p}' \in \mathbb{R}^n_+$ and $\overline{\boldsymbol{p}}$ is a stable extension of $\boldsymbol{p}'$, which means that $\boldsymbol{p}' \in \mathbb{EC}^+(G)$.

Now, for each $\boldsymbol{p} \in \mathbb{EC}^+(G)$ let $E_{\boldsymbol{p}} = \{\boldsymbol{q} \mid \boldsymbol{q} \geq \boldsymbol{p}, \boldsymbol{q} \in \mathbb{C}(G(\Delta))\}$ and define the following linear program LP($\boldsymbol{p}$) with variables $q_1, \dots, q_n, x_1, \dots, x_n$:

$$
\begin{aligned}
&\max \sum_{i \in N} x_i \\
\text{subject to: } & x_i \leq q_i - p_i \\
& x_i \leq \Delta/n \\
& \boldsymbol{q} \in E_{\boldsymbol{p}}
\end{aligned}
$$

Note that LP($\boldsymbol{p}$) is a linear program because $E_{\boldsymbol{p}}$ is defined by a set of linear constraints. Moreover, if $(q_1, \dots, q_n, x_1, \dots, x_n)$ is an optimal solution to LP($\boldsymbol{p}$) then we have $x_i = \min\{q_i - p_i, \Delta/n\}$ for all $i \in N$ and hence the value of LP($\boldsymbol{p}$), which we will denote by $\ell(\boldsymbol{p})$, is equal to $\max_{\boldsymbol{q} \in E_{\boldsymbol{p}}} \sum_{i \in N} \min\{q_i - p_i, \Delta/n\}$.

Further, let LP* be the linear program with variables $p_1, \dots, p_n, q_1, \dots, q_n, x_1, \dots, x_n$ given by:

$$
\begin{aligned}
&\max \sum_{i \in N} x_i \\
\text{subject to: } & x_i \leq q_i - p_i \\
& x_i \leq \Delta/n \\
& \boldsymbol{p} \in \mathbb{EC}^+(G) \\
& \boldsymbol{q} \in E_{\boldsymbol{p}}
\end{aligned}
$$

Again, LP* is a linear program because $\mathbb{EC}^+(G)$ and $E_{\boldsymbol{p}}$ are defined by sets of linear constraints. Let $(p_1^*, \dots, p_n^*, q_1^*, \dots, q_n^*, x_1^*, \dots, x_n^*)$ be an optimal solution to LP*; we will argue that $\boldsymbol{p}^* = (p_1^*, \dots, p_n^*)$ is in $\mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$. Note that, by construction, $\boldsymbol{p}^* \in \mathbb{EC}^+(G)$ and $\ell(\boldsymbol{p}^*) = \sum_{i \in N} \min\{q_i^* - p_i^*, \Delta/n\} \geq \ell(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathbb{EC}^+(G)$. Let

$$
\begin{aligned}
N^+ &= \{i \in N \mid q_i^* - p_i^* > \Delta/n\}, \\
N^0 &= \{i \in N \mid q_i^* - p_i^* = \Delta/n\}, \\
N^- &= \{i \in N \mid q_i^* - p_i^* < \Delta/n\}.
\end{aligned}
$$

We claim that for every $i \in N^-$ it holds that $p_i^* = 0$. Indeed, if $N^- = \emptyset$, this claim is trivially true, so suppose that $N^- \neq \emptyset$ and, hence, $N^+ \neq \emptyset$. Now suppose that there is an agent $i \in N^-$ with $p_i^* > 0$. Pick an agent $j \in N^+$ and set

$$\delta = \min\{p_i^*,\ \Delta/n - (q_i^* - p_i^*),\ q_j^* - p_j^* - \Delta/n\};$$

note that $\delta > 0$. Construct a pre-imputation $\boldsymbol{r}$ by setting

$$
r_k = \begin{cases} p_k^* - \delta/2 & \text{if } k = i, \\ p_k^* + \delta/2 & \text{if } k = j, \\ p_k^* & \text{if } k \in N \setminus \{i, j\}. \end{cases}
$$

We have

$$0 < q_i^* - r_i = q_i^* - p_i^* + \delta/2 < \Delta/n \quad \text{and} \quad q_j^* - r_j = q_j^* - p_j^* - \delta/2 > \Delta/n.$$

Thus $\boldsymbol{q}^* \in E_{\boldsymbol{r}}$ and

$$\begin{aligned}\ell(\boldsymbol{r}) &= \max_{\boldsymbol{q}\in E_{\boldsymbol{r}}} \sum_{i\in N} \min\{q_i - r_i, \Delta/n\} \geq \sum_{i\in N} \min\{q_i^* - r_i, \Delta/n\} \\ &= \sum_{i\in N} \min\{q_i^* - p_i^*, \Delta/n\} + \delta/2 = \ell(\boldsymbol{p}^*) + \delta/2,\end{aligned}$$

a contradiction with our choice of $\boldsymbol{p}^*$. Hence, $p^*(N^-) = 0$.

Now, consider an arbitrary coalition S. If $N^+ \cup N^0 \subseteq S$, we have

$$p^*(S) = p^*(N) = v(N) \geq v(S) > v(S) - \frac{n-1}{n}\Delta.$$

On the other hand, suppose that $N^+ \cup N^0 \not\subseteq S$. Consider some agent $i \in (N^+ \cup N^0) \setminus S$, and let $\Delta_i = q_i^* - p_i^*$; note that $\Delta_i \geq \Delta/n$. Since $\boldsymbol{p}^* \in \mathbb{EC}(G)$, we have $q^*(N) = p^*(N) + \Delta$ and $q^*(S) \leq p^*(S) + \Delta - \Delta_i \leq p^*(S) + \frac{n-1}{n}\Delta$, so

$$p^*(S) \geq q^*(S) - \frac{n-1}{n}\Delta \geq v(S) - \frac{n-1}{n}\Delta.$$

Thus $\boldsymbol{p}^* \in \mathrm{SC}_{\frac{n-1}{n}\Delta}(G)$, and since $\boldsymbol{p}^*$ is also in $\mathbb{R}^n_+$, we obtain $\boldsymbol{p}^* \in \mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$. Consequently, $\varepsilon_{\mathbf{PS}} \leq \frac{n-1}{n}\Delta$, or, equivalently, $\Delta = \mathrm{CoS}^+(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{PS}}$.

Tightness follows immediately from Proposition 14. ❑

We emphasize that monotonicity is a necessary condition both in Proposition 12 and in Theorem 16, as illustrated by the following example.

Example 3 *Consider a three player-game $G = \langle N, v\rangle$, where $v(\{1\}) = 100$, $v(\{2\}) = v(\{3\}) = 0$, $v(\{2,3\}) = 1$, and $v(S) = v(S \setminus \{1\})$ for every coalition S with $1 \in S$, $S \neq \{1\}$; by construction, this game is not monotone. We have $v(N) = 1$ and $\mathrm{CoS}^+(G) = 100$: If $\boldsymbol{p}$ is a stable payoff vector for G, we have $p_1 \geq 100$ and $p_2 + p_3 \geq 1$, so $p(N) \geq 101$, and, conversely, $(100, 1, 0)$ is a stable payoff vector for G. On the other hand, if $\boldsymbol{q}$ is a pre-imputation for G and $q_i \geq 0$ for all $i \in N$ then $q_1 \leq 1$ and hence $v(\{1\}) - q_1 \geq 99$ and therefore $\varepsilon_{\mathbf{PW}}(G) \geq 99$ and $\varepsilon_{\mathbf{PS}}(G) \geq 99$. In particular, we have neither $\varepsilon_{\mathbf{PW}}(G) \leq \frac{1}{2}\mathrm{CoS}^+(G)$ nor $\varepsilon_{\mathbf{PW}}(G) \leq \frac{n-1}{n}\mathrm{CoS}^+(G)$.*

6. Computing the Cost of Stability: The Case of Weighted Voting Games

Our next goal is to understand the complexity of computing the cost of stability, either exactly or approximately. However, here we face the usual difficulty associated with the complexity-theoretic analysis of cooperative games: We are interested in algorithms whose running time is polynomial in the number of players, yet the generic representation of an n-player game is given by $2^n - 1$ numbers. There are two standard routes that can be used to circumvent this issue (Chalkiadakis et al., 2011): One option is to assume that the game in question admits a compact encoding, i.e., there is an "oracle" that, given a coalition, computes its value in polynomial time, and another option is to focus on a specific succinctly representable class of cooperative games. In this section we explore both of these routes, using the class of weighted voting games as our case study for the second approach. We hope that the insights offered by our analysis of weighted voting games and the techniques we develop provide useful intuition for other classes of cooperative games.

6.1 General Observations

The linear program LP_G discussed in Section 3.2 provides a way of computing $\mathrm{CoS}^+(G)$ for any coalitional game G. However, this linear program contains exponentially many constraints (one for each subset of N). Therefore, representing it explicitly and then solving it directly would be too time-consuming for most games.[4] Thus, in what follows, we restrict our attention to games with compactly representable characteristic functions.

Specifically, we consider games that can be described by polynomial-size boolean circuits (see the book by Chalkiadakis et al. (2011) for details), where the size of a circuit is the number of its gates. Formally, we say that a class $\mathcal{G}$ of games has a *compact circuit representation* if there exists a polynomial p such that for every $G \in \mathcal{G}$, $G = \langle N, v\rangle$, $|N| = n$, there exists a circuit $\mathcal{C}$ of size $p(n)$ with n binary inputs that on input $(b_1, \dots, b_n)$ outputs $v(C)$, where $C = \{i \in N \mid b_i = 1\}$.

Unfortunately, it turns out that having a compact circuit representation does not guarantee efficient computability of $\mathrm{CoS}^+(G)$. Indeed, weighted voting games with integer weights have such a representation (again, see the book by Chalkiadakis et al. (2011) for details). However, in the next section we will show that computing $\mathrm{CoS}^+(G)$ for such games is computationally intractable (Theorem 18). We can, however, provide a *sufficient* condition for $\mathrm{CoS}^+(G)$ to be efficiently computable. To do so, we will first formally state the relevant computational problems.

Super-Imputation-Stability	
Given:	A coalitional game G (compactly represented by a circuit), a supplemental payment Δ, and an imputation $\boldsymbol{p} = (p_1, \dots, p_n)$ in the adjusted game $G(\Delta)$.
Question:	Is it true that $\boldsymbol{p} \in \mathbb{C}(G(\Delta))$?

Cost-of-Stability	
Given:	A coalitional game G (compactly represented by a circuit) and a parameter Δ.
Question:	Is it true that $\mathrm{CoS}^+(G) \leq \Delta$, i.e., that $\mathbb{C}(G(\Delta)) \neq \emptyset$?

Consider first Super-Imputation-Stability. Fix a game $G = \langle N, v\rangle$. Given a super-imputation $\boldsymbol{p}$ for G, let $d(G, \boldsymbol{p}) = \max_{C \subseteq N}(v(C) - p(C))$ be the maximum deficit of a coalition under $\boldsymbol{p}$. Clearly, $\boldsymbol{p}$ is stable if and only if $d(G, \boldsymbol{p}) \leq 0$. Thus a polynomial-time algorithm for computing $d(G, \boldsymbol{p})$ can be converted into a polynomial-time algorithm for Super-Imputation-Stability. Further, we can decide Cost-of-Stability via solving LP_G by the ellipsoid method. The ellipsoid method runs in polynomial time given a polynomial-time *separation oracle*, i.e., a procedure that takes as input a candidate feasible solution, checks whether it is indeed feasible, and if this is not the case, returns a violated constraint. Now, given a vector $\boldsymbol{p}$ and a parameter Δ, we can easily check if they satisfy the constraints of LP_G that check whether $\boldsymbol{p}$ is an imputation for $G(\Delta)$. To verify the stability constraints (i.e., $p(S) \geq v(S)$ for all $S \subseteq N$), we need to check if $\boldsymbol{p}$ is in the core of $G(\Delta)$. As argued above, this can be done by checking whether $d(G, \boldsymbol{p}) \leq 0$. We summarize these results as follows.

[4] A framework to deal with succinctly specified linear programs, i.e., linear programs with exponentially many constraints represented in polynomial space, has recently been proposed and analyzed by Greco, Malizia, Palopoli, and Scarcello (2014).

Theorem 17 *Consider a class of coalitional games $\mathcal{G}$ with a compact circuit representation. If there is an algorithm that for any $G \in \mathcal{G}$, $G = \langle N, v\rangle$, $|N| = n$, and for any super-imputation $\boldsymbol{p}$ for G computes $d(G, \boldsymbol{p})$ in time* $\mathrm{poly}(n, |\boldsymbol{p}|)$*, where $|\boldsymbol{p}|$ is the number of bits in the binary representation of $\boldsymbol{p}$, then for any $G \in \mathcal{G}$ the problems* Super-Imputation-Stability *and* Cost-of-Stability *are polynomial-time solvable.*

We mention in passing that for games with polynomial-time computable characteristic functions both problems are in coNP. For Super-Imputation-Stability, membership is trivial; for Cost-of-Stability, it follows from the fact that the game $G(\Delta)$ has a polynomial-time computable characteristic function as long as G does, and hence we can apply the results of Greco, Malizia, Palopoli, and Scarcello (2011b) (see the proof of Theorem 18 for details).

6.2 Weighted Voting Games: Hardness Results

In this section, we establish the computational hardness of the problems defined above for weighted voting games. In what follows, unless specified otherwise, we assume that all weights and the threshold are integers given in binary, whereas all other numeric parameters, such as the supplemental payment Δ and the entries of the payoff vector $\boldsymbol{p}$, are rationals given in binary. Further, given a weighted voting game $[w_1, \dots, w_n; q]$, we set $w_{\max} = \max\{w_1, \dots, w_n\}$. Standard results on linear threshold functions (Muroga, 1971) imply that weighted voting games with integer weights have a compact circuit representation. Thus we can define the computational problems Super-Imputation-Stability-WVG and Cost-of-Stability-WVG by specializing the problems Super-Imputation-Stability and Cost-of-Stability to weighted voting games. Both of the resulting problems turn out to be computationally hard.

Theorem 18 Super-Imputation-Stability-WVG *and* Cost-of-Stability-WVG *are both* coNP*-complete.*

Proof. Both of our reductions are from Partition, a well-known NP-complete problem (Garey & Johnson, 1979), which is defined as follows: Given a list $A = (a_1, \dots, a_n)$ of nonnegative integers such that $\sum_{i=1}^{n} a_i = 2K$, decide whether there is a sublist A' of A such that $\sum_{a_i \in A'} a_i = K$.

We first show that Cost-of-Stability-WVG is coNP-hard. Given an instance $A = (a_1, \dots, a_n)$ of Partition, we construct a weighted voting game G by setting $N = \{1, \dots, n\}$, $w_i = a_i$ for each i, $1 \le i \le n$, and $q = K$. Set $\Delta = \frac{K-1}{K+1}$. We claim that (G, Δ) is a yes-instance of Cost-of-Stability-WVG if and only if A is a no-instance of Partition.

Indeed, suppose that A is a yes-instance of Partition, and let A' be the corresponding sublist. Set $N' = \{i \mid a_i \in A'\}$ and $N'' = N \setminus N'$. Suppose for the sake of contradiction that $G(\Delta)$ has a nonempty core, and let $\boldsymbol{p}$ be an imputation in the core of $G(\Delta)$. We have $p(N) = \frac{2K}{K+1} < 2$, and hence either $p(N') < 1$ or $p(N'') < 1$ (or both). On the other hand, since $\sum_{i \in N'} a_i = K$, we have $w(N') = w(N'') = K = q$, i.e., at least one of the coalitions N' and N'' is blocking, a contradiction.

On the other hand, suppose that A is a no-instance of Partition, and consider a vector $\boldsymbol{p}^* = (p_1^*, \dots, p_n^*)$, where $p_i^* = \frac{w_i}{K+1}$. We have $p^*(N) = \frac{2K}{K+1}$, and hence $p^*(N) - v(N) = \frac{K-1}{K+1}$.

That is, $\boldsymbol{p}^*$ is an imputation for $G(\Delta)$. We now show that $\boldsymbol{p}^*$ is in the core of $G(\Delta)$ (and thus that $G(\Delta)$ has a nonempty core). Indeed, consider any coalition $C \subset N$ such that $v(C) = 1$. We have $w(C) \geq q$. Moreover, as A is a no-instance of Partition, there is no coalition $C \subset N$ whose weight is exactly q, so we have $w(C) \geq q + 1 = K + 1$. Thus we have $p^*(C) = \frac{w(C)}{K+1} \geq 1$. Hence, C is not a blocking coalition for $\boldsymbol{p}^*$, and therefore $\boldsymbol{p}^* \in \mathbb{C}(G(\Delta))$.

We can use the same construction to show that Super-Imputation-Stability-WVG is coNP-hard. Indeed, consider G, $\Delta = \frac{K-1}{K+1}$, and the payoff vector $\boldsymbol{p}^*$ defined in the previous paragraph. It follows from our proof that $\boldsymbol{p}^*$ is in the core of $G(\Delta)$ if and only if A is a no-instance of Partition. Moreover, Super-Imputation-Stability-WVG is clearly in coNP: To verify that a given super-imputation $\boldsymbol{p}$ is unstable, it suffices to guess a coalition C and check that it is winning, i.e., $w(C) \geq q$, but $p(C) < 1$. Finally, to see that Cost-of-Stability-WVG is in coNP, observe that this problem is equivalent to deciding whether the corresponding game $G(\Delta)$ has a nonempty core. Furthermore, it is easy to see that $G(\Delta)$ has a *polynomial-time compact representation* in the sense of Greco et al. (2011b), Section 3. Thus Theorem 6.8 in the work of Greco et al. (2011b) implies that deciding whether the core of $G(\Delta)$ is nonempty is in coNP. Hence, Cost-of-Stability-WVG is also in coNP. ❑

The reductions in the proof of Theorem 18 are from Partition. Consequently, our hardness results depend in an essential way on the weights being given in binary. Thus it is natural to ask what happens if the agents' weights are polynomially bounded (or given in unary). It turns out that in this case the results of Section 6.1 imply that Super-Imputation-Stability-WVG and Cost-of-Stability-WVG are in P, since for weighted voting games with small weights one can compute $d(G, \boldsymbol{p})$ in polynomial time.

Theorem 19 Super-Imputation-Stability-WVG *and* Cost-of-Stability-WVG *are in* P *when the agents' weights are polynomially bounded (or given in unary).*

Proof. As argued in Section 6.1, it suffices to show that, given a weighted voting game $G = [\boldsymbol{w}; q]$ and a super-imputation $\boldsymbol{p}$ for G, we can compute $d(G, \boldsymbol{p})$ in time $\mathrm{poly}(n, w_{\max}, |\boldsymbol{p}|)$, where $|\boldsymbol{p}|$ denotes the number of bits in the binary representation of $\boldsymbol{p}$.

For any i, $1 \leq i \leq n$, and any w, $1 \leq w \leq w(N)$, let

$$X_{i,w} = \min\{p(C) \mid C \subseteq \{1, \ldots, i\}, w(C) = w\}.$$

We can compute the quantities $X_{i,w}$ inductively as follows. For $i = 1$, we have $X_{i,w} = p_1$ if $w = w_1$, and $X_{i,w} = +\infty$ otherwise. Now, suppose that we have computed $X_{i',w}$ for each i', $1 \leq i' \leq i$. We can then compute $X_{i+1,w}$ as $X_{i+1,w} = \min\{X_{i,w}, p_i + X_{i,w-w_i}\}$. Observe that $p^* = \min\{X_{n,w} \mid w \geq q\}$ is the minimum payment that a winning coalition in G can receive under $\boldsymbol{p}$. As $p_i \geq 0$ for all i, $1 \leq i \leq n$, we have $d(G, \boldsymbol{p}) = 1 - p^*$.

Clearly, the running time of this algorithm is polynomial in n, $w_{\max}$, and $|\boldsymbol{p}|$. Observe that one can construct a similar algorithm that runs in polynomial time even if the weights are large, as long as all entries of $\boldsymbol{p}$ are known to be multiples of $\frac{1}{M}$, where M is a positive integer bounded by a polynomial function of n. ❑

6.3 Approximating the Cost of Stability in Weighted Voting Games

For large weights, the algorithms described at the end of the previous section may not be practical. Thus the center may want to trade off its payment and computation time, i.e., provide a slightly higher supplemental payment for which the corresponding stable super-imputation can be computed efficiently. It turns out that this is indeed possible, i.e., $\mathrm{CoS}^+(G)$ can be efficiently approximated to an arbitrary degree of precision. We also observe (Theorem 23) that simply paying each agent in proportion to her weight results in a 2-approximation to the adjusted gains, i.e., to the quantity $v(N) + \mathrm{CoS}^+(G)$.

Theorem 20 *There exists an algorithm $\mathcal{A}(G, \varepsilon)$ that, given a weighted voting game $G = [\boldsymbol{w}; q]$ in which the weights of all players are nonnegative integers given in binary and a parameter $\varepsilon > 0$, outputs a value Δ that satisfies $\mathrm{CoS}^+(G) \le \Delta \le (1+\varepsilon)\mathrm{CoS}^+(G)$ and runs in time $\mathrm{poly}(n, \log w_{\max}, 1/\varepsilon)$. That is, there exists a fully polynomial-time approximation scheme (FPTAS) for computing $\mathrm{CoS}^+(G)$ in weighted voting games.*

Proof. We start by stating the classic characterization of the outcomes in the core of a simple game (see, e.g., Theorem 3.2 in the book chapter by Elkind and Rothe (2015) and the subsequent remarks).

Lemma 21 *Let $G = \langle N, v\rangle$ be a simple coalitional game. If there are no veto agents in G, then the core of G is empty. Otherwise, let $N' = \{i_1, \dots, i_m\}$ be the set of veto agents in G. Then the core of G is the set of imputations that distribute all the gains among the veto agents only, i.e., $\mathbb{C}(G) = \{\boldsymbol{p} \in \mathbb{I}(G) \mid p(N') = 1, p_i \ge 0$ for all $i \in N\}$.*

Next, we prove a simple lemma that will be useful for the analysis of our algorithm.

Lemma 22 *For every simple game G such that $\mathrm{CoS}^+(G) \neq 0$ we have $\mathrm{CoS}^+(G) \ge 1/n$.*

Proof. Consider a simple game G with $\mathrm{CoS}^+(G) \neq 0$; note that G does not have a veto player. Suppose for the sake of contradiction that $\mathrm{CoS}^+(G) = \Delta < 1/n$, that is, the game $G(\Delta)$ has a nonempty core. Let $\boldsymbol{p} = (p_1, \dots, p_n)$ be an imputation in the core of $G(\Delta)$. As we have $v'(N) = \Delta + 1 > 1$, there must be at least one player i such that $p_i > 1/n$. Hence, $p(N \setminus \{i\}) < 1 + \Delta - 1/n < 1$. Therefore, the coalition $N \setminus \{i\}$ satisfies $v(N \setminus \{i\}) = 1$ (since i is not a veto player), $p(N \setminus \{i\}) < 1$, and hence $\boldsymbol{p}$ is not stable, a contradiction. ❑

Our proof is inspired by the FPTAS for computing the value of the least core in weighted voting games due to Elkind, Goldberg, Goldberg, and Wooldridge (2009).

We first give an additive fully polynomial-time approximation scheme for $\mathrm{CoS}^+(G)$, i.e., an algorithm $\mathcal{A}'(G, \varepsilon)$ that, given a weighted voting game $G = [w_1, \dots, w_n; q]$ and $\varepsilon > 0$, can compute a value Δ satisfying $\mathrm{CoS}^+(G) \le \Delta \le \mathrm{CoS}^+(G) + \varepsilon$ and runs in time $\mathrm{poly}(n, \log w_{\max}, 1/\varepsilon)$. We will then show how to convert it into an FPTAS using Lemma 22.

Set $X = 2\lceil 1/\varepsilon\rceil$, and let $\varepsilon' = 1/X$. We have $\varepsilon/4 \le \varepsilon' \le \varepsilon/2$.

Consider the linear program LP_G given in Section 3.2. Instead of solving LP_G directly, we consider a family of linear feasibility programs (LFP) $(\mathcal{L}_i)_{i=1,\dots,nX}$, where the kth LFP

$\mathcal{L}_k$ is given by

$$\begin{aligned} p_i &\geq 0 \quad \text{for } i = 1, \ldots, n, \\ p_1 + \cdots + p_n &\leq 1 + \varepsilon' k, \\ \sum_{i \in C} p_i &\geq 1 \quad \text{for all } C \subseteq N \text{ such that } \sum_{i \in C} w_i \geq q. \end{aligned}$$

As $\varepsilon' n X = n$, it follows that at least one of these LFPs has a feasible solution. Now, let k^* be the smallest value of k for which $\mathcal{L}_k$ has a feasible solution. We have $\varepsilon'(k^* - 1) < \mathrm{CoS}^+(G) \leq \varepsilon' k^*$, or, equivalently, $\mathrm{CoS}^+(G) \leq \varepsilon' k^* < \mathrm{CoS}^+(G) + \varepsilon'$. Hence, by computing k^* we can obtain an additive ε'-approximation to $\mathrm{CoS}^+(G)$. Now, while it is not clear if we could find k^* in polynomial time, we will now show how to find a value k that is guaranteed to be in the set $\{k^*, k^* + 1\}$.

It is natural to approach this problem by trying to successively solve $\mathcal{L}_1, \ldots, \mathcal{L}_{nX}$. However, just as the linear program LP_G, the LFP $\mathcal{L}_k$ has exponentially many constraints (one for each winning coalition of G). Moreover, an implementation of the separation oracle for $\mathcal{L}_k$ would involve solving Knapsack, which is an NP-hard problem when weights are given in binary. Hence, we will now take a somewhat different approach. Namely, we will show how to design an algorithm $\mathcal{S}$ that, given a candidate solution $(p_1, \ldots, p_n)$ for $\mathcal{L}_k$, either outputs a constraint that is violated by this solution or finds a feasible solution for $\mathcal{L}_{k+1}$. The running time of $\mathcal{S}(p_1, \ldots, p_n)$ is $\mathrm{poly}(n, \log w_{\max}, 1/\varepsilon)$.

The algorithm $\mathcal{S}$ first checks if the candidate solution $(p_1, \ldots, p_n)$ satisfies the first $n+1$ constraints of the LFP. If no violated constraint is discovered at this step, it rounds up the payoffs by setting

$$p'_i = \min \left\{ \frac{\varepsilon' t}{n} \,\middle|\, t \in \mathbb{N}, \frac{\varepsilon' t}{n} \geq p_i \right\} \qquad \text{for each } i,\ 1 \leq i \leq n.$$

Note that for each i, $1 \leq i \leq n$, we have $p_i \leq p'_i \leq p_i + \frac{\varepsilon'}{n}$, and the rounded payoff p'_i can be represented as $p'_i = \frac{\varepsilon'}{n} t_i$, where $t_i \in \{0, \ldots, nX\}$. We can now use a variant of the dynamic programming algorithm used in the proof of Theorem 19 to decide whether there is a subset of agents C that satisfies $\sum_{i \in C} w_i \geq q$ and $\sum_{i \in C} p'_i < 1$ (see the remark at the end of that proof). If there is such a subset, the rounded vector $(p'_1, \ldots, p'_n)$ violates the constraint that corresponds to C, and hence the original vector $(p_1, \ldots, p_n)$, which satisfies $p_i \leq p'_i$ for all $i \in N$, violates it, too. Hence, $\mathcal{S}$ outputs the corresponding constraint and stops. Otherwise, it follows that $(p'_1, \ldots, p'_n)$ satisfies all constraints of $\mathcal{L}_k$ that correspond to the winning coalitions of G. Moreover, we have

$$\sum_{i=1}^{n} p'_i \leq \sum_{i=1}^{n} p_i + n \frac{\varepsilon'}{n} \leq 1 + \varepsilon' k + \varepsilon'.$$

Hence, $(p'_1, \ldots, p'_n)$ is a feasible solution for $\mathcal{L}_{k+1}$, so $\mathcal{S}$ outputs it and stops.

We are now ready to describe our algorithm $\mathcal{A}'(G, \varepsilon)$. It tries to solve $\mathcal{L}_1, \mathcal{L}_2, \ldots$ (in this order). To solve $\mathcal{L}_k$, it runs the ellipsoid algorithm on its input. Whenever the ellipsoid algorithm makes a call to the separation oracle, $\mathcal{A}'$ passes this request to $\mathcal{S}$, which either identifies a violated constraint, in which case $\mathcal{A}'$ continues simulating the ellipsoid algorithm, or outputs a feasible solution for $\mathcal{L}_{k+1}$, in which case $\mathcal{A}'$ stops and outputs $\varepsilon'(k+1)$. If the ellipsoid algorithm terminates and decides that the current LFP does not have a feasible solution, $\mathcal{A}'$ proceeds to the next LFP in its list. If the ellipsoid algorithm outputs a feasible solution for $\mathcal{L}_k$, $\mathcal{A}$ outputs $\varepsilon' k$.

Recall that we denote by k^* the smallest value of k for which $\mathcal{L}_k$ has a feasible solution. Clearly, $\mathcal{A}$ will correctly report that neither of $\mathcal{L}_1, \dots, \mathcal{L}_{k^*-2}$ has a feasible solution. When solving $\mathcal{L}_{k^*-1}$, it will either solve it correctly (i.e., report that it has no feasible solutions) and move on to $\mathcal{L}_{k^*}$, or discover a feasible solution for $\mathcal{L}_{k^*}$. In the former case, $\mathcal{A}'$ will either solve $\mathcal{L}_{k^*}$ correctly, i.e., find a feasible solution, or discover a feasible solution to $\mathcal{L}_{k^*+1}$. In either case, the output $\varepsilon' k$ of our algorithm satisfies $k \in \{k^*, k^*+1\}$.

We have shown that $\mathrm{CoS}^+(G) \le \varepsilon' k^* \le \mathrm{CoS}^+(G) + \varepsilon'$. Consequently, we have $\mathrm{CoS}^+(G) \le \varepsilon' k \le \varepsilon'(k^*+1) \le \mathrm{CoS}^+(G) + 2\varepsilon' \le \mathrm{CoS}^+(G) + \varepsilon$. This proves that $\mathcal{A}'$ is an additive fully polynomial-time approximation scheme for the cost of stability.

We will now show how to convert $\mathcal{A}'$ into an FPTAS $\mathcal{A}$. Our algorithm $\mathcal{A}$ is given a game $G = [\boldsymbol{w}; q]$ and a parameter ε. It first tests if $\mathrm{CoS}^+(G) = 0$ (equivalently, if G has a nonempty core). By Lemma 21, this can be done by checking if G has a veto player, i.e., whether $w(N \setminus \{i\}) < q$ for some i, $1 \le i \le n$.

If $\mathrm{CoS}^+(G) \neq 0$, $\mathcal{A}$ runs $\mathcal{A}'$ on input $(G, \varepsilon/n)$. Let Δ be the output of $\mathcal{A}'(G, \varepsilon/n)$; we have $\mathrm{CoS}^+(G) \le \Delta \le \mathrm{CoS}^+(G) + \varepsilon/n$. On the other hand, by Lemma 22 we have $\mathrm{CoS}^+(G) \ge 1/n$, and therefore

$$\mathrm{CoS}^+(G) + \varepsilon/n \le \mathrm{CoS}^+(G) + \varepsilon \mathrm{CoS}^+(G) = (1+\varepsilon)\mathrm{CoS}^+(G).$$

Hence, Δ satisfies $\mathrm{CoS}^+(G) \le \Delta \le (1+\varepsilon)\mathrm{CoS}^+(G)$, as required. ❑

We will now discuss the approximation guarantees of a simple procedure that pays each agent in proportion to her weight.

Theorem 23 *For every weighted voting game $G = [\boldsymbol{w}; q]$ with $\mathrm{CoS}^+(G) = \Delta$, the super-imputation $\boldsymbol{p}^*$ given by $p_i^* = \min\left\{1, \frac{w_i}{q}\right\}$ is stable. Moreover, for any super-imputation $\boldsymbol{p} \in \mathbb{C}(G(\Delta))$ we have $p^*(N) \le 2p(N)$.*

Proof. First, it is easy to see that $\boldsymbol{p}^*$ is stable, as we have $p^*(C) \ge \min\left\{1, \frac{w(C)}{q}\right\}$.

Now, set $\Delta = \mathrm{CoS}^+(G)$ and fix a super-imputation $\boldsymbol{p}$ in the core of $G(\Delta)$. Let $N' = \{i \mid w_i \ge q\}$ and set $k = |N'|$. Clearly, if $i \in N'$, for any stable super-imputation $\boldsymbol{p}'$ we have $p'_i \ge 1 = p^*_i$. On the other hand, it is clear that paying any agent more than 1 is suboptimal, so $p_i = 1$ for any $i \in N'$.

Sort the agents in $N \setminus N'$ in the order of nonincreasing weight, and partition them into sets $C_1, \dots, C_m$ in the following way:

$j \leftarrow 0$
while there are unallocated agents **do**
  $j \leftarrow j + 1$
  Add agents to C_j until $w(C_j) \ge q$ or until there are no more agents.
end while
if $w(C_j) < q$ **then**

$m \leftarrow j$
else
$m \leftarrow j+1$, $C_m = \emptyset$
end if

Note that this procedure guarantees that $w(C_m) < q$, i.e., the last coalition C_m loses. In particular, if $m = 1$ then $w(C_1) < q$, and since $w(N) \geq q$, this means that $k \geq 1$ and $C_1 = N \setminus N'$. In this case, we have

$$p(N) \geq k, \quad p^*(N) = k + \sum_{i \in C_1} \frac{w_i}{q} < k + \frac{q}{q} = k+1,$$

and hence $p^*(N)/p(N) < (k+1)/k \leq 2$. Therefore, throughout the remainder of the proof we can assume that $m > 1$.

Set $j' = \text{argmax}_{1 \leq j \leq m} w(C_j)$, that is, j' is the index of a maximum-weight coalition among $C_1, \ldots, C_m$. Observe that since $w(C_1) \geq q$ and $w(C_m) < q$, we have $j' \neq m$. To finish the proof, we consider two cases and show that $p^*(N) \leq 2p(N)$ in each case.

Case 1: $w(C_{j'}) + w(C_m) \leq 2q$. For each $j \leq m-1$, we have $w(C_j) \geq q$, and therefore $p(C_j) \geq 1$. Thus we have

$$p(N) \geq k + \sum_{j \neq m} p(C_j) \geq k + m - 1.$$

On the other hand, we have $w(C_j) \leq 2q$ for all j, $1 \leq j \leq m$, so

$$\begin{aligned} p^*(N) &= p^*(N') + \sum_{j \neq j', m} p^*(C_j) + p^*(C_{j'}) + p^*(C_m) \\ &= k + \sum_{j \neq j', m} \frac{w(C_j)}{q} + \frac{w(C_{j'}) + w(C_m)}{q} \\ &\leq k + 2(m-2) + 2 \leq 2(k+m-1) \leq 2p(N). \end{aligned}$$

Case 2: $w(C_{j'}) + w(C_m) > 2q$. We begin by bounding $p^*(N)$, as it may be slightly larger in this case:

$$\begin{aligned} p^*(N) &= k + \sum_{j \neq m} \frac{w(C_j)}{q} + \frac{w(C_m)}{q} \\ &\leq k + \frac{(m-1)2q + q}{q} = k + 2m - 1. \end{aligned}$$

Fortunately, we can provide a better lower bound for $p(N)$. Let A_1 be the set that contains the last player in $C_{j'}$ only, and set $A_2 = C_{j'} \setminus A_1$ and $A_3 = C_m$. We have $w(A_1) < q$, since A_1 has just one agent, and we have already removed all agents whose weight is at least q. Furthermore, we have $w(A_2) < q$, since we move on to the next set as soon as a total weight of at least q is reached in the current set. On the other hand, we have $A = A_1 \cup A_2 \cup A_3 = C_{j'} \cup C_m$. As $w(C_{j'}) + w(C_m) > 2q$, we have

$w(A_1) + w(A_3) = w(A) - w(A_2) \geq 2q - q = q$ and $w(A_2) + w(A_3) = w(A) - w(A_1) \geq 2q - q = q$.

Therefore, we have $p(A_1 \cup A_2) \geq 1$, $p(A_1 \cup A_3) \geq 1$, $p(A_2 \cup A_3) \geq 1$, and hence $p(A_1 \cup A_2 \cup A_3) \geq 3/2$. Thus we have

$$\begin{aligned} p(N) &= \sum_{i \in N'} p_i + \sum_{j \neq j', m} p(C_j) + p(C_{j'}) + p(C_m) \\ &\geq k + (m-2) + p(C_{j'} \cup C_m) \\ &= k + m - 2 + p(A_1 \cup A_2 \cup A_3) \\ &\geq k + m - 2 + \frac{3}{2} = \frac{2k + 2m - 1}{2} \geq \frac{p^*(N)}{2}. \end{aligned}$$

This completes the proof of Theorem 23. ❑

7. Related Work

The term "cost of stability" was introduced in a preliminary version of this work by Bachrach, Meir, Zuckerman, Rothe, and Rosenschein (2009b), Bachrach, Elkind, Meir, Pasechnik, Zuckerman, Rothe, and Rosenschein (2009a), who proved several bounds on the additive cost of stability, and presented computational complexity results for problems related to the cost of stability in some classes of games. Two other papers, by subsets of the current set of authors (Meir, Bachrach, & Rosenschein, 2010; Meir, Rosenschein, & Malizia, 2011), studied the cost of stability in cost-sharing games and games with restricted cooperation, as well as the relationship between the cost of stability and the least core. The current paper covers all of these previous results as well as improved bounds and a number of results that do not appear in the earlier papers.

7.1 Recent Research on Subsidies and the Cost of Stability

Since the first of these papers has been published, several groups of researchers studied the cost of stability, focusing mainly on computational questions. Resnick, Bachrach, Meir, and Rosenschein (2009) examined the cost of stability in threshold network flow games, a family of simple games played on flow networks where a coalition of edges wins if it can guarantee a sufficient flow from the source to the sink. Aziz, Brandt, and Harrenstein (2010) studied the complexity of computing the cost of stability and the least core in a variety of coalitional games, comparing games with thresholds (such as threshold network flow games and weighted voting games) to their variants without a threshold. Aadithya, Michalak, and Jennings (2011) showed that for coalitional games represented by algebraic decision diagrams the cost of stability can be computed in polynomial time. Greco, Malizia, Palopoli, and Scarcello (2011a) proved bounds on the complexity of computing the cost of stability, for games with and without coalition structures. Persien, Rey, and Rothe (2016) studied the cost of stability in relation to the least core in path-disruption games.

Meir, Zick, Elkind, and Rosenschein (2013) studied bounds on the cost of stability when cooperation is restricted by a network of social connections in the model of Myerson (1977), rather than by the size of the coalition. These bounds where further improved by

Bousquet, Li, and Vetta (2015). Chalkiadakis, Greco, and Markakis (2016) studied various computational questions related to core stability on such network-restricted games, and in particular showed that computing the cost of stability is easy for some networks (cycles, trees) but computationally hard for other networks. The exact complexity class also depends on whether the game is superadditive.

A model for subsidies in coalitional games was independently suggested by Bejan and Gómez (2009), who focused (as we do in Section 5) on the relationship between subsidies and other solution concepts. However, in their work the additional payment required to stabilize a game is collected from the participating agents by means of a specific *taxation* system, rather than injected into the game by an external authority, whereas we do not assume any form of taxation. The taxation approach was extended by Zick, Polukarov, and Jennings (2013), who also studied the connections between taxes and the cost of stability. The relation between the cost of stability and another property of TU games (which aims to measure how far a game is from being a weighted voting game) was studied by Freixas and Kurz (2014).

7.2 Approximate Core

Several other researchers studied subsidies and other incentive issues in cost-sharing games using different terminology. Specifically, Deng, Ibaraki, and Nagamochi (1999) show that a coalitional game whose characteristic function is given by an integer program of a certain form has a nonempty core if and only if the linear relaxation of this problem has an integer solution. Their argument can be used to relate the multiplicative cost of stability in such games and the integrality gap of the respective program (see Appendix A). The connection between the integrality gap and the multiplicative cost of stability is made by Goemans and Skutella (2004) in the context of facility location games.

A number of other authors have studied the cost of stability in a variety of combinatorial optimization games, sometimes adding other requirements on top of minimization of subsidies. A common assumption is that players gain some *private utility* from participating in the game; in contrast, our model assumes that participation is mandatory, or, equivalently, that the utility derived from participation is sufficiently high to guarantee participation at any cost. Specifically, Devanur, Mihail, and Vazirani (2005) suggested a mechanism that is strategyproof and recovers at least a fraction of $\frac{1}{\ln n+1}$ of the total cost in set cover games, and a constant fraction (namely, 0.462) in metric facility location games. Our results in Appendix A imply that for set cover games this bound is tight, even if the strategyproofness requirement is dropped.

An application that has drawn much attention is routing in networks, which was initially formulated as a minimum spanning tree (MST) game (Claus & Kleitman, 1973). In an MST game the agents are nodes of a graph, and each edge is a connection that has a fixed price. The cost of a coalition is the price of the cheapest tree that connects all participating nodes to the source node. The multiplicative cost of stability in this model is always 1, as the core of an MST is never empty (Granot & Huberman, 1981). However, there is a more realistic variant of this game known as the Steiner tree game, where nodes are allowed to route through nodes that are not part of their coalition. Megiddo (1978) showed that the core of a Steiner tree game may be empty, and therefore its cost of stability is nontrivial. Jain and

Vazirani (2001) proposed a mechanism for the Steiner tree game with multiplicative cost of stability of $1/2$, under the stronger requirements of group strategyproofness.[5] Könemann, Leonardi, Schäfer, and van Zwam (2008, 2005) put forward mechanisms for the more general Steiner forest game that have the same cost of stability, and suggest that this bound is tight.

It is interesting to note that the value of the optimal Steiner tree (i.e., the value of the grand coalition) can be written as an integer linear program whose integrality gap is lower-bounded by $1/2$ as well (Vazirani, 2003, Example 22.10, p. 206). However, in contrast to set cover games, for Steiner tree games we do not know if the integrality gap is always equal to the inverse of the multiplicative cost of stability, or if better cost-sharing mechanisms are possible when the strategyproofness constraint is relaxed. In fact, a different line of research by Skorin-Kapov (1995) suggested a cost-sharing mechanism for Steiner trees that does not guarantee strategyproofness, and showed *empirically* that it allocates at least 92% of the cost on all tested instances.

Other cost-sharing mechanisms have been suggested for many different games. For example, Moulin and Shenker (2001) studied the tradeoff between efficiency and the cost of stability in subadditive games; further results are provided by Pál and Tardos (2003) and Immorlica, Mahdian, and Mirrokni (2005). Some of the proposed mechanisms impose strong requirements such as group strategyproofness, in addition to stability. Therefore, it is an interesting question whether tighter bounds on the cost of stability for specific families of games can be derived once these requirements are relaxed.

7.3 Subsidies in Normal-Form Games

The idea of providing subsidies to ensure stability has also been explored in the context of normal-form games. Monderer and Tennenholtz (2004) investigate the setting where an interested party wishes to influence the behavior of agents in a game not under its control. In spirit, their approach is close to the one we take here: The interested party may commit to making nonnegative payments to the agents if certain strategy profiles are selected. Payments are given to agents individually, but they are dependent on the strategies selected by all agents. As in our work, it is assumed that the interested party wishes to minimize its expenses. Determining the optimal monetary offers to be made in order to implement a desired outcome is shown to be NP-hard in general, but becomes tractable under certain constraints. Also, it is sometimes possible for the external party to stabilize a particular outcome without paying anything, which is clearly impossible in our setting.

Another closely related paper is that by Buchbinder, Lewin-Eytan, Naor, and Orda (2010), who study subsidies in a normal-form version of the set cover game. However, the focus of this paper is on efficiency rather than stability, and the subsidy is financed by taxes collected from the users.

8. Conclusions and Future Work

We have examined the possibility of stabilizing a coalitional game by offering the agents additional payments in order to discourage them from deviating. We provided bounds on the

[5] More precisely, Jain and Vazirani (2001) demanded full cost recovery and relaxed stability constraints. The bound on the cost of stability is achieved if we divide their proposed payments by 2.

Table 1: The multiplicative cost of stability for various classes of TU games. The corresponding theorem or proposition appears in brackets. All bounds are either exactly or asymptotically tight.

$\mathrm{CoS}^{\times}(G)$	any	super/sub-additive		
		all	WVG	anonymous
Profit-sharing (upper bound)	n	$\sqrt{n}$ (Thm. 7)	2 (Thm. 10)	$\frac{2n}{n+1}$ (Thm. 9)
Cost-sharing (lower bound)	0	$\frac{1}{\ln n+1}$ (Prop. 29)	-	$\frac{n}{2(n-1)}$ (Thm. 31)

Table 2: The ratio $\frac{\mathrm{CoS}^{+}(G)}{\varepsilon_{\mathbf{X}}(G)}$ for a profit-sharing game G, where $\varepsilon_{\mathbf{X}}$ varies according to the variant of the least core (weak/strong, with/without positive payoff requirement). Results with (*) are due to Bejan and Gómez (2009). All bounds are either exactly or asymptotically tight. **SA** and **M** stand for the additional requirements of superadditivity and monotonicity, respectively.

$\mathrm{CoS}^{+}(G)/\varepsilon(G)$	WLC	PWLC	SLC	PSLC
Upper bound	n (*)	$\Rightarrow$ n	$\sqrt{n}$ (**SA**, Thm. 13)	$\Rightarrow$ $\sqrt{n}$ (**SA**)
Lower bound	n (*)	2 (**M**, Prop. 12)	$\frac{n-1}{n}$ (Thm. 14)	$\frac{n-1}{n}$ (**M**, Thm. 16)

cost of stability both for general games and under various restrictions on the characteristic function, such as superadditivity and anonymity, and our results are summarized in Table 1.

It should be noted that our results naturally extend to the case where we drop the superadditivity requirement, but allow agents to form coalition structures rather than the grand coalition. By using the superadditive cover, as defined by Aumann and Dréze (1974) (or the subadditive cover for cost-sharing games), we get the same tight bounds on the multiplicative cost of stabilizing the *optimal coalition structure*.

We have also explored the relationship between the cost of stability, the (strong and weak) least core, and its variants such as the positive least core (Table 2). Finally, we have studied the complexity of computing the cost of stability in weighted voting games and have obtained both hardness results and efficient approximation algorithms.

There are several lines of possible future research. First, it would be interesting to study the cost of stability in other restricted classes of games, for example games defined by limited MC-nets (Li & Conitzer, 2014; Lesca, Perny, & Yokoo, 2017), or games with a bounded number of player types. For the latter, it has been shown that the nucleolus can be computed in polynomial time, because, given the bounded number of different player types, the linear program needed for nucleolus computation, which in general requires exponentially many constraints, can be replaced by a linear program with only polynomially many constraints (Greco, Malizia, Scarcello, & Palopoli, 2012; Greco et al., 2014). It would be interesting to verify whether this approach can be extended to the computation of the cost of stability. Another direction is to explore how the cost of stability is affected by additional constraints on the cost/profit-sharing mechanisms, such as truthfulness, fairness, and efficiency. For example, we could require the core of the adjusted game to contain a "fair" imputation (e.g., the Shapley imputation), rather than just be nonempty. Finally, the notion of the cost of stability could be extended to games with nontransferable utility and to partition function games, where subsidies may assume more complicated forms.

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Appendix A. Cost-Sharing Games

We denote a cost-sharing game by $G = \langle N, c\rangle$. The definition of the cost of stability for cost-sharing games is similar to the one for profit-sharing games: The additive cost of stability specifies the minimum subsidy that keeps the grand coalition stable, and the multiplicative cost of stability specifies the fraction of the cost of the grand coalition that the agents can cover without subsidy (the cost recovery ratio):

$$\mathrm{CoS}^{+}(G) = \inf\left\{c(N) - p(N) \mid p(S) \leq c(S) \text{ for all } S \subseteq N\right\}, \tag{4}$$

$$\mathrm{CoS}^{\times}(G) = \sup\left\{\frac{p(N)}{c(N)} \;\middle|\; p(S) \leq c(S) \text{ for all } S \subseteq N\right\}. \tag{5}$$

A.1 Cost-Profit Duality

There is a natural mapping between profit-sharing games and cost-sharing games that is known to preserve core emptiness/nonemptiness (Potters & Sudhölter, 1999). Thus one may wonder if we really need to derive bounds on the cost of stability for cost-sharing games from scratch, rather than simply infer them from the bounds in Section 4. We briefly describe this dual mapping (not to be confused with linear duality) and show that the cost of stability of a game reveals almost no information on the cost of stability of its dual.

The *dual* of a game $G = \langle N, v \rangle$ (where v is either value or cost) is denoted by $G^* = \langle N, v^* \rangle$ where

$$v^*(S) = \begin{cases} v(S) & \text{if } S = N \\ v(N) - v(N \setminus S) & \text{if } S \neq N. \end{cases}$$

Clearly, the dual of G^* is G again. Suppose that G is a profit-sharing game and G^* is a cost-sharing game (or vice versa). It is known that the core of G is empty if and only if the core of G^* is empty; Bilbao (2000) provides a simple proof (p. 4). Hence, $\mathrm{CoS}^{\times}(G) = 1$ if and only if $\mathrm{CoS}^{\times}(G^*) = 1$. Also, G^* is monotone if and only if G is. One might therefore expect that there exists a function f such that $\mathrm{CoS}^{\times}(G) = f(\mathrm{CoS}^{\times}(G^*))$ for every cost-sharing game G. Unfortunately, the following proposition shows that the cost of stability of a game does not reveal much about the cost of stability of its dual (as long as the core is known to be empty).

Proposition 24 *For every positive integer n there are monotone anonymous cost-sharing n-player games G_a and G_b with* $\mathrm{CoS}^{\times}(G_a^*) = \mathrm{CoS}^{\times}(G_b^*)$, $\mathrm{CoS}^{\times}(G_a) > \frac{1}{2}$, $\mathrm{CoS}^{\times}(G_b) = 0$.

Proof. We will proceed by explicitly defining profit-sharing games G_a^* and G_b^* and then computing their duals. For $x \in \{a, b\}$ we set $v_x(N) = n$ and $v_x(\{i\}) = n/2$ for all $i \in N$. For a coalition S, $|S| \neq 0, 1, n$, we set $v_a(S) = n/2$ and $v_b(S) = n$. Then $\mathrm{CoS}^{\times}(G_a^*) = \mathrm{CoS}^{\times}(G_b^*) = n/2$, since each agent has to be paid $n/2$ to cooperate.

In the dual cost-sharing game G_a, we have $c_a(N) = n$ and, moreover, $c_a(S) = n - v_a(N \setminus S) = n - (n/2) = n/2$ for $S \neq \emptyset, N$. Note that the payoff vector with $p_i = \frac{n}{2(n-1)}$ for each $i \in N$ is stable: For each $S \subsetneq N$, we have $p(S) = \frac{n}{2(n-1)}|S| \leq \frac{n}{2} = c_a(S)$. Thus $\mathrm{CoS}^{\times}(G_a) \geq \frac{p(N)}{c_a(N)} = \frac{n}{2(n-1)} > \frac{1}{2}$.

In G_b, we have $c_b(N) = n$, $c_b(S) = n/2$ if $|S| = n-1$, and $c_b(S) = 0$ if $|S| < n-1$. In particular, $c_b(\{i\}) = 0$ for all singletons. Since each agent can get the service for free, $\mathrm{CoS}^{\times}(G_b) = 0$. ❑

Another approach to transform a cost-sharing game into a profit-sharing game is by considering the *cost-savings game* associated with the given cost game (Young, 1985). Formally, given a cost-sharing game $G = \langle N, c \rangle$, we define the associated game $G' = \langle N, v_c \rangle$ by setting

$$v_c(S) = \sum_{i \in S} c(\{i\}) - c(S) \qquad \text{for all } S \subseteq N.$$

Intuitively, the value of a coalition S in the game G' is the cost savings that agents in S realize by staying together. We note that G' may fail to be a profit-sharing game, as it may be the case that $\sum_{i \in S} c(\{i\}) < c(S)$ for some $S \subseteq N$; however, if G is subadditive, we have $v_c(S) \geq 0$ for all $S \subseteq N$. We will now show that the additive cost of stability of a cost-sharing game is equal to the additive cost of stability of the associated cost-savings game whenever the latter is well-defined.

Proposition 25 *Consider a cost-sharing game $G = \langle N, c\rangle$ such that $\sum_{i\in S} c(\{i\}) \geq c(S)$ for all $S \subseteq N$. Then* $\mathrm{CoS}^+(G) = \mathrm{CoS}^+(G')$*.*

Proof. The assumption that $\sum_{i\in S} c(\{i\}) \geq c(S)$ for all $S \subseteq N$ ensures that G' is a profit-sharing game. Let $\Delta = \mathrm{CoS}^+(G')$. We will show that $\mathrm{CoS}^+(G) \leq \Delta$. Consider a payoff vector $\boldsymbol{p}$ for G' that satisfies $p(N) = v_c(N) + \Delta$, $p(S) \geq v_c(S)$ for all $S \subseteq N$. Define a payoff vector $\boldsymbol{q}$ for G by setting $q_i = c(\{i\}) - p_i$ for $i \in N$. Note that for any $S \subseteq N$ we have

$$q(S) = \sum_{i\in S} c(\{i\}) - p(S) \leq \sum_{i\in S} c(\{i\}) - v_c(S) = c(S)$$

and

$$q(N) = \sum_{i\in N} c(\{i\}) - p(N) = \sum_{i\in N} c(\{i\}) - v_c(N) - \Delta = c(N) - \Delta.$$

This establishes that $\mathrm{CoS}^+(G) \leq \Delta$. Conversely, suppose that $\mathrm{CoS}^+(G) = \Delta$ and consider a payoff vector $\boldsymbol{q}$ for G that satisfies $q(N) = c(N) - \Delta$, $q(S) \leq c(S)$ for all $S \subseteq N$. Set $p_i = c(\{i\}) - q_i$. Then for any $S \subseteq N$ we have

$$p(S) = \sum_{i\in S} c(\{i\}) - q(S) \geq \sum_{i\in S} c(\{i\}) - c(S) = v_c(S)$$

and

$$p(N) = \sum_{i\in N} c(\{i\}) - q(N) = \sum_{i\in N} c(\{i\}) - c(N) + \Delta = v_c(N) + \Delta.$$

Hence, $\mathrm{CoS}^+(G') \leq \Delta$. Thus, we have established that $\mathrm{CoS}^+(G') = \mathrm{CoS}^+(G)$. ❑

However, the following result demonstrates that it is difficult to relate the multiplicative cost of stability of G to that of G' (and hence we cannot use the mapping from G to G' in order to import our bounds for profit-sharing games to the cost-sharing setting).

Proposition 26 *For each positive integer n there are anonymous subadditive cost-sharing n-player games G_a and G_b with* $\mathrm{CoS}^\times(G_a) = \mathrm{CoS}^\times(G_b)$ *and* $\mathrm{CoS}^\times(G'_a) \neq \mathrm{CoS}^\times(G'_b)$*.*

Proof. We define three-player cost-sharing games G_a and G_b; the construction can be easily extended to any number of players by adding dummy players. Both in G_a and in G_b the cost of every coalition of size two is 10 and the cost of the grand coalition is 17. In G_a the cost of every singleton coalition is 10, whereas in G_b it is 7. Note that both G_a and G_b are subadditive, and by construction both of these games are anonymous. Further, $\mathrm{CoS}^\times(G_a) = \mathrm{CoS}^\times(G_b) = \frac{15}{17}$: $(5, 5, 5)$ is a stable payoff vector for both games, and for every stable payoff vector $\boldsymbol{p}$ we have $p_1 + p_2 \leq 10$, $p_1 + p_3 \leq 10$, $p_2 + p_3 \leq 10$ and hence $p(N) \leq 15$.

Now, in G'_a the value of the grand coalition is 13 and the value of every coalition of size two is 10, whereas in G'_b the value of the grand coalition is 4 and the value of every coalition of size two is 4 as well; by Theorem 8 we have $\mathrm{CoS}^\times(G'_a) = \frac{3}{13} \cdot \frac{10}{2} = \frac{15}{13}$ and $\mathrm{CoS}^\times(G'_b) = \frac{3}{4} \cdot \frac{4}{2} = \frac{3}{2}$. ❑

A.2 Cost-of-Stability Bounds for Cost-Sharing Games

An interesting special class of cost-sharing games is the class of *set cover games (SCGs)*. Briefly, an SCG is described by an instance of the set cover problem: The agents are elements of the ground set, and the cost of a coalition S is the cost of a cheapest collection of subsets that cover all elements of S. More formally, a set cover game is a cost-sharing game given by a tuple $\langle N, \mathcal{F}, w\rangle$, where $N = \{1, \dots, n\}$ is a set of agents, $\mathcal{F}$ is a collection of subsets of N that satisfies $\bigcup_{F\in\mathcal{F}} F = N$, and $w : \mathcal{F} \to \mathbb{R}_+$ is a mapping that assigns a nonnegative weight to each set in $\mathcal{F}$. The cost of a coalition $S \subseteq N$ is given by

$$c(S) = \min\left\{ \sum_{F\in\mathcal{F}'} w(F) \;\middle|\; \mathcal{F}' \subseteq \mathcal{F}, S \subseteq \bigcup_{F\in\mathcal{F}'} F \right\}.$$

Theorem 1 of Deng et al. (1999) can be adapted to show that the cost recovery ratio of SCGs can be derived directly from the underlying set cover problem.

Theorem 27 (Deng et al. (1999)) *Let G be an SCG. Then $\mathrm{CoS}^{\times}(G) = \frac{1}{\mathrm{IG}(G)}$, where $\mathrm{IG}(G)$ is the integrality gap of G, i.e., the ratio between the values of the optimal integer and fractional solutions of the set cover problem associated with G.*

The integrality gap of the set cover problem is well-studied in the literature: It is known to be bounded from above by $H_n = \sum_{i=1}^{n} \frac{1}{i} < \ln n + 1$. Moreover, this bound is essentially tight, even when the sets are nonweighted (Chvátal, 1979; Slavík, 1997). We will now show that *any* monotone subadditive cost-sharing game can be represented as an SCG.

Lemma 28 *Set cover games are monotone and subadditive. Furthermore, every monotone and subadditive cost-sharing game can be described as a set cover game.*

Proof. Fix a set cover game G given by a tuple $\langle N, \mathcal{F}, w\rangle$. Clearly, G is monotone. Further, for every pair of subsets $S, T \subseteq N$ we have $S \cup T \subseteq \mathcal{F}^*(S) \cup \mathcal{F}^*(T)$. Therefore, $c(S \cup T) \le c(S) + c(T)$, i.e., G is subadditive.

Conversely, given a monotone subadditive cost-sharing game $G = \langle N, c\rangle$, we construct a set cover game by setting $\mathcal{F} = 2^N$ and $w(F) = c(F)$ for every $F \in \mathcal{F}$. We will now argue that the resulting game $G' = \langle N, c'\rangle$ is equivalent to G. Indeed, consider a set S and its cheapest cover $\mathcal{F}^*(S)$. We have $c'(S) = \sum_{F\in\mathcal{F}^*(S)} c(F)$. Since G is monotone, we can assume that the sets in $\mathcal{F}^*(S)$ are pairwise disjoint: If we have $F_1 \cap F_2 \ne \emptyset$ for some $F_1, F_2 \in \mathcal{F}^*(S)$, we can replace F_2 with $F_2 \setminus F_1$ without increasing the overall cost. Now, set $F' = \cup_{F\in\mathcal{F}^*(S)} F$. Subadditivity of G implies that $c(F') \le \sum_{F\in\mathcal{F}^*(S)} c(F) = c'(S)$. Further, since S is a subset of F', we have $c(S) \le c(F')$ and hence $c(S) \le c'(S)$. On the other hand, $\{S\}$ is a cover of S, so we have $c'(S) \le c(S)$. Thus $c'(S) = c(S)$. Since this holds for every set $S \subseteq N$, the games G and G' are equivalent. ❑

Thus we obtain the following bound as a corollary.

Proposition 29 *Let $G = \langle N, c\rangle$ be a monotone subadditive cost-sharing game with $|N| = n$. Then $\mathrm{CoS}^{\times}(G) \ge \frac{1}{\ln n+1}$ and this bound is asymptotically tight.*

A.3 Anonymous Cost-Sharing Games

Just as for profit-sharing games, an anonymous cost-sharing game $G = \langle N, c\rangle$ can be described by a list of n numbers $c_1, \dots, c_n$, where $c_k = c(\{1, \dots, k\})$ for $k = 1, \dots, n$.

Theorem 30 *Let $G = \langle N, c\rangle$ be an anonymous cost-sharing game. Then* $\mathrm{CoS}^{\times}(G) = \frac{n}{c_n} \cdot \min_{k \le n} \frac{c_k}{k}$.

Proof. Pick $k^* \in \mathrm{argmin}_{k \le n} c_k/k$, and let $\boldsymbol{p}$ be a payoff vector given by $p_i = c_{k^*}/k^*$ for all $i \in N$. Clearly, $\boldsymbol{p}$ is stable: For every $S \subseteq N$, we have $p(S) = |S|c_{k^*}/k^* \le c(S)$ by our choice of k^*.

Now, suppose that there is a stable payoff vector $\boldsymbol{q}$ with $q(N) > p(N)$. Renumber the players so that $q_1 \ge \dots \ge q_n$ and set $S^* = \{1, \dots, k^*\}$. Clearly, we have $q(S^*)/k^* \ge q(N)/n$, and hence

$$q(S^*) \ge \frac{k^*}{n} q(N) > \frac{k^*}{n} p(N) = c_{k^*},$$

which means that $\boldsymbol{q}$ is not stable. Hence,

$$\mathrm{CoS}^{\times}(G) = \frac{p(N)}{c(N)} = \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*},$$

which completes the proof. ❑

Without further assumptions, the multiplicative cost of stability of an anonymous cost-sharing game can still be as low as 0: Consider, for instance, the game $G = \langle N, c\rangle$ given by $c(N) = 1$ and $c(S) = 0$ for every $S \subsetneq N$. However, if we assume both anonymity and subadditivity, then the subsidy is always less than half the cost.

Theorem 31 *Let $G = \langle N, c\rangle$ be an anonymous subadditive cost-sharing game. Then* $\mathrm{CoS}^{\times}(G) \ge \frac{1}{2} + \frac{1}{2n-2}$, *and this bound is tight.*

Proof. For $n = 2$ the theorem is trivially true, so assume $n \ge 3$. Fix some $k^* \in \mathrm{argmin}\, c_k/k$. If $k^* = n$, the theorem follows immediately from Theorem 30, so assume $k^* \le n - 1$. Let $q = n/k^*$. We have $q \ge n/(n-1) > 1$, and thus $\lceil q\rceil \ge 2$. We will argue that $q/\lceil q\rceil \ge n/(2n-2)$. We consider the following cases:

- $n \ge 4$, $q > 2$. Then $q/\lceil q\rceil \ge q/(q+1) > 2/3 \ge n/(2n-2)$.
- $n = 3$, $q > 2$. Then it has to be the case that $k^* = 1$, so $q = n = 3$ and hence we obtain $q/\lceil q\rceil = 3/3 > n/(2n-2)$.
- $n = 3$, $q \le 2$. Then it has to be the case that $k^* = 2$, so $q = 3/2$, $\lceil q\rceil = 2$, and $q/\lceil q\rceil = 3/4$, whereas $n/(2n-2) = 3/4$.

We showed that in all cases $q/\lceil q\rceil \ge n/(2n-2)$. Further, since G is subadditive, we have $c_n \le \lceil n/k^*\rceil c_{k^*}$. Combining this with Theorem 30, we obtain

$$\begin{aligned}\mathrm{CoS}^{\times}(G) &= \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*} \ge \frac{n}{k^*} \cdot \frac{1}{\lceil \frac{n}{k^*}\rceil} = \frac{q}{\lceil q\rceil} \\ &\ge \frac{n}{2n-2} = \frac{1}{2} + \frac{1}{2n-2}.\end{aligned}$$

To see that this bound is tight, for each $n \geq 2$ we define a game $G_n = \langle N, c \rangle$ with $|N| = n$ by setting $c(N) = 2$ and $c(S) = 1$ for every $S \subset N$ with $S \neq \emptyset, N$. In this game, we have $k^* = n - 1$, so by Theorem 30 we obtain

$$\mathrm{CoS}^{\times}(G_n) = \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*} = \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2n-2},$$

which completes the proof. ❑

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