

Graph Width Measures for CNF-Encodings with Auxiliary Variables

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Abstract

We consider bounded width CNF-formulas where the width is measured by popular graph width measures on graphs associated to CNF-formulas. Such restricted graph classes, in particular those of bounded treewidth, have been extensively studied for their uses in the design of algorithms for various computational problems on CNF-formulas. Here we consider the expressivity of these formulas in the model of clausal encodings with auxiliary variables. We first show that bounding the width for many of the measures from the literature leads to a dramatic loss of expressivity, restricting the formulas to those of low communication complexity. We then show that the width of optimal encodings with respect to different measures is strongly linked: there are two classes of width measures, one containing primal treewidth and the other incidence cliquewidth, such that in each class the width of optimal encodings only differs by constant factors. Moreover, between the two classes the width differs at most by a factor logarithmic in the number of variables. Both these results are in stark contrast to the setting without auxiliary variables where all width measures we consider here differ by more than constant factors and in many cases even by linear factors.

1. Introduction

Graph width measures like treewidth and cliquewidth have been studied extensively in the context of propositional satisfiability. The general idea is to assign graphs to CNF-formulas and compute their width with respect to different width measures. Then, if the resulting width is small, there are algorithms that solve SAT, but also more complex problems like #SAT or MAX-SAT or even QBF efficiently (see e.g. Samer & Szeider, 2010a; Fischer, Makowsky, & Ravve, 2008; Slivovsky & Szeider, 2013; Paulusma, Slivovsky, & Szeider, 2016; Sæther, Telle, & Vatshelle, 2015; Chen, 2004). There is also a considerable body of work on reasoning problems from artificial intelligence restricted to knowledge encoded by CNF-formulas with restricted underlying graphs: for example, treewidth restrictions have been studied for abduction, closed world reasoning, circumscription, disjunctive logic programming (Gottlob, Pichler, & Wei, 2010) and answer set programming (Jakl, Pichler, & Woltran, 2009). There is thus by now a large body of work on how problems can be solved on bounded width CNF-formulas for different graph width measures.

Curiously, however, there seems to be very little work on the natural question of what we can actually encode with these restricted CNF-formulas. This question is pertinent because good algorithms for problems are less attractive if they cannot deal with interesting

instances. We make two main contributions on the expressivity of bounded width CNF-formulas here.

As a first main contribution, we show, for a wide class of width measures, that one can give width lower bounds of any encoding of a function by means of communication complexity (Theorem 9). Such lower bounds were known for treewidth (Briquel, Koiran, & Meer, 2011), but with our general approach, we extend them for many different width measures, in particular (signed and unsigned) cliquewidth (Fischer et al., 2008; Slivovsky & Szeider, 2013), modular treewidth (Paulusma et al., 2016) and MIM-width (Sæther et al., 2015). As a consequence, in a sense, for all these measures, formulas of bounded width can only encode simple functions.

All these lower bounds not only work for *representations* of functions as CNF-formulas but also on *clausal encodings*, i.e. CNF-formulas using auxiliary variables. It is folklore that adding auxiliary variables can decrease the size of an encoding: for example the parity function has no subexponential CNF-representations but there is an easy linear size encoding using auxiliary variables. We here observe a similar effect for the example of treewidth: we show that any CNF-representation of the **AtMostOne**_{*n*}-function of *n* inputs without auxiliary variables has primal treewidth $n - 1$ which is the highest possible. But when authorizing the use of auxiliary variables, **AtMostOne**_{*n*} can be computed with formulas of bounded treewidth easily. This shows that lower bounds for clausal encodings are far stronger than those of CNF-representations. Considering that **AtMostOne**_{*n*} is arguably a very easy function, we feel that encodings with auxiliary variables are the more interesting notion in our setting so we focus on them here.

We remark that this is of course not the first time that communication complexity has been used to show lower bounds on the size or width of representations for Boolean functions. In fact, this is one of the motivations for the development of the area and there is a large literature on this (see e.g. Kushilevitz & Nisan, 1997; Hromkovic, 1997; Jukna, 2012). In particular, there are many results for showing lower bounds on different forms of branching programs by means of communication complexity (see e.g. Wegener, 2000; Duris, Hromkovic, Jukna, Sauerhoff, & Schnitger, 2004). More recently, this approach has been generalized to more general languages considered in knowledge compilation (Pipatsrisawat & Darwiche, 2010; Bova, Capelli, Mengel, & Slivovsky, 2016). However, beyond the lower bounds on treewidth already discussed by Briquel et al. (2011), we are not aware of any use of communication complexity to show bounds on width measures of CNF-formulas.

In a second main contribution, we focus on the *relative* expressive power of different graph width measures for clausal encodings. For the graph width measures studied in the literature, it is known that without auxiliary variables the expressivity of bounded width CNF-formulas is different for all notions and they form a partial order with so-called MIM-width as the most general notion (see e.g. Brault-Baron, Capelli, & Mengel, 2014, Section 5). Somewhat surprisingly, the situation changes completely when one allows auxiliary variables: in this setting, the commonly considered width notions are all up to constant factors equivalent to either primal treewidth or to incidence cliquewidth (Theorem 23). This is true for every individual function. We remark that for the parameters primal treewidth, dual treewidth and incidence treewidth, it was already known that the width of encodings minimizing the respective width measures differs only by constant factors (Samer & Szeider,

2010b; Briquel et al., 2011; Lampis, Mengel, & Mitsou, 2018). All other relationships are new.

We also show that, assuming that an optimal encoding of a function has at least primal treewidth $\log(n)$ where n is the number of variables, incidence cliquewidth and primal treewidth differ exactly by a factor of $\Theta(\log(n))$ for optimal encodings. So, up to a logarithmic scaling, in fact all the commonly used width measures (Samer & Szeider, 2010a; Fischer et al., 2008; Slivovsky & Szeider, 2013; Paulusma et al., 2016; Sæther et al., 2015) coincide when allowing auxiliary variables. Note that this scaling exactly corresponds to the runtime differences of many algorithms: while treewidth based algorithms often have runtimes of the type $2^{O(k)}n^c$ for treewidth k and a constant c , cliquewidth based algorithms typically give runtimes roughly $n^{O(k')}$ for cliquewidth k' . These runtimes coincide exactly when treewidth and cliquewidth differ by a logarithmic factor which, as we show here, they do generally for encodings with auxiliary variables.

We finally use our main results for several applications. In particular, we answer an open question of Briquel et al. (2011) on the cliquewidth of the permutation function \mathbf{PERM}_n and generalize a classical theorem on planar circuits by Lipton and Tarjan (1980), see Section 6 for details.

Most of our results use machinery recently developed in the area of knowledge compilation. In particular, we use a combination of the algorithm proposed by Bova, Capelli, Mengel, and Slivovsky (2015), the width notion for DNNF developed by Capelli and Mengel (2019) and the lower bound techniques introduced by Pipatsrisawat and Darwiche (2010) and Bova et al. (2016). Relying on these building blocks, most of our proofs become rather simple.

2. Preliminaries

2.1 CNF-Formulas and their Graphs

We use standard notations for CNF-formulas as the ones used by Biere, Heule, van Maaren, and Walsh (2009). Let X be a set of variables. A *CNF-representation* of a Boolean function f in variables X is a CNF-formula F on the variable set X that has as models exactly the assignments on which f evaluates to true. A *clausal encoding* of f is a CNF-formula F' on a variable set $X \cup Y$ such that

- for every assignment $a : X \rightarrow \{0, 1\}$ on which f evaluates to true, there is an extension a' of a to Y that is a model of F' , and
- for every assignment $a : X \rightarrow \{0, 1\}$ on which f evaluates to false, no extension a' of a to Y is a model of F' .

The variables in Y are called *auxiliary variables*. An auxiliary variable y is called *dependent* if and only if in the first item above all extensions a' satisfying F' take the same value on y (Giunchiglia, Maratea, & Tacchella, 2002). We say that a clausal encoding has *dependent auxiliary variables* if all its auxiliary variables are dependent. Note that for such an encoding the extension a' is unique.

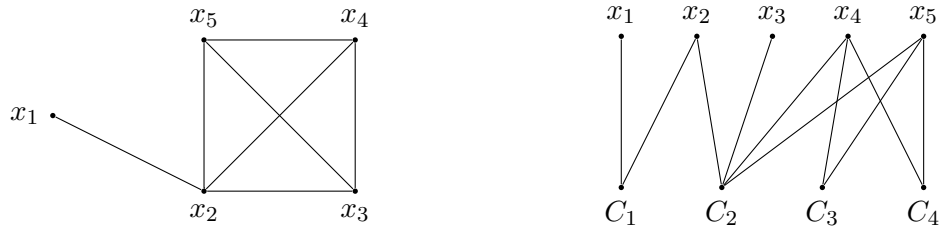


Figure 1: Graphs associated to the CNF-formula F in Example 1: primal graph (left) and incidence graph (right).

We use standard notations from graph theory and assume the reader to have a basic background in the area (Diestel, 2012). By $N(v)$ we denote the open neighborhood of a vertex in a graph.

In some parts of this paper, we will also deal with Boolean circuits. We assume that the reader is familiar with basic definitions in the area. As it is common when considering circuits with structurally restricted underlying graphs, we assume that every input variable appears in only one input gate. This property is sometimes called the *read-once property*.

To every CNF-formula F , we assign two graphs. The *primal graph* of F has as vertices the variables of F and two variables x, y are connected by an edge if and only if there is a clause C such that a literal in x and a literal in y appear in C . The *incidence graph* of F has as vertex set the union of the variable set and the clause set of F . Edges in the incidence graph are exactly the pairs x, C where x is a variable and C a clause that contains a literal in x .

Example 1. Let us consider the clauses $C_1 := x_1 \vee \neg x_2$, $C_2 := x_2 \vee x_3 \vee \neg x_4 \vee \neg x_5$, $C_3 := \neg x_4 \vee x_5$ and $C_4 := x_4 \vee x_5$, and let the CNF-formula F be defined as $F := C_1 \wedge C_2 \wedge C_3 \wedge C_4$. Its primal and incidence graphs are given in Figure 1.

2.2 Graph Width Measures

In this section, we will introduce several graph width measures we will consider throughout this paper. A *tree decomposition* $(T, (B_t)_{t \in V(T)})$ of a graph $G = (V, E)$ consists of a tree T and, for every node t of T , a set $B_t \subseteq V$ called *bag* such that:

- $\bigcup_{t \in V(T)} B_t = V$,
- for every edge $uv \in E$, there is a bag B_t such that $\{u, v\} \subseteq B_t$, and
- for every $v \in V$, the set $\{t \in V(T) \mid v \in B_t\}$ is connected in T .

The *width* of a tree decomposition is defined as $\max\{|B_t| \mid t \in V(T)\} - 1$. The *treewidth* $\text{tw}(G)$ of G is defined as the minimum width taken over all tree decompositions of G . The *primal treewidth* $\text{tw}_p(F)$ of a CNF-formula F is defined as the treewidth of its primal graph and the *incidence treewidth* $\text{tw}_i(F)$ of F is defined as that of the incidence graph.

Example 2. Let us again consider the formula F of Example 1. Figure 2 shows a tree decomposition of the primal graph and the incidence graph of F . Both of these decompositions

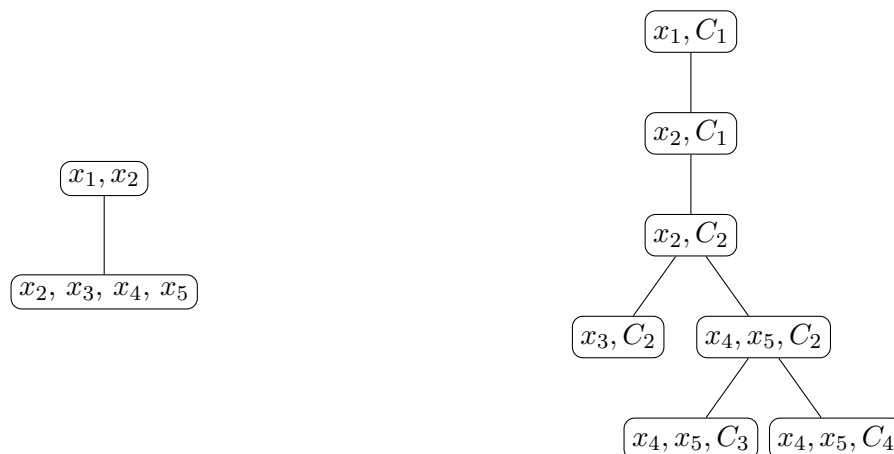


Figure 2: Tree decompositions of the graphs associated to the CNF-formula F in Example 2: primal graph (left) and incidence graph (right).

are optimal: it is well-known that for every tree decomposition of a graph G , the vertices of every clique must be contained in a common bag. So, in this case, x_2, x_3, x_4, x_5 must be in one bag for every tree decomposition of the primal graph of F and thus $\text{tw}_p(F) \geq 3$ which shows that the decomposition of Figure 2 is optimal and $\text{tw}_p(F) = 3$. Concerning the treewidth of the incidence graph, remark that this graph has a cycle and is thus not a tree. Since trees are well-known to be the only graphs of treewidth 1, it follows that $\text{tw}_i(F) \geq 2$ and thus the decomposition in Figure 2 is optimal and $\text{tw}_i(F) = 2$.

We say that two vertices u, v in a graph $G = (V, E)$ have the same *neighborhood type* if and only if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. It can be shown that having the same neighborhood type is an equivalence relation on V . A generalization of treewidth is *modular treewidth* which is defined as follows: from a graph G we construct a new graph G' by contracting all vertices sharing a neighborhood type, i.e., from every equivalence class we delete all vertices but one. The modular treewidth of G is then defined to be the treewidth of G' . The modular treewidth $\text{mtw}(F)$ of a CNF-formula F is defined as the modular treewidth of its incidence graph.

Example 3. Let us consider again the formula F from Example 1. Figure 3 shows a contraction of all vertices sharing a neighborhood type in the incidence graph of F . This contraction resulting in a tree, we have that $\text{mtw}(F) = 1$.

The *cliquewidth* $\text{cw}(G)$ of a graph G is defined as the minimum number of labels needed to construct G with the following operations:

- creating a new vertex with label i ,
- taking the disjoint union of two labeled graphs,
- joining all vertices with a label i to all vertices with a label j for $i \neq j$, and
- renaming a label i to j for $i \neq j$.

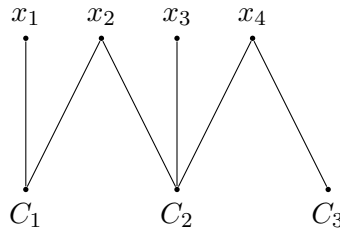


Figure 3: A contraction of the incidence graph of the CNF-formula F in Example 3. In the original graph, x_4 and x_5 have the same neighborhood type, as do C_3 and C_4 . We thus get the shown contraction by deleting x_5 and C_4 . Note that the obtained graph is a tree.

The *incidence cliquewidth* $\text{cw}(F)$ of a formula F is defined as the cliquewidth of the incidence graph of F (Slivovsky & Szeider, 2013).

Finally, we consider the adaption of cliquewidth to signed graphs. To this end, let us make some additional definitions. The *signed incidence graph* G' of a CNF-formula F is the graph we get from the incidence graph $G = (V, E)$ by labeling the edges with $\{+, -\}$ as follows:

- every edge xC such that x appears positively in C is labeled by $+$, and
- every edge xC such that x appears negatively in C is labeled by $-$.

The signed cliquewidth of a graph G' is defined as the minimum number of labels needed to construct G' with the following operations:

- creating a new vertex with label i ,
- taking the disjoint union of two labeled graphs,
- joining all vertices with a label i to all vertices with a label j for $i \neq j$ by an edge with label $+$,
- joining all vertices with a label i to all vertices with a label j for $i \neq j$ by an edge with label $-$, and
- renaming a label i to j for $i \neq j$.

The *signed incidence cliquewidth* $\text{scw}(F)$ of F is defined as the signed cliquewidth of its signed incidence graph (Fischer et al., 2008).

We will deal with several other graph width measures for a CNF-formula in the remainder of this paper, in particular dual treewidth $\text{tw}_d(F)$ and MIM-width $\text{mimw}(F)$. Since for those notions we will only use some of their properties, we will refrain from overwhelming the reader by giving their definitions and refer to the literature (see e.g. Samer & Szeider, 2010a; Fischer et al., 2008; Vatshelle, 2012; Sæther et al., 2015; Slivovsky & Szeider, 2013). We also consider the treewidth $\text{tw}(C)$ and the cliquewidth $\text{cw}(C)$ of Boolean circuits C .

2.3 Communication Complexity

Here we give some very basic notions of communication complexity, focusing only on so-called combinatorial rectangles, which are an important object in the field. For more details, the reader is referred to the very readable textbook by Kushilevitz and Nisan (1997).

Let X be a set of variables and $\Pi = (Y, Z)$ a partition of X . A *combinatorial rectangle* respecting Π is a Boolean function $r(X)$ that can be written as a conjunction $r(X) = r_1(Y) \wedge r_2(Z)$. For a Boolean function f on X , a *rectangle cover of size s* respecting Π is defined to be a representation

$$f(X) = \bigvee_{i=1}^s r^i(X) = \bigvee_{i=1}^s r_1^i(Y) \wedge r_2^i(Z),$$

where all $r^i(X) = r_1^i(Y) \wedge r_2^i(Z)$ are combinatorial rectangles respecting Π . The non-deterministic communication complexity $\text{cc}(f, \Pi) = \text{cc}(f, (Y, Z))$ of f is defined as $\log(s_{\min})$ where s_{\min} is the minimum size of any rectangle cover of f respecting Π .

Example 4. By definition, all formulas in disjunctive normal forms are rectangle covers of the functions they compute respecting all possible partitions. For example,

$$F = (\neg x \wedge \neg y \wedge z) \vee (x \wedge y \wedge z) \vee (x \wedge \neg y \wedge \neg z)$$

is a rectangle cover of size 3 respecting every partition of $\{x, y, z\}$. However, for example for the partition $(\{x, y\}, \{z\})$, there is the smaller rectangle cover

$$(((\neg x \wedge \neg y) \vee (x \wedge y)) \wedge z) \vee (x \wedge \neg y \wedge \neg z)$$

of size 2. It is not hard to see that there is no smaller rectangle cover of F for this partition.

The best-case non-deterministic communication complexity with $\frac{1}{3}$ -balance $\text{cc}_{\text{best}}^{1/3}(f)$ is defined as $\text{cc}_{\text{best}}^{1/3}(f) := \min_{\Pi}(\text{cc}(f, \Pi))$ where the minimum is over all partitions $\Pi = (Y, Z)$ of X with $\min(|Y|, |Z|) \geq |X|/3$.

Example 5. Consider the function $\mathbf{EQ}_n(x_1, \dots, x_n, y_1, \dots, y_n)$ which is true if and only if for every $i \in [n]$ we have $x_i = y_i$. It is well-known that for the partition $\Pi_1 = (\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\})$ we have $\text{cc}(\mathbf{EQ}_n, \Pi_1) = n$ (see e.g. Kushilevitz & Nisan, 1997, Chapter 2). However, for the partition

$$\Pi_2 = (\{x_1, y_1, \dots, x_{\lceil n/2 \rceil}, y_{\lceil n/2 \rceil}\}, \{x_{\lceil n/2 \rceil+1}, y_{\lceil n/2 \rceil+1}, \dots, x_n, y_n\})$$

we have that

$$\mathbf{EQ}_n(x_1, \dots, x_n, y_1, \dots, y_n) = \left(\bigwedge_{i=1}^{\lceil n/2 \rceil} x_i = y_i \right) \wedge \left(\bigwedge_{i=\lceil n/2 \rceil+1}^n x_i = y_i \right)$$

is a rectangle cover of size 1 respecting Π_2 . Thus, we have $\text{cc}_{\text{best}}^{1/3}(\mathbf{EQ}_n) = \text{cc}(\mathbf{EQ}_n, \Pi_2) = 0$.

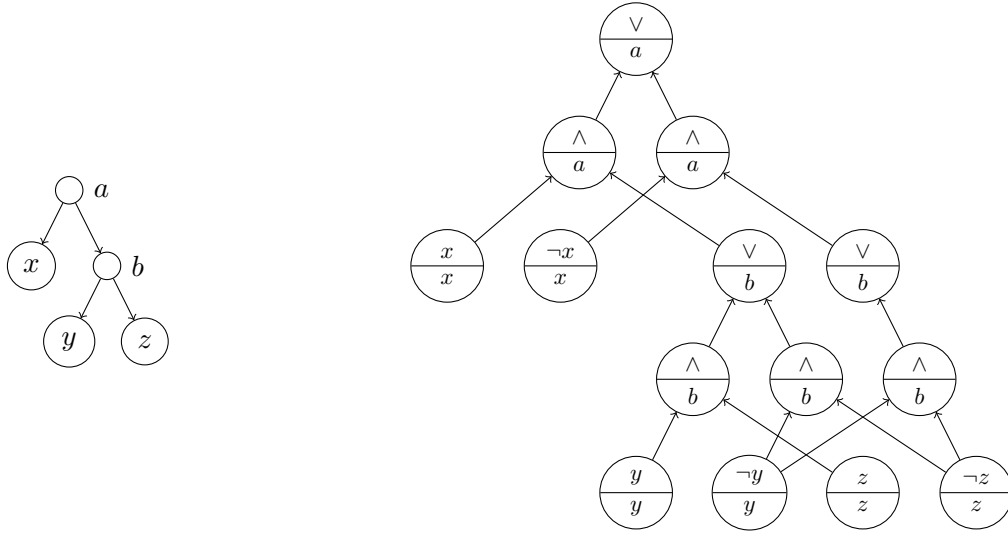


Figure 4: A v-tree on the left and a complete structured DNNF structured by this v-tree. For the internal nodes of the v-tree, we give node names on the right of the nodes whereas for leaves we assume that the name is the label. All gates of the complete structured DNNF show the operation of the gate (on top) and the name t of the node in the v-tree for which this gate is in $\mu(t)$ (on bottom).

2.4 Structured Deterministic DNNF

Out of the rich landscape of representations from knowledge compilation (Darwiche & Marquis, 2002; Pipatsrisawat & Darwiche, 2008), we only introduce one that we will use in the remainder of this paper. For all circuits in this section, we assume that \wedge -gates have exactly two inputs while the number of \vee -gates may be arbitrary.

A *v-tree* T for a variable set X is a full binary tree whose leaves are in bijection with X . We call the variable assigned by this bijection to a leaf v the *label* of v . For a node $t \in T$, we denote by T_t the subtree of T that has t as its root and by $\text{var}(T_t)$ the variables that are labels of leaves in T_t .

Example 6. We give a v-tree for the variable set $\{x, y, z\}$ on the left of Figure 4.

We give some definitions introduced by Capelli and Mengel (2019). A *complete structured DNNF* D structured by a v-tree T is a Boolean circuit with the following properties: there is a labeling μ of the nodes in T with subsets of gates of D such that:

- For every gate g of D there is a unique node t_g of T with $g \in \mu(t_g)$.
- If t is a leaf labeled by a variable x , then $\mu(t)$ may only contain x and $\neg x$. Moreover, for every input gate g , the node t_g is a leaf.
- For every \vee -gate g , all inputs are \wedge -gates in $\mu(t_g)$.
- Every \wedge -gate g has exactly two inputs g_1, g_2 that are both \vee -gates or input gates. Moreover, t_{g_1} and t_{g_2} are the children of t_g in T and in particular $t_{g_1} \neq t_{g_2}$.

The *width* $wi(D)$ of D is defined as the maximal number of \vee -gates in any set $\mu(t)$. We often speak of complete structured DNNF without mentioning the v -tree by which it is structured in cases where the form of the v -tree is unsubstantial. Intuitively, a complete structured DNNF is a Boolean circuit in negation normal form in which the gates are organized into blocks $\lambda(t)$ which form a tree shape. In every block one then computes a 2-DNF whose inputs are gates from the blocks that are the children of $\lambda(t)$ in the tree shape.

Example 7. On the right side of Figure 4, we give a complete structured DNNF structured by the v -tree of Example 6. There are 3 \vee -gates in $\mu(b)$, so the width of the given complete structured DNNF is 3.

A complete structured DNNF is called *deterministic* if and only if for every assignment and for every \vee -gate, at most one input evaluates to true. Note that we do not allow constant input gates here. We remark that if we allowed those, we could always get rid of them in the circuit by propagation without changing any other properties of the circuit (Capelli & Mengel, 2019, Section 4). We also remark that in a complete structured DNNF D , we can *forget* a variable x , i.e., construct a complete structured DNNF D' computing $\exists x D$, by setting all occurrences of x and $\neg x$ to 1 and propagating the constants in the obvious way. This operation does not increase the width (Capelli & Mengel, 2019). However, if D is deterministic, this is generally not the case for D' .

3. The Effect of Auxiliary Variables

In this section, we will motivate the use of auxiliary variables when considering width measures of CNF-encodings. To this end, we will show with an example that auxiliary variables may arbitrarily reduce the treewidth of encodings. Note that this is not very surprising since it is not too hard to see that CNF-representations of, say, the parity function, are of high treewidth. However, in this case the *size* of the representation is exponential, so in a sense parity is a hard function for CNF-representations anyway. Here we will show that even for functions that have small CNF-representations there can be a large gap between the treewidth of representations and clausal encodings with auxiliary variables. That is why we think it is useful to systematically study width measures for clausal encodings.

As an example for a function where auxiliary variables have a dramatic impact on width, consider the **AtMostOne** $_n$ -function on variables x_1, \dots, x_n which accepts exactly those assignments in which at most one variable is assigned to 1. There is an obvious quadratic size representation as

$$\mathbf{AtMostOne}_n(x_1, \dots, x_n) = \bigwedge_{i,j \in [n], i < j} \neg x_i \vee \neg x_j.$$

However, this representation has as primal graph the clique K_n which is of treewidth $n - 1$. We will see that in fact there is no representation of **AtMostOne** $_n$ that is of smaller primal treewidth unless one adds auxiliary variables, in which case there is a simple encoding of primal treewidth 2.

Theorem 1. *Any CNF-representation of the **AtMostOne** $_n$ -function of n inputs without auxiliary variables has primal treewidth $n - 1$. However, there is a clausal encoding of **AtMostOne** $_n$ of primal treewidth 2.*

To prove Theorem 1, we split the statement into two lemmas.

Lemma 2. *Any CNF-representation of the **AtMostOne**_n-function of n inputs without auxiliary variables has primal treewidth $n - 1$.*

Proof. Let x_1, \dots, x_n be the variables of **AtMostOne**_n. We proceed with two claims.

Claim 3. *Every non-tautological clause C of any CNF-representation of **AtMostOne**_n must contain at least the negation of two variables from x_1, \dots, x_n .*

Proof. Suppose that a clause C does not contain two such literals. Then, there are two possible cases: either C contains no negated variables, or exactly one. In the first case, the model of **AtMostOne**_n setting all variables to 0 does not satisfy C , so C cannot be part of the CNF-representation. In the second case, let x_i be the (only) variable of **AtMostOne**_n appearing negatively in C . Then, the model of **AtMostOne**_n setting only x_i to 1 and all other variables to 0 does not satisfy C , so C cannot be part of the CNF-representation, either. Hence, at least two negated variables must appear in C . \square

From Claim 3, we will deduce that all pairs of variables must appear conjointly in at least one clause.

Claim 4. *For each pair of variables x_i, x_j from x_1, \dots, x_n with $i \neq j$, there is a clause in the CNF-representation of **AtMostOne**_n containing both $\neg x_i$ and $\neg x_j$.*

Proof. Suppose that, for a pair x_i, x_j , such a clause does not exist. Let a be the assignment that sets exactly the variables x_i, x_j to 1 and all other variables to 0. Let C be a clause from the CNF-representation. By our previous claim, C contains two negated variables from x_1, \dots, x_n . Because of our assumption, at least one of these literals is neither $\neg x_i$ nor $\neg x_j$, and this literal is satisfied by a . Thus C is satisfied by a . Since this is true for every clause C , it follows that a satisfies all the clauses of the representation, so it is one of its models. However, a is not a model of **AtMostOne**_n. As a consequence, a clause containing both $\neg x_i$ and $\neg x_j$ must exist, which is also true for every pair x_i, x_j . \square

Claim 4 shows that for each pair of variables, there is a clause containing both of them. It follows that all variables are connected to all other variables in the primal graph of the representation. So the primal graph is a clique which has treewidth $n - 1$. \square

We now prove the second part of Theorem 1, which shows that if we allow the use of auxiliary variables, we may decrease the treewidth dramatically.

Lemma 5. *There is a clausal encoding of **AtMostOne**_n of primal treewidth 2.*

Proof. We use the well-known ladder encoding presented by Gent and Nightingale (2004) and Biere et al. (2009, Section 2.2.5). We introduce the auxiliary variables y_0, \dots, y_n . The encoding consists of the following clauses, for every $i \in [n]$:

- the validity clauses $\neg y_{i-1} \vee y_i$, and
- clauses representing the constraint $x_i \leftrightarrow (\neg y_{i-1} \wedge y_i)$

It is easy to see that this encoding is correct: the auxiliary variables y_i encode if one of the variables x_j for $j \leq i$ is assigned to 1. Concerning the treewidth bound, we construct for every index $i \in [n]$ the bag $B_i := \{y_{i-1}, y_i, x_i\}$. Then $(P_n, (B_i)_{i \in [n]})$ where P_n has nodes $[n]$ and edges $\{(i, i + 1) \mid i \in [n - 1]\}$ is a tree decomposition of the encoding of width 2. \square

4. Width vs. Communication

In this section, we show that from communication complexity we get lower bounds for the various width notions of Boolean functions. The main building block is the following result that is an application of the main result of Pipatsrisawat and Darwiche (2010) to complete structured DNNF.

Theorem 6. *Let D be a complete structured DNNF structured by a v-tree T computing a function f in variables X . Let t be a node of T and let $Y := \text{var}(T_t)$ and $Z = X \setminus \text{var}(T_t)$. Finally, let ℓ be the number of \vee -gates in $\mu(t)$. Then there is a rectangle cover of f respecting (Y, Z) of size at most ℓ .*

Note that Pipatsrisawat and Darwiche (2010) consider models that are structured DNNF which are not necessarily complete, a slightly more general model than ours. Thus their statement is slightly different. However, it is easy to see that in our restricted setting, their proof shows the statement we give above (see also Bova et al., 2016, Section 5). Since Theorem 6 is somewhat technical, it will be more convenient here to use the following easy consequence.

Proposition 7. *Let D be a complete structured DNNF structured by a v-tree T computing a function f in variables X . Let t be a node of T and let $Y := \text{var}(T_t)$ and $Z = X \setminus \text{var}(T_t)$. Then*

$$\log(\text{wi}(D)) \geq \text{cc}(f, (Y, Z)).$$

Proof. From Theorem 6 and the definition of width, it follows directly that the size of any rectangle cover of f respecting (Y, Z) is upper bounded by the width of D . Taking the logarithm on both sides yields the claim. \square

In many cases, instead of considering explicit v-trees, it is more convenient to simply use best-case communication complexity.

Corollary 8. *Let f be a Boolean function in variables X . Then, for every complete structured DNNF computing f , we have*

$$\text{wi}(D) \geq 2^{\text{cc}_{\text{best}}^{1/3}(f)}.$$

Proof. Note that for every v-tree with X on the leaves, there is a node t such that $|X|/3 \leq |\text{var}(T_t)| \leq 2|X|/3$. Plugging this into Proposition 7 directly yields the result. \square

We will use Corollary 8 to turn compilation algorithms that produce complete structured DNNF based on a parameter of the input (see Amarilli, Capelli, Monet, & Senellart, 2018a; Bova & Szeider, 2017) into inexpressivity bounds based on this parameter. We first give an abstract version of this result that we will instantiate for concrete measures later on.

Theorem 9. *Let \mathcal{C} be a (fully expressive) representation language for Boolean functions. Let \mathfrak{p} be a parameter $\mathfrak{p} : \mathcal{C} \rightarrow \mathbb{N}$. Assume that there is for every Boolean function f and every $C \in \mathcal{C}$ that encodes f a complete structured DNNF with*

$$\text{wi}(D) \leq 2^{\mathfrak{p}(C)}.$$

Then we have

$$\mathfrak{p}(C) \geq \text{cc}_{\text{best}}^{1/3}(f).$$

Proof. From the assumption, we get $\mathfrak{p}(C) \geq \log(\text{wi}(D))$. Then we apply Corollary 8 to directly get the result. \square

Intuitively, it is exactly the algorithmic usefulness of parameters that makes the resulting instances inexpressive. Note that it is not surprising that instances whose expressiveness is severely restricted allow for good algorithmic properties. However, here we see that the inverse of this statement is also true in a quite harsh way: if a parameter has good algorithmic properties allowing efficient compilation into DNNF, then this parameter puts strong restrictions on the complexity of the expressible functions.

Note that instead of Corollary 8 we could have used Proposition 7 in the proof of Theorem 9 to get a slightly stronger result. We chose to go with a simpler statement here but note that we will use the extended strength of Proposition 7 later on in Section 6.

From Theorem 9, we directly get lower bounds for many of the remaining width measures (as those from Paulusma, Slivovsky, & Szeider, 2013; Samer & Szeider, 2010a; Fischer et al., 2008; Slivovsky & Szeider, 2013; Sæther et al., 2015). The first result considers the parameters with respect to which SAT is fixed-parameter tractable.

Corollary 10. *There is a constant $b > 0$ such that for every Boolean function f and every CNF C encoding f we have*

$$\min\{\text{tw}_i(C), \text{tw}_p(C), \text{tw}_d(C), \text{scw}(C)\} \geq b \cdot \text{cc}_{\text{best}}^{1/3}(f).$$

Proof. This follows directly from Theorem 9 and the fact that for all these parameters there are algorithms that, given an input CNF of parameter value k , construct an equivalent complete structured DNNF of width $2^{O(k)}$. \square

Using the compilation algorithm by Amarilli et al. (2018a) and Amarilli, Monet, and Senellart (2018b), we get essentially the same result for circuit representations.

Corollary 11. *There is a constant $b > 0$ such that for every Boolean function f and every circuit C encoding f we have*

$$\min\{\text{tw}(C), \text{cw}(C)\} \geq b \cdot \text{cc}_{\text{best}}^{1/3}(f).$$

We remark that for treewidth 1 the circuits of Corollary 11 boil down to so-called read-once functions which have been studied extensively (see e.g. Crama & L. Hammer, 2011, Chapter 10).

Finally, we give a version for parameters that allow polynomial time algorithms when fixed but no fixed-parameter algorithms.

Corollary 12. *There is a constant $b > 0$ such that for every Boolean function f in n variables and every CNF C encoding f we have*

$$\min\{\text{mimw}(C), \text{cw}(C), \text{mtw}(C)\} \geq b \cdot \frac{\text{cc}_{\text{best}}^{1/3}(f)}{\log(n)}.$$

Proof. All of the width measures in the statement allow compilation into complete structured DNNF of size – and thus also width – $n^{O(k)}$ for parameter value k and n variables (Bova et al., 2015). Thus, with Theorem 9, for each measure there is a constant b' with $\log(n^k) = k \log(n) \geq b' \text{cc}_{\text{best}}^{1/3}(f)$ which completes the proof. \square

Note that the bounds of Corollary 12 are lower by a factor of $\log(n)$ than those of Corollary 10. We will see in the next section that in a sense this difference is unavoidable.

5. Relations between Different Width Measures of Encodings

In this section, we will show that the different width measures for optimal clausal encodings are strongly related. To this end, in different subsections, we will show the relation of treewidth to all other width measures we consider. We will then combine these relationships between treewidth and other width measures to analyze the relationships between all width measures we consider.

5.1 From Treewidth to Modular Treewidth and Cliquewidth

We will start by proving that primal treewidth bounds imply bounds for modular treewidth and cliquewidth.

Theorem 13. *Let k be a positive integer and f be a Boolean function of n variables that has a CNF-encoding F of primal treewidth at most $k \log(n)$. Then f also has a CNF-encoding F' of modular incidence treewidth and cliquewidth $O(k)$. Moreover, if F has dependent auxiliary variables, then so has F' .*

Before we prove Theorem 13, let us here discuss this result a little. It is well known that the modular treewidth and the cliquewidth of a CNF formula can be much smaller than its treewidth (Slivovsky & Szeider, 2013). Theorem 13 strengthens this by saying essentially that for *every* function we can gain a factor logarithmic in the number of variables.

In particular, this shows that the lower bounds we can get from Corollary 12 are the best possible: the maximal lower bounds we can show are of the form $n/\log(n)$ and since there is always an encoding of every function of treewidth n , by Theorem 13 there is always an encoding of cliquewidth roughly $n/\log(n)$. Thus the maximal lower bounds of Corollary 12 are tight up to constants.

Note that for Theorem 13, it is important that we are allowed to change the encoding. For example, the primal graph of the formula $F = \bigwedge_{i,j \in [n]} (x_{i,j} \vee x_{i+1,j}) \wedge (x_{i,j} \vee x_{i,j+1})$ has the $n \times n$ -grid as a minor and thus treewidth n (see e.g. Diestel, 2012, Chapter 12). But the incidence graph of F has no modules and also has the $n \times n$ -grid as a minor, so F has modular incidence treewidth at least n as well. So we gain nothing by going from primal treewidth to modular treewidth without changing the encoding. What Theorem 13 tells us

is that there is a different formula F' that encodes the function of F , potentially with some additional variables, such that the treewidth of F' is at most $O(n/\log(n))$.

Let us note that encodings with dependent auxiliary variables are often useful, e.g. when considering counting problems. In fact, for such clausal encodings, the number of models is the same as for the function they encode. It is thus interesting to see that dependence of the auxiliary variables can be maintained by the construction of Theorem 13. We will see that this is also the case for most other constructions we make.

Proof (of Theorem 13). The basic idea is that we do not treat the variables in the bags of the tree decomposition individually but organize them in groups of size $\log(n)$. We then simulate the clauses of the original formula by clauses that work on the groups. Since for every group there are only a linear number of assignments, all encoding sizes stay polynomial. We now give the details of the proof.

Let $(T, (B_t)_{t \in T})$ be a tree decomposition of F of width at most $k \log(n)$. For every clause C of F there is a bag $\lambda(C)$ that contains the variables of C . By adding some copies of bags, we may assume w.l.o.g. that for every bag B there is at most one clause with $\lambda(C) = B$ and call this clause $\lambda^{-1}(B)$.

In a first step, we construct a coloring $\mu : \text{var}(F) \rightarrow [k + 1]$ such that in every bag there are at most $\log(n)$ variables of every color. This can be done iteratively as follows: first split the bag B_r at the root r into color classes as required. Since there are at most $k \log(n) + 1$ variables in B_r by assumption, we can split them into $k + 1$ color classes of size at most $\log(n)$ arbitrarily. Now let t be a node of T with parent t' . By the coloring of the variables in $B_{t'}$, some of the variables in B_t are already colored. We simply add the variables not appearing in $B_{t'}$ arbitrarily to color classes such that no color class is too big. Again, since B_t contains at most $k \log(n) + 1$ variables, this is always possible. Moreover, due to the connectivity condition, there is for every variable x a unique node t_x that is closest to the root under the bags containing x . Consequently, we can make no contradictory decisions during this coloring process, so μ is well-defined.

We now construct F' . To this end, we first introduce for every variable x and every node t such that $x \in B_t$ a new variable x_t . Now for every node t with parent t' and every color i , we add a set $C_{t',t,i}$ of clauses in all variables $x_t, x_{t'}$ with $\mu(x) = i$. We construct these clauses in such a way that they are satisfied by exactly the assignments in which for each pair $x_t, x_{t'}$ such that both these variables exist, both variables take the same value. Note that the clauses in $C_{t',t,i}$ have at most $2 \log(n)$ variables, so there are at most n^2 of them. Moreover, they contain all the same variables. The result is a formula in which all x_t for a variable x take the same value in all satisfying assignments.

In a next step, we do for each clause C the following: let $t = \lambda(C)$. For every color i , we define $X_{i,t}$ to be the set of variables x_t such that $\mu(x) = i$. We add a fresh variable $y_{C,i}$ and clauses $C_{C,i}$ in the variables $X_{i,t} \cup \{y_{C,i}\}$ that accept exactly the assignments a with

- $a(y_{C,i}) = 1$ and there is an $x_t \in X_{i,t}$ such that setting x to $a(x_t)$ satisfies C , or
- $a(y_{C,i}) = 0$ and there is no $x_t \in X_{i,t}$ such that setting x to $a(x_t)$ satisfies C .

Next, we add the clause $C' = \bigvee_{i \in [k+1]} y_{C,i}$. Finally, for every variable x , rename one arbitrary variable x_t to x . This completes the construction of F' .

We claim that F' is an encoding of f . To see this, first note that, as discussed before, for every variable x of F , in the satisfying assignments of F' , all x_t and x take the same value. So, we define for every assignment a of F a partial assignment a' of F' as an extension of a by setting $a'(x_t) = a(x)$ for every x_t . a satisfies a clause C if and only if there is at least one variable x of C such that $a(x)$ makes C true. Let $\mu(x) = i$, then a satisfies C if and only if $\mathcal{C}_{C,i}$ is satisfied by the extension of a' that sets $y_{C,i}$ to 1. So a satisfies C if and only if there is an extension of a' that satisfies $\mathcal{C}_{C,i}$. Consequently, a satisfies F if and only if there is an extension a'' of a' that satisfies F' , so F' is an encoding of f as claimed.

To see that the construction maintains dependence of auxiliary variables, observe first that the auxiliary variables already present in F are still in F' and they are still dependent. We claim that all the new variables depend on those of F . For the variables x_t , this is immediate since they must take the same value as x in every model. Moreover, the variables $y_{C,i}$ depend on the x_t by definition. As a consequence, all auxiliary variables are dependent

We now show that the modular treewidth of F' is at most $O(k)$. First note that all sets $X_{i,t}$ are modules as are the clause sets $\mathcal{C}_{t,t',i}$ and $\mathcal{C}_{C,i}$. W.l.o.g. we may assume that for every t , there is at most one clause C with $\lambda(C) = t$ and that T is a binary tree. We construct a tree decomposition $(T, (B_t)_{t \in V(T)})$ as follows: we put a representant of $X_{i,t}$, $\mathcal{C}_{t,t',i}$, $\mathcal{C}_{t',t,i}$ and $\mathcal{C}_{C,i}$ into B'_t . Moreover, we add $y_{C,i}$ and C' to B'_t . It is easy to see that constructed like this, $(T, (B_t)_{t \in V(T)})$ is a tree decomposition of width at most $O(k)$.

Finally, we will show that the incidence graph of the formula i can be constructed with $O(k)$ labels. In this construction, the relabeling operation will only ever be used to *forget* labels, i.e., we change a label i into a global dummy label d such that vertices labeled by d are never used in joining operations.

In a first step, we color T with 4 colors such that for every node t , the node t , its at most two children and its parent all have different colors. We denote the color of t by $\eta(t)$. Then, for every t individually, we create the nodes in $\mathcal{C}_{C,i}$, $X_{t,i}$ where C is such that $\lambda(C) = t$. The clauses in $\mathcal{C}_{C,i}$ get label $(i, \eta(t), 0)$ and the variables in $X_{t,i}$ get label $(i, \eta(t), 1)$. By joining the vertices with labels $(i, \eta(t), 0)$ with those with $(i, \eta(t), 1)$, we connect the variables in $X_{t,i}$ with the clauses in $\mathcal{C}_{C,i}$. We then create the $y_{C,i}$, each with individual labels and connect them to the clauses with label $(i, \eta, 0)$. Finally, we create the clause vertex C' with an individual label and connect it to the $y_{C,i}$. We then forget the labels of all vertices except the $X_{t,i}$. We call the resulting graph G_t .

Note that at this point, the only thing that remains to do is to introduce the clauses in the $\mathcal{C}_{t,t',i}$ and connect them to the variables in G_t and $G_{t'}$. To do so, we work in a bottom-up fashion along T . For the leaves of T , there is nothing to do. So let t be an internal node of T with children t_1, t_2 ; the case in which t only has one child is treated analogously. By induction, we assume that we have graphs G'_{t_1} and G'_{t_2} containing G_{t_1} and G_{t_2} as respective subgraphs such that:

- all variables appearing in G'_{t_j} are already connected to all clauses, except the variables in the $X_{t_j,i}$ which are not yet connected to the clauses $\mathcal{C}_{t,t_1,i}$,
- all vertices in G'_{t_j} except for those in the X_{t_i} have the dummy label d .

We proceed as follows: we make a disjoint union of G_t , G'_{t_1} and G'_{t_2} . Then we create nodes for all clauses in the $\mathcal{C}_{t,t_1,i}$ giving them the label $(i, \eta(t), 2)$. Then we connect all nodes

with label $(i, \eta(t_1), 1)$ to those with label $(i, \eta(t), 2)$, i.e., we connect the nodes in $X_{t_1, i}$ with the clauses in $\mathcal{C}_{t, t_1, i}$. Then we connect all nodes with label $(i, \eta(t), 1)$ to those with label $(i, \eta(t), 2)$, i.e., we connect the nodes in $X_{t, i}$ with the clauses in $\mathcal{C}_{t, t_1, i}$. We proceed analogously with t_2 . Finally, we forget all labels but those for the $X_{t, i}$. This completes the construction.

Verifying the clauses in F' , one can see that the resulting graph is indeed the incidence graph of F' . Moreover, we have only used $O(k)$ clauses by construction. This completes the proof. \square

5.2 Back to Treewidth

We now show that the reverse of Theorem 13 is also true: upper bounds for many width measures imply also bounds for the primal treewidth of clausal encodings. Note that this is at first sight surprising since without auxiliary variables many of those width measures are known to be far stronger than primal treewidth.

Theorem 14. *Let f be a Boolean function of n variables.*

- a) *If F has a clausal encoding of modular treewidth, cliquewidth or mim-width k then f also has a clausal encoding F' of primal treewidth $O(k \log(n))$ with $O(kn \log(n))$ auxiliary variables and $n^{O(k)}$ clauses.*
- b) *If F has a clausal encoding of incidence treewidth, dual treewidth, or signed incidence cliquewidth k , then f also has a clausal encoding F' of primal treewidth $O(k)$ with $O(nk)$ auxiliary variables and $2^{O(k)}n$ clauses.*

To show Theorem 14 and several similar results for other width measures in this section, we make a detour through DNNF. The idea is to show that from certain DNNF representations of functions, we can get clausal encodings of primal treewidth strongly related to the width of the DNNF. Since many width measures can be used to construct small width DNNFs, we get small width clausal encodings for these width measures. We now give a precise statement of the relation between DNNF and treewidth of clausal encodings.

Lemma 15. *Let f be a Boolean function in n variables that is computed by a complete structured DNNF of width k . Then f has a clausal encoding F of primal treewidth $9 \log(k)$ with $O(n \log(k))$ variables and $O(nk^3)$ clauses. Moreover, if D is deterministic then F has dependent auxiliary variables.*

The proof of Lemma 15 will rely on so-called proof trees in DNNF, a concept that has found wide application in circuit complexity and in particular also in knowledge compilation. To this end, we make the following definition: a *proof tree* \mathcal{T} of a complete structured DNNF D is a circuit constructed as follows:

1. The output gate of D belongs to \mathcal{T} .
2. Whenever \mathcal{T} contains an \vee -gate, we add exactly one of its inputs.
3. Whenever \mathcal{T} contains an \wedge -gate, we add both of its inputs.
4. No other gates are added to \mathcal{T} .

Note that the choice in Step 2 is non-deterministic, so there are in general many proof trees for D . Observe also that due to the structure of D given by its v -tree, every proof tree is in fact a tree which justifies the name. Moreover, letting T be the v -tree of D , every proof tree of D has exactly one \vee -gate and one \wedge -gate in the set $\mu(t)$ for every non-leaf node t of T . For every leaf t , every proof tree contains an input gate x or $\neg x$ where x is the label of t in T .

The following simple observation that can easily be shown by using distributivity is the main reason for the usefulness of proof trees.

Observation 16. *Let D be a complete structured DNNF and a an assignment to its variables. Then a satisfies D if and only if it satisfies one of its proof trees. Moreover, if D is deterministic, then every assignment a that satisfies D satisfies exactly one proof tree of D .*

Proof (of Lemma 15). Let D be the complete structured DNNF computing f and let T be the v -tree of D . The idea of the proof is to use auxiliary variables to “guess” for every t an \vee -gate and an \wedge -gate. Then we use clauses along the v -tree T to verify that the guessed gates in fact form a proof tree and check in the leaves of T if the assignment to the variables of f satisfies the encoded proof tree. We now give the details of the construction.

We first note that, as shown by Capelli and Mengel (2019), in complete structured DNNF of width k , one may assume that every set $\mu(t)$ contains at most k^2 \wedge -gates so we assume this to be the case for D . For every node t of T , we introduce a set X_t of $3 \log(k)$ auxiliary variables to encode one \vee -gate and one \wedge -gate of $\mu(t)$ if t is an internal node. If t is a leaf, X_t encodes one of the at most 2 input gates in $\mu(t)$. We now add clauses that verify that the gates chosen by the variables X_t encode a proof tree by doing the following for every t that is not a leaf: first, add clauses in X_t that check if the chosen \wedge -gate is in fact an input of the chosen \vee -gate. Since X_t has at most $3 \log(k)$ variables, this introduces at most k^3 clauses. Let t_1 and t_2 be the children of t in T . Then we add clauses that verify if the \wedge -gate chosen in t has as input either the \vee -gate chosen in t_1 if t_1 is not a leaf, or the input gate chosen in t_1 if t_1 is a leaf. Finally, we add analogous clauses for t_2 . Each of these clause sets is again in $3 \log(k)$ variables, so there are at most $2k^3$ clauses in them overall. The result is a CNF-formula that accepts an assignment if and only if it encodes a proof tree of D .

We now show how to verify if the chosen proof tree is satisfied by an assignment to f . To this end, for every leaf t of T labeled by a variable x , add clauses that check if an assignment to x satisfies the corresponding input gate of D . Since $\mu(t)$ contains at most 2 gates, this only requires at most 4 clauses. This completes the construction of the clausal encoding. Overall, since T has n internal nodes, the CNF has $n(3 \log(k) + 1)$ variables and $3nk^3 + 4n$ clauses.

It remains to show the bound on the primal treewidth. To this end, we construct a tree decomposition $(T, (B_t)_{t \in V(T)})$ with the v -tree T as underlying tree as follows: for every internal node $t \in V(T)$, we set $B_t := X_t \cup X_{t_1} \cup X_{t_2}$ where t_1 and t_2 are the children of t . Note that for every clause that is used for checking if the chosen nodes form a proof tree, the variables are thus in a bag B_t . For every leaf t , set $B_t := X_t \cup \{x\}$ where x is the variable that is the label of t . This covers the remaining clauses. It follows that all edges of the primal graph are covered. To check the third condition of the definition of a tree decomposition, note that every auxiliary variable in a set X_t appears only in B_t and

potentially in $B_{t'}$ where t' is the parent of t in T . Thus $(T, (B_t)_{t \in V(T)})$ constructed in this way is a tree decomposition of the primal graph of C . Obviously, the width is bounded by $9 \log(k)$ since every X_t has size $3 \log(k)$, which completes the proof. \square

Proof (of Theorem 14). We first prove a). As shown by Bova et al. (2015), whenever the function f has a clausal encoding F with one of the width measures from this statement bounded by k , then there is also a complete structured DNNF D of width $n^{O(k)}$ computing F . Now forget all auxiliary variables of F to get a DNNF representation D' of f . Note that since forgetting does not increase the width (Capelli & Mengel, 2019), D' also has width at most $n^{O(k)}$. We then simply apply Lemma 15 to get the result.

To see b), just observe that, following the same construction, the width of D is $2^{O(k)}$ for all considered width measures (Bova et al., 2015). \square

Remark that the construction of Theorem 14 has a surprising property: the size and the number of auxiliary variables of the constructed encoding F' does *not* depend on the size of the initial encoding at all. Both depend only on the number of variables in f and the width.

To maintain dependence of the auxiliary variables in the above construction, we have to work some more than for Theorem 14. We start with some definitions. We call a complete structured DNNF *reduced* if from every gate there is a directed path to the output gate. Note that every complete structured DNNF can be turned into a reduced DNNF in linear time by a simple graph traversal and that this transformation maintains determinism and structuredness by the same v-tree. The following property will be useful.

Lemma 17. *Let D be a reduced complete structured DNNF and let g be a gate in D . Let a_g be an assignment to $\text{var}(g)$, the variables in the subcircuit rooted in g , that satisfies g . Then, a_g can be extended to an assignment a that satisfies D .*

Proof. We use the fact that an assignment to D is satisfying if and only if there is a proof-tree that witnesses this. So let \mathcal{T}_g be a proof tree that witnesses a_g satisfying g . We extend it to a proof tree for an extension a of a_g as follows: first add a path from g to the output gate to \mathcal{T}_g and then iteratively add more gates as required by the definition of proof trees where the choices in \vee -gates are performed arbitrarily. The result is an extension \mathcal{T} of \mathcal{T}_g which witnesses that an assignment a that extends a_g satisfies D . \square

Let f be a function in variables $X \cup \{z\}$. We say that z is *definable* in X with respect to f if there is a function g such that for all assignments a with $f(a) = 1$ we have $a(z) = g(a|_X)$ where $a|_X$ is the restriction of a to X .

Lemma 18. *Let f be a function in variables $X \cup \{z\}$ such that z is definable in X with respect to f . Let D be a reduced complete structured deterministic DNNF computing f . Then the complete structured DNNF D' we get from D by forgetting z is deterministic as well.*

Proof. By way of contradiction, assume this were not the case. Then there is an \vee -gate g in D' and an assignment a' to X such that two children g_1 and g_2 are satisfied by a' . By Lemma 17, we may assume that a' satisfies D' . Then there are extensions a_1 and a_2 of a' that assign a value to z such that a_1 satisfies g_1 and a_2 satisfies g_2 in D . Note that both a_1

and a_2 satisfy D and thus, by definability, a_1 and a_2 assign the same value to z . So $a_1 = a_2$ and hence a_1 satisfies both g_1 and g_2 in D which contradicts the determinism of D . \square

Theorem 19. *Let f be a Boolean function of n variables.*

- a) *If F has a clausal encoding with dependent auxiliary variables of modular treewidth, cliquewidth or mim-width k then f also has a clausal encoding F' with dependent auxiliary variables of primal treewidth $O(k \log(n))$ with $O(kn \log(n))$ auxiliary variables and $n^{O(k)}$ clauses.*
- b) *If F has a clausal encoding with dependent auxiliary variables of incidence treewidth, dual treewidth, or signed incidence cliquewidth k , then f also has a clausal encoding F' with dependent auxiliary variables of primal treewidth $O(k)$ with $O(nk)$ auxiliary variables and $2^k n$ clauses.*

Proof. The proof is essentially the same as that of Theorem 14 with some additional twists. First observe that the complete structured DNNF D constructed with the construction of Bova et al. (2015) is deterministic. Then we use Lemma 18 when forgetting the auxiliary variables and get a D' that is deterministic without increasing the width. Then, since D' is deterministic, we can construct a clausal encoding with dependent auxiliary variables using Lemma 15. \square

Next we will show that signed incidence cliquewidth is linearly related to primal treewidth when allowing auxiliary variables. We will state a result similar to Lemma 15.

To do so, we will start with a special case for which we introduce some more definitions: a *special tree decomposition* of a graph G is defined as a tree decomposition $(T, (B_t)_{t \in V(T)})$ in which for every vertex $x \in V(G)$ the set $\{t \in V(T) \mid x \in B_t\}$ lies on a leaf-root path in T (Courcelle, 2012). The *special treewidth* is defined as the smallest width of any special tree decomposition of G . Finally, we define the primal special treewidth of a CNF-formula as the special treewidth of its primal graph.

Lemma 20. *Every CNF-formula of primal special treewidth k has signed incidence cliquewidth at most $k + 1$.*

Proof. Let $(T, (B_t)_{t \in V(T)})$ be a special tree decomposition of the primal graph of F . It is well known that for every clause C there is a node $t = \lambda(C)$ of T such that all variables of C are in B_t . By adding copies of some bags B_t along a root-leaf path in T , we may assume that $\lambda(C) \neq \lambda(C')$ for every pair C, C' of clauses with $C \neq C'$.

We will show how to construct the signed incidence graph G' of F with the operations in the definition of signed cliquewidth along the tree T . In a first step, we label every variable x of F with a color $\mu(x)$ from $\{1, \dots, k + 1\}$ such that in every bag B_t there are no two variables with the same label $\mu(x)$. This can be done similarly to the first step of the proof of Theorem 13 by descending from the root to the leaves and labeling the variables in the bags along this way. The label $\mu(x)$ will be the label that the variable gets when it is created in the construction of G' . As in the proof of Theorem 13, the only renamings of labels that we will perform will be forget operations, i.e., renaming a label to a dummy label d .

For the construction of G' , we will iteratively construct for every $t \in V(T)$ a graph G_t that contains all variables in $S_t := \bigcup_{t' \in V(T_t)} B_{t'}$ where T_t is the subtree of T rooted in t . Moreover, G_t contains all clauses such that $\lambda(C)$ lies in T_t and all signed edges connecting them to their variables.

If t is a leaf, then we create all variables in B_t and if there is a clause C with $\lambda(C) = t$, we introduce it with color $k + 2$. Since all variables of C have different colors, we can then introduce all signed edges individually. This completes the construction for the leaf case.

Let now t be an internal node with children t_1, \dots, t_ℓ . By assumption, we have already constructed $G_{t_1}, \dots, G_{t_\ell}$. Note that for every i the variables in G_{t_i} that are not in B_t are by construction already connected to all their clauses in G_{t_i} , so we can safely forget their label in a first step. Now we take the disjoint union of all G_{t_i} . Note that this union is in fact disjoint, because, since we start from a special tree decomposition, no node appears in more than one G_{t_i} . Now we create the variables which appear in B_t but not in any G_{t_i} . Note that at this point the vertices with non-dummy labels are exactly those in B_t . If there is no clause C with $\lambda(C) = t$, we are done. Otherwise, we create C and connect it to all its variables by signed edges as in the leaf case. This completes the construction of G_t .

For the root r of T we have $G_r = G'$ by definition. Moreover, we have used at most $k + 2$ labels. This completes the proof. \square

With Lemma 20, we can give a version of Lemma 15 for signed incidence cliquewidth easily.

Lemma 21. *Let f be a Boolean function in n variables that is computed by a structured DNNF of width k . Then f has a clausal encoding F of signed incidence cliquewidth and primal special treewidth $O(\log(k))$ with $O(n \log(k))$ variables and $O(nk^3)$ clauses. Moreover, if D is deterministic then F has dependent auxiliary variables.*

Proof. We only have to observe that in fact the tree decomposition in the proof of Lemma 15 is special and apply Lemma 20. \square

Corollary 22. *Let f be a function with a CNF-representation of primal treewidth k . Then f has a clausal encoding of signed incidence cliquewidth and special treewidth $O(k)$.*

5.3 Putting Things Together

We can now state the main result of this section.

Theorem 23. *Let $A = \{\text{tw}_p, \text{tw}_d, \text{tw}_i, \text{scw}\}$ and $B = \{\text{mtw}, \text{cw}, \text{mimw}\}$. Let f be a Boolean function in n variables.*

- a) *Let $w_1 \in A$ and $w_2 \in B$. Then there are constants c_1 and c_2 such that the following holds: let F_1 and F_2 be clausal representations for f with minimal w_1 -width and w_2 -width, respectively. Then*

$$w_1(F_1) \leq k \log(n) \Rightarrow w_2(F_2) \leq c_1 k$$

and

$$w_2(F_2) \leq k \Rightarrow w_1(F_1) \leq c_2 k \log(n).$$

b) Let $w_1 \in A$ and $w_2 \in A$ or $w_1 \in B$ and $w_2 \in B$. Then there are constants c_1 and c_2 such that the following holds: let F_1 and F_2 be clausal representations for f of minimal w_1 -width and w_2 -width, respectively. Then

$$w_1(F_1) \leq k \Rightarrow w_2(F_2) \leq c_1 k$$

and

$$w_2(F_2) \leq k \Rightarrow w_1(F_1) \leq c_2 k.$$

Proof. Assume first that $w_1 = \text{tw}_p$. For a) we get the second statement directly from Theorem 14 a). For cw and mtw we get the first statement by Theorem 13. For mimw it follows by the fact that for every graph $\text{mimw}(G) \leq c \cdot \text{cw}(G)$ for some absolute constant c (Vatshelle, 2012, Section 4).

For b), the second statement is Theorem 14 b). Since for every formula F we have $\text{tw}_i(F) \leq \text{tw}_p(F) + 1$, (see e.g. Fischer et al., 2008), the first statement for tw_i is immediate. For scw it is shown in Corollary 22, while for tw_d it has been shown by Samer and Szeider (2010b). All other combinations of w_1 and w_2 can now be shown by an intermediate step using tw_p . \square

6. Applications

6.1 Cardinality Constraints

In this section, we consider cardinality constraints, i.e., constraints of the form $\sum_{i \in [n]} x_i \leq k$ in the Boolean variables x_1, \dots, x_n . The value k is commonly called the *degree* or the *threshold* of the constraint. Let us denote by C_n^k the cardinality constraint with n variables and degree k . Cardinality constraints have been studied extensively and many encodings are known (see e.g. Sinz, 2005a). Here we add another perspective on cardinality constraint encodings by determining their optimal treewidth. We remark that we could have studied cardinality constraints in which the relation is \geq instead of \leq with essentially the same results. We start with an easy observation:

Observation 24. C_n^k has an encoding of primal treewidth $O(\log(\min(k, n - k)))$

Proof. First assume that $k < n/2$. We iteratively compute the partial sums of $S_j := \sum_{i \in [j]} x_i$ and encode their values in $\log(k) + 1$ bits $Y^j := \{y_1^j, \dots, y_{\log(k)+1}^j\}$. We cut these sums off at $k+1$ (if we have seen at least $k+1$ variables set to 1, this is sufficient to compute the output). In the end we encode a comparator comparing the last sum S_n to k .

Since the computation of S_{j+1} can be done from S_j and x_{j+1} , we can compute the partial sums with clauses containing only the variables in $Y^j \cup Y^{j+1} \cup \{x_{j+1}\}$, so $O(\log(k))$ variables. The resulting CNF-formula can easily be seen to be of treewidth $O(\log(k))$.

If $k > n/2$, we proceed similarly but count variables assigned to 0 instead of those set to 1. \square

We remark that our construction is the “basic approach” described by Biere et al. (2009, Section 8.6.7). It has some similarity with the sequential counter introduced by Sinz (2005b). The main difference is that we encode the partial sums S_j in binary whereas in the

sequential counter, they are encoded in unary. This latter encoding has better properties with respect to unit propagation, whereas our encoding has smaller treewidth, which is the parameter we are optimizing for. We now show that Observation 24 is essentially optimal.

Proposition 25. *Let $k < n/2$. Then*

$$\text{cc}_{\text{best}}^{1/3}(C_n^k) = \Omega(\log(\min(k, n/3))).$$

Proof. Let $s = \min(k, \frac{n}{3})$. Consider an arbitrary partition Y, Z with $\frac{n}{3} \leq |Y| \leq \frac{2n}{3}$. We show that every rectangle cover of C_n^k must have s rectangles. To this end, choose assignments $(a_0, b_0), \dots, (a_s, b_s)$ such that $a_i : Y \rightarrow \{0, 1\}$ assigns i variables to 1 and $b_i : Z \rightarrow \{0, 1\}$ assigns $k - i$ variables to 1. Note that every (a_i, b_i) satisfies C_n^k . We claim that no rectangle $r_1(Y) \wedge r_2(Z)$ in a rectangle cover of C_n^k can have models (a_i, b_i) and (a_j, b_j) for $i \neq j$. To see this, assume that such a model exists and that $i < j$. Then the assignment (a_j, b_i) is also a model of the rectangle since a_j satisfies $r_1(Y)$ and b_i satisfies $r_2(Z)$. But (a_j, b_i) contains more than k variables assigned to 1, so the rectangle $r_1(Y) \wedge r_2(Z)$ cannot appear in a rectangle cover of C_n^k . Thus, every rectangle cover of C_n^k must have a different rectangle for every model (a_i, b_i) and thus at least s rectangles. This completes the proof for this case. \square

A symmetric argument shows that for $k > n/2$ we have the lower bound $\text{cc}_{\text{best}}^{1/3}(C_n^k) = \Omega(\log(\min(n - k, n/3)))$. Observing that $k < n$ for non-trivial cardinality constraints, we get the following from Theorem 6.

Corollary 26. *Clausal encodings of smallest primal treewidth for C_n^k have primal treewidth $\Theta(\log(s))$ for $s = \min(k, n - k)$. The same statement is true for dual and incidence treewidth and signed incidence cliquewidth. For incidence cliquewidth, modular treewidth and mim-width, there are clausal encodings of C_n^k of constant width.*

6.2 The Permutation Function

We now consider the permutation function PERM_n which has the n^2 input variables $X_n = \{x_{ij} \mid i, j \in [n]\}$ thought of as a matrix in these variables. PERM_n evaluates to 1 on an input a if and only if a is a permutation matrix, i.e., in every row and in every column of a there is exactly one 1.

Example 8. The function PERM_2 has the variables $x_{11}, x_{12}, x_{21}, x_{22}$ which we interpret organized as the matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. The only inputs on which PERM_2 evaluates to 1 are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Inputs on which PERM_2 evaluates to 0 are for example $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (the first row has more than one 1-entry) and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (the first column has no 1-entry).

PERM_n is known to be hard in several versions of branching programs (Wegener, 2000). Briquel et al. (2011) showed that clausal encodings of PERM_n require treewidth $\Omega(n/\log(n))$. We here give an improvement by a logarithmic factor.

Lemma 27. *For every v -tree T on variables X_n , there is a node t of T such that*

$$\text{cc}(\mathbf{PERM}_n, Y, Z) = \Omega(n)$$

where $Y = \text{var}(T_t)$ and $Z = X \setminus Y$.

Proof. The proof is a variation of arguments used by Briquel et al. (2011), Krause (1988) and (Wegener, 2000, Section 4.12). Since all models of \mathbf{PERM}_n assign exactly n variables to 1, for every model a of \mathbf{PERM}_n there is a node t_a in T such that T_{t_a} contains between $n/3$ and $2n/3$ variables assigned to 1 by a . Since T has n internal nodes and \mathbf{PERM}_n has $n!$ models, there must be a node t such that for at least $(n-1)!$ of the models a we have $t = t_a$. We will show in the remainder that t has the desired property.

Denote by A the set of models of \mathbf{PERM}_n for which $t_a = t$. Let $Y = \text{var}(T_t)$ and $Z = X_n \setminus Y$ as in the statement of the lemma. Every model a of \mathbf{PERM}_n corresponds to a permutation π_a on $[n]$ that assigns every $i \in [n]$ to the j such that $a(x_{ij}) = 1$. Note that because of the properties of a , π_a is well-defined and indeed a permutation.

Let $R(X) = r_1(Y) \wedge r_2(Z)$ be a rectangle in a rectangle cover of \mathbf{PERM}_n with partition (Y, Z) . We will show that $R(X)$ contains few models from A . To this end, fix a model $a \in A$ of $R(X)$ and define $I(a) = \{i \mid x_{i, \pi_a(i)} \in Y\}$. Note that $k := |I(a)|$ is the number of variables in Y that are assigned to 1 by a and thus $n/3 \leq k \leq 2n/3$. Let a' be another model of $R(X)$. Then $I(a') = I(a)$ because otherwise $a|_Y \cup a'|_Z$ does not encode a permutation where $a|_Y$ denotes the restriction of a to Y and $a'|_Z$ that of a' to Z . Letting $I'(a) = \{\pi_a(i) \mid i \in I(a)\}$, we get similarly that for all models a' of $R(X)$ we have $I'(a) = I'(a')$. It follows that the models of $r_1(Y)$ are all bijections between $I(a)$ and $I'(a)$ and thus $r_1(Y)$ has at most $k!$ models.

By a symmetric argument, one sees that $r_2(Z)$ has at most $(n-k)!$ models. Thus, the number of models of R is bounded by $k!(n-k)! \leq (\frac{n}{3})!(\frac{2n}{3})!$. As a consequence, to cover all $(n-1)!$ models in a , one needs at least

$$\frac{(n-1)!}{(\frac{n}{3})!(\frac{2n}{3})!} = \frac{1}{n} \binom{n}{\frac{n}{3}} \geq \frac{1}{n} \left(\frac{n}{3}\right)^{\frac{n}{3}} = \frac{1}{n} \sqrt[3]{3}^n$$

rectangles, which completes the proof. □

As a consequence of Lemma 27, we get an asymptotically tight treewidth bound for encodings of \mathbf{PERM}_n .

Corollary 28. *Clausal encodings of smallest primal treewidth for C_n^k have primal treewidth $\Theta(n)$.*

Proof (sketch). The lower bound follows by using Lemma 27 and Proposition 7 and then arguing as in the proof of Theorem 9.

For the upper bound, observe that checking if out of n variables exactly one has the value 1 can easily be done with n variables. We apply this for every row in a bag of a tree decomposition. We perform these checks for one row after the other and additionally use variables for the columns that remember if in a column we have seen a variable assigned 1 so far. Overall, to implement this, one needs $O(n^2)$ auxiliary variables and gets a formula of treewidth $O(n)$. □

From Corollary 28 we get the following bound by applying Theorem 13. This answers an open problem of Briquel et al. (2011), who showed only conditional lower bounds for the incidence cliquewidth of encodings of \mathbf{PERM}_n .

Corollary 29. *Clausal encodings of smallest incidence cliquewidth for C_n^k have width $\Theta(n/\log(n))$.*

6.3 Improved Lower Bounds for Minor-Free Graphs

In this section, we show how our approach can be used to improve lower bounds for structurally restricted classes of circuits. We recall that a *minor* H of a graph G is a graph that we can get from G by deleting vertices, deleting edges and contracting edges. For a graph H , the class of H -minor-free graphs is defined as the class of graphs consisting of all graphs that do not have H as a minor. H -minor-free graphs have been studied extensively. In particular, it is known that for planar graphs, and more generally for all graphs embeddable in a fixed surface, there is a graph H such that those graphs are H -minor free. For example, planar graphs are K_5 -minor-free and $K_{3,3}$ -minor-free.

We say that a Boolean circuit C is H -minor-free if the underlying undirected graph of C is H -minor-free. Remember that we assume that every input variable is the label of at most one input gate. There have long been quadratic lower bounds for planar circuits (Lipton & Tarjan, 1980). Those were generalized to almost quadratic lower bounds of the order $\Omega(n^2/\log(n)^2)$ for H -minor-free circuits shown by de Oliveira Oliveira (2018). We show here that with our techniques it is easy to improve these bounds to quadratic lower bounds.

The basic building block for our lower bound will be the following result on the treewidth of H -minor-free graphs (as in de Oliveira Oliveira, 2018).

Theorem 30 (Alon, Seymour, & Thomas, 1990). *For every graph H there is a constant h such that every H -minor-free graph G has treewidth at most $h\sqrt{|V(G)|}$.*

Corollary 31. *For every graph H there is a constant h' such that for every function f , every H -minor-free circuit C computing f has at least $cc_{\text{best}}^{1/3}(f)^2$ gates.*

Proof. By Corollary 11, any circuit computing f must have treewidth $\Omega(cc_{\text{best}}^{1/3}(f))$. By Theorem 30, the treewidth of C is at most \sqrt{s} where s is the number of gates in C . Thus $\sqrt{s} \geq cc_{\text{best}}^{1/3}(f)$ and the claim follows. \square

To show a quadratic lower bound, consider the function $\Delta\text{-free}_n$ in variables X_{ij} with $1 \leq i < j \leq n$ which is defined as follows: interpret the input as the adjacency matrix of a graph G and return 1 if and only if G does not have a triangle as a subgraph. We note that $\Delta\text{-free}_n$ is a classical function, considered in communication complexity essentially since the creation of the field (Papadimitriou & Sipser, 1984). Here, we will use the following result:

Theorem 32 (Jukna & Schnitger, 2002). *The best-case non-deterministic communication complexity with $\frac{1}{3}$ -balance of $\Delta\text{-free}_n$ is quadratic in n , i.e.,*

$$cc_{\text{best}}^{1/3}(\Delta\text{-free}_n) = \Omega(n^2).$$

We directly get the following generalization of the quadratic lower bound shown by Lipton and Tarjan (1980), which improves that of de Oliveira Oliveira (2018).

Theorem 33. *For every fixed graph H there is a constant h' such that every H -minor-free circuit computing Δ -free $_n$ has $\Omega(n^4)$ gates, i.e., quadratic in the number of inputs.*

7. Conclusion

We have shown several results on the expressivity of clausal encodings with restricted underlying graphs. In particular, we have seen that many graph width measures from the literature put strong restrictions on the expressivity of encodings. We have also seen that, contrary to the case of representations by CNF-formulas, in the case where auxiliary variables are allowed, all width measures we have considered are strongly related to primal treewidth and never differ by more than a logarithmic factor. Moreover, most of our results are also true while maintaining dependence of auxiliary variables.

From a practical standpoint, one point of our results might be that formulas solved with width-based algorithms as those from the theoretical literature can likely only deal with quite simple formulas. Otherwise, for example if formulas contain big cardinality constraints or pseudo-Boolean constraints, the width of the formulas might be infeasibly high. This is because all those algorithms are at least exponential in the width of the input. An implementation of such algorithms would thus likely have to implement heuristics and optimizations not presented in the theory literature. For example, Fichte, Hecher, and Zisser (2019) showed that one can use parallelism of GPUs to improve the efficiency of treewidth-based counting and thus scale to higher treewidth.

To close the paper, let us discuss several questions. First, the number of clauses of the encodings guaranteed by Theorem 14 is very high. In particular, it is exponential in the width k . It would be interesting to understand if this can be avoided, i.e., if there are encodings of roughly the same primal treewidth whose size is polynomial in k .

It would also be interesting to see if our results can be extended to other classes of CNF-formulas on which SAT is tractable. Interesting classes to consider would e.g. be the classes described by Ganian and Szeider (2017). In this paper, the authors define another graph for CNF-formulas for which bounded treewidth yields tractable model counting. It is not clear if the classes characterized that way allow small complete structured DNNF so our framework does not apply directly. It would still be interesting to see if one can show similar expressivity results to those here. Other interesting classes one could consider are those defined by backdoors (see e.g. Gaspers & Szeider, 2013).

Acknowledgements

The authors are grateful to the anonymous reviewers for their comments, which greatly helped to improve the presentation of the paper. The first author would like to thank David Mitchell for asking the right question at the right moment. This paper grew largely out of an answer to this question.

References

- Alon, N., Seymour, P. D., & Thomas, R. (1990). A Separator Theorem for Graphs with an Excluded Minor and its Applications. In *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing, STOC 1990*.
- Amarilli, A., Capelli, F., Monet, M., & Senellart, P. (2018a). Connecting Knowledge Compilation Classes and Width Parameters. *CoRR*, *abs/1811.02944*.
- Amarilli, A., Monet, M., & Senellart, P. (2018b). Connecting Width and Structure in Knowledge Compilation. In *21st International Conference on Database Theory, ICDT 2018, March 26-29, 2018, Vienna, Austria*.
- Biere, A., Heule, M., van Maaren, H., & Walsh, T. (Eds.). (2009). *Handbook of Satisfiability*, Vol. 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press.
- Bova, S., Capelli, F., Mengel, S., & Slivovsky, F. (2015). On Compiling CNFs into Structured Deterministic DNNFs. In *Theory and Applications of Satisfiability Testing - SAT 2015 - 18th International Conference*.
- Bova, S., Capelli, F., Mengel, S., & Slivovsky, F. (2016). Knowledge Compilation Meets Communication Complexity. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016*.
- Bova, S., & Szeider, S. (2017). Circuit Treewidth, Sentential Decision, and Query Compilation. In *Proceedings of the 36th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2017*, pp. 233–246.
- Braut-Baron, J., Capelli, F., & Mengel, S. (2014). Understanding model counting for β -acyclic CNF-formulas. *CoRR*, *abs/1405.6043*.
- Briquel, I., Koiran, P., & Meer, K. (2011). On the expressive power of CNF formulas of bounded tree- and clique-width. *Discrete Applied Mathematics*, *159*(1), 1–14.
- Capelli, F., & Mengel, S. (2019). Tractable QBF by Knowledge Compilation. In *36th International Symposium on Theoretical Aspects of Computer Science, STACS 2019*, Vol. 126, pp. 18:1–18:16.
- Chen, H. (2004). Quantified Constraint Satisfaction and Bounded Treewidth. In *Proceedings of the 16th European Conference on Artificial Intelligence, ECAI 2004*.
- Courcelle, B. (2012). On the model-checking of monadic second-order formulas with edge set quantifications. *Discrete Applied Mathematics*, *160*(6), 866–887.
- Crama, Y., & Hammer, P. (2011). *Boolean Functions: Theory, Algorithms, and Applications*.
- Darwiche, A., & Marquis, P. (2002). A Knowledge Compilation Map. *J. Artif. Intell. Res.*, *17*, 229–264.

- de Oliveira Oliveira, M. (2018). Size-Treewidth Tradeoffs for Circuits Computing the Element Distinctness Function. *Theory Comput. Syst.*, 62(1), 136–161.
- Diestel, R. (2012). *Graph Theory, 4th Edition*, Vol. 173 of *Graduate texts in mathematics*. Springer.
- Duris, P., Hromkovic, J., Jukna, S., Sauerhoff, M., & Schnitger, G. (2004). On multipartition communication complexity. *Inf. Comput.*, 194(1), 49–75.
- Fichte, J. K., Hecher, M., & Zisser, M. (2019). An Improved GPU-Based SAT Model Counter. In Schiex, T., & de Givry, S. (Eds.), *Principles and Practice of Constraint Programming - 25th International Conference, CP 2019, Stamford, CT, USA, September 30 - October 4, 2019, Proceedings*, Vol. 11802 of *Lecture Notes in Computer Science*, pp. 491–509. Springer.
- Fischer, E., Makowsky, J., & Ravve, E. (2008). Counting truth assignments of formulas of bounded tree-width or clique-width. *Discrete Applied Mathematics*, 156(4), 511–529.
- Ganian, R., & Szeider, S. (2017). New Width Parameters for Model Counting. In *20th International Conference Theory and Applications of Satisfiability Testing, SAT 2017*.
- Gaspers, S., & Szeider, S. (2013). Strong Backdoors to Bounded Treewidth SAT. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013*.
- Gent, I. P., & Nightingale, P. (2004). A new encoding of alldifferent into SAT. In *International Workshop on Modelling and Reformulating Constraint Satisfaction Problems*.
- Giunchiglia, E., Maratea, M., & Tacchella, A. (2002). Dependent and Independent Variables in Propositional Satisfiability. In *Logics in Artificial Intelligence, JELIA 2002*.
- Gottlob, G., Pichler, R., & Wei, F. (2010). Bounded treewidth as a key to tractability of knowledge representation and reasoning. *Artif. Intell.*, 174(1), 105–132.
- Hromkovic, J. (1997). *Communication Complexity and Parallel Computing*. Texts in Theoretical Computer Science. An EATCS Series. Springer.
- Jakl, M., Pichler, R., & Woltran, S. (2009). Answer-Set Programming with Bounded Treewidth. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence, IJCAI 2009*.
- Jukna, S. (2012). *Boolean Function Complexity - Advances and Frontiers*, Vol. 27 of *Algorithms and combinatorics*. Springer.
- Jukna, S., & Schnitger, G. (2002). Triangle-Freeness Is Hard To Detect. *Combinatorics, Probability & Computing*, 11(6), 549–569.
- Krause, M. (1988). Exponential Lower Bounds on the Complexity of Local and Real-time Branching Programs. *Elektronische Informationsverarbeitung und Kybernetik*, 24(3), 99–110.

- Kushilevitz, E., & Nisan, N. (1997). *Communication complexity*. Cambridge University Press.
- Lampis, M., Mengel, S., & Mitsou, V. (2018). QBF as an Alternative to Courcelle’s Theorem. In *21st International Conference Theory and Applications of Satisfiability Testing, SAT 2018*.
- Lipton, R. J., & Tarjan, R. E. (1980). Applications of a Planar Separator Theorem. *SIAM J. Comput.*, 9(3), 615–627.
- Papadimitriou, C. H., & Sipser, M. (1984). Communication Complexity. *J. Comput. Syst. Sci.*, 28(2), 260–269.
- Paulusma, D., Slivovsky, F., & Szeider, S. (2013). Model Counting for CNF Formulas of Bounded Modular Treewidth. In *30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013*.
- Paulusma, D., Slivovsky, F., & Szeider, S. (2016). Model Counting for CNF Formulas of Bounded Modular Treewidth. *Algorithmica*, 76(1), 168–194.
- Pipatsrisawat, K., & Darwiche, A. (2008). New Compilation Languages Based on Structured Decomposability. In *Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence, AAAI 2008*.
- Pipatsrisawat, T., & Darwiche, A. (2010). A Lower Bound on the Size of Decomposable Negation Normal Form. In *Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2010*.
- Sæther, S. H., Telle, J. A., & Vatshelle, M. (2015). Solving #SAT and MAXSAT by Dynamic Programming. *J. Artif. Intell. Res.*, 54, 59–82.
- Samer, M., & Szeider, S. (2010a). Algorithms for propositional model counting. *J. Discrete Algorithms*, 8(1), 50–64.
- Samer, M., & Szeider, S. (2010b). Constraint satisfaction with bounded treewidth revisited. *J. Comput. Syst. Sci.*, 76(2), 103–114.
- Sinz, C. (2005a). Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. In *Principles and Practice of Constraint Programming - CP 2005, 11th International Conference, CP 2005*.
- Sinz, C. (2005b). Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. In *Proceedings of the 11th International Conference on Principles and Practice of Constraint Programming, CP’05*, pp. 827–831 Berlin, Heidelberg. Springer-Verlag.
- Slivovsky, F., & Szeider, S. (2013). Model Counting for Formulas of Bounded Clique-Width. In *Algorithms and Computation - 24th International Symposium, ISAAC 2013*.
- Vatshelle, M. (2012). *New Width Parameters of Graphs*. Ph.D. thesis, University of Bergen.
- Wegener, I. (2000). *Branching Programs and Binary Decision Diagrams*. SIAM.