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Classifying Inconsistency Measures Using Graphs

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Abstract

The aim of measuring inconsistency is to obtain an evaluation of the imperfections in a set of formulas, and this evaluation may then be used to help decide on some course of action (such as rejecting some of the formulas, resolving the inconsistency, seeking better sources of information, etc). A number of proposals have been made to define measures of inconsistency. Each has its rationale. But to date, it is not clear how to delineate the space of options for measures, nor is it clear how we can classify measures systematically. To address these problems, we introduce a general framework for comparing syntactic measures of inconsistency. It is based on the notion of an inconsistency graph for each knowledgebase (a bipartite graph with a set of vertices representing formulas in the knowledgebase, a set of vertices representing minimal inconsistent subsets of the knowledgebase, and edges representing that a formula belongs to a minimal inconsistent subset). We then show that various measures can be computed using the inconsistency graph. Then we introduce abstractions of the inconsistency graph and use them to construct a hierarchy of syntactic inconsistency measures. Furthermore, we extend the inconsistency graph concept with a labeling that extends the hierarchy to include some other types of inconsistency measures.

1. Introduction

Inconsistency is a key issue for operating in the real world. Routinely, we are faced with inconsistencies when we deal with information, opinions, requirements, desires, plans, etc. So if we are to build computing systems that are inconsistency capable (i.e. systems that can handle inconsistency in information, opinions, requirements, desires, plans, etc., for instance when making decisions), then we need technologies for assessing and acting on inconsistency (Gabbay & Hunter, 1991; Bertossi, Hunter, & Schaub, 2004; Grant & Hunter, 2011).

A key aspect of inconsistency capability is the ability to measure inconsistency so that we can obtain a better assessment of the nature of the inconsistencies. Application areas being investigated for inconsistency measures include software engineering (Zhu & Jin, 2005; Mu, Jin, Liu, Zowghi, & Wei, 2013), network intrusion detection (McAreavey, Liu, Miller, & Mu, 2011), ontology systems (Zhou, Huang, Qi, Ma, Huang, & Qu, 2009), knowledgebase systems (Mu, Liu, Jin, & Bell, 2011b; Potyka, 2014; Mu, Wang, & Wen, 2016), databases (Decker & Martinenghi, 2011; Bertossi, 2018), analysing spatial and temporal information (Condotta, Raddaoui, & Salhi, 2016), and answer set programming (Madrid & Ojeda-Aciego, 2011; Ulbricht, Thimm, & Brewka, 2016).

Numerous proposals for inconsistency measures have been made (Grant, 1978; Knight, 2002; Hunter, 2002; Konieczny, Lang, & Marquis, 2003; Hunter & Konieczny, 2004; Grant & Hunter, 2006; Ma, Qi, Hitzler, & Lin, 2007; Qi & Hunter, 2007; Grant & Hunter, 2008; Zhou et al., 2009; Jabbour, Ma, & Raddaoui, 2014) and some inter-relationships established (e.g. Grant & Hunter, 2011; Thimm, 2016b). Furthermore, some axioms have been proposed for the minimal properties of measures of (Hunter & Konieczny, 2006, 2010), and some alternatives have been proposed (e.g. Besnard, 2014), which offer some groupings of approaches. See Grant and Martinez (2018) for a summary of the state-of-the-art on measuring inconsistency in information. Nevertheless, our understanding of measures lacks a general framework in which to position and compare different measures of inconsistency. To address this shortcoming, in this paper we propose a general framework for inconsistency measures based on minimal inconsistent subsets of the knowledgebase.

The majority of the syntactic inconsistency measures are based on minimal inconsistent subsets. Each measure focuses on a different aspect of these sets. To illustrate this point, consider the following knowledgebases that have the same number and size of minimal inconsistent subsets. For some key inconsistency measures, these knowledgebases are indistinguishable.

- For $K_1 = \{a, \neg a, b, \neg b\}$, there are two minimal inconsistent subsets $M_1 = \{a, \neg a\}$ and $M_2 = \{b, \neg b\}$. There is no overlap between these minimal inconsistent sets.
- For $K_2 = \{a, \neg a \wedge \neg b, b\}$, there are two minimal inconsistent subsets $M_1 = \{a, \neg a \wedge \neg b\}$ and $M_2 = \{b, \neg a \wedge \neg b\}$. There is an overlap between these minimal inconsistent sets.

In this paper, we analyse inconsistency in terms of the minimal inconsistent subsets of the knowledgebase. For this purpose, we introduce inconsistency graphs that capture information about the minimal inconsistent subsets in graphical form. This provides substantial information about the inconsistency in a knowledgebase and a number of existing inconsistency measures can be captured this way. Conversely, we show how and when functions on inconsistency graphs yield inconsistency measures, and new measures based on these functions are defined.

This paper extends substantially our previous paper (De Bona, Grant, Hunter, & Konieczny, 2018). In addition to providing all the proofs, we also include further results concerning the nature of inconsistency graphs, computing inconsistency measures from consistency graphs, devising inconsistency measures from inconsistency graphs, constructing inconsistency graphs from inconsistency measures, and creating a hierarchy of non-syntactic measures.

The plan of this paper is as follows. In Section 2 we provide general definitions needed in the paper. In Section 3 we review some basic work on inconsistency measures including properties and examples of inconsistency measures. We introduce inconsistency and consistency graphs and their fundamental properties in Section 4. Then, in Section 5 we investigate in detail the relationship of inconsistency and consistency graphs to inconsistency measures. Section 6 shows how abstractions of inconsistency graphs can be used to create a hierarchy of syntactic inconsistency measures. Furthermore, in Section 7 we enlarge the previous hierarchy in a way that allows us to capture some semantic inconsistency measures as well. The paper is concluded in Section 8.

2. Preliminaries

We assume a propositional language $\mathcal{L}$ of formulas composed from a countable set of propositional variables (atoms) $\mathcal{P}$ and the logical connectives $\wedge$, $\vee$, $\neg$. We use φ and ψ for arbitrary formulas and $a, b, c, \ldots$ for atoms. A **knowledgebase** K is a finite set of formulas. We write $\mathcal{K}$ for the set of all knowledgebases (defined from the language $\mathcal{L}$). We let $\vdash$ denote the classical consequence relation, and write $K \vdash \perp$ to denote that K is inconsistent. Logical equivalence is defined in the usual way: $K \equiv K'$ iff $K \vdash K'$ and $K' \vdash K$. We write $\mathbb{R}^{\geq 0}$ for the set of nonnegative real numbers, $\mathbb{R}^{\geq 0}_{\infty}$ for $\mathbb{R}^{\geq 0} \cup \{\infty\}$, and 2^X for the set of all subsets (the power set) of any set X.

φ	T	T	T	B	B	B	F	F	F
ψ	T	B	F	T	B	F	T	B	F
$\varphi \vee \psi$	T	T	T	T	B	B	T	B	F
$\varphi \wedge \psi$	T	B	F	B	B	F	F	F	F
$\neg\varphi$	F	F	F	B	B	B	T	T	T

Table 1: Truth table for Priest's three valued logic (3VL). This semantics extends the classical semantics with a third truth value, B, denoting "inconsistency".

For a knowledgebase K, $\mathsf{MI}(K)$ is the set of minimal inconsistent subsets of K, i.e. $\mathsf{MI}(K) = \{K' \mid K' \subseteq K, K' \vdash \bot, \forall K'' \subset K' \; K'' \nvdash \bot\}$, and $\mathsf{MC}(K)$ is the set of maximal consistent subsets of K, i.e. $\mathsf{MC}(K) = \{K' \mid K' \subseteq K, K' \nvdash \bot, \forall K'' s.t. \; K' \subset K'' \; K'' \vdash \bot\}$. Also, if $\mathsf{MI}(K) = \{M_1, ..., M_n\}$ then $\mathsf{Problematic}(K) = M_1 \cup ... \cup M_n$, and $\mathsf{Free}(K) = K \setminus \mathsf{Problematic}(K)$. So $\mathsf{Free}(K)$ contains the formulas in K that are not involved in any minimal inconsistency and $\mathsf{Problematic}(K)$ contains the formulas in K that are involved in at least one minimal inconsistency. The set of formulas in K that are individually inconsistent is given by the function $\mathsf{Selfcontradictions}(K) = \{\varphi \in K \mid \{\varphi\} \vdash \bot\}$. In the next section we will use these functions in definitions for syntactic inconsistency measures.

For one of the inconsistency measures we need to use Priest's three valued logic (3VL) (Priest, 1979) with the classical two valued semantics augmented by a third truth value denoting inconsistency. The truth values for the connectives are defined in Table 1. The inconsistency truth value B denotes that a formula is both true and false (i.e. we have conflicting information about it). We can assume a ranking over the truth tables from T to B to F (i.e. T is the maximum and F is the minimum), and as can be seen from the truth table, conjunction takes the minimum truth value of the two conjuncts, disjunction takes the maximum truth value of the two disjuncts, and negation is an involution. An **interpretation** i is a function that assigns to each atom that appears in K one of three truth values: $i : \mathsf{Atoms}(K) \to \{F, B, T\}$, where $\mathsf{Atoms}(K)$ denotes the atoms occurring in knowledgebase K. An interpretation i whose range is $\{F, T\}$ is said to be **classical**. For an interpretation i, $\mathsf{Conflictbase}(i) = \{a \in \mathsf{Atoms}(K) \mid i(a) = B\}$ denotes the set of atoms that are assigned the non-classical truth value B, and $\mathsf{Truebase}(i, K) = \{\varphi \in K \mid i(\varphi) = T\}$ denotes the set of formulas evaluated as T in knowledgebase K. For a knowledgebase K we define the (3VL) models as the set of interpretations where no formula in K is assigned the truth value F: $\mathsf{Models}(K) = \{i \mid \text{ for all } \varphi \in K, i(\varphi) = T \text{ or } i(\varphi) = B\}$ Then, as a measure of inconsistency for K we define

$$\mathsf{Contension}(K) = \mathsf{Min}\{|\mathsf{Conflictbase}(i)| \mid i \in \mathsf{Models}(K)\}$$

So contension gives the minimum number of atoms that need to be assigned B in order to get a 3VL model of K.

Example 1. *For $K = \{a, \neg a, a \vee b, \neg b\}$, there are two models of K, i_1 and i_2, where $i_1(a) = B$, $i_1(b) = B$, $i_2(a) = B$, and $i_2(b) = F$. Therefore,* $\mathsf{Conflictbase}(i_1) = 2$ *and* $\mathsf{Conflictbase}(i_2) = 1$. *Hence,* $\mathsf{Contension}(K) = 1$.

When a set of classical interpretations H is such that, for each $\varphi \in K$, there is an $i \in H$ such that $i(\varphi) = T$, H is called a **hitting set** of K.[1]

Finally, we define the **probabilistic satisfiability** problem (PSAT), on which inconsistency measures can be based. A **PSAT instance** is a set $\Gamma = \{P(\varphi_i) \geq p_i \mid 1 \leq i \leq m\}$, where

1. Note that this concept of hitting sets differs from Reiter's definition (Reiter, 1987).

$\varphi_1, \ldots, \varphi_m \in \mathcal{L}$ are formulas and $p_1, \ldots, p_m \in [0,1]$ are real numbers. Intuitively, a PSAT instance assigns probability lower bounds[2] to formulas. To define the semantics, let $\mathbb{I}_C$ denote the set of classical interpretations. A **probability function** over a given set X is a function $\pi : X \to [0,1]$ such that $\sum_{x \in X} \pi(x) = 1$. For each probability function $\pi : \mathbb{I}_C \to \mathbb{R}^{\geq 0}$, let $P_\pi : \mathcal{L} \to \mathbb{R}$ be the function defined for all $\varphi \in \mathcal{L}$ as $P_\pi(\varphi) = \sum\{\pi(i) \mid i \in \mathbb{I}_C, \, i(\varphi) = T\}$. That is, the probability of a formula φ according to π is the sum of the probabilities assigned to the interpretations assigning T to φ. A PSAT instance $\Gamma = \{P(\varphi_i) \geq p_i \mid 1 \leq i \leq m\}$ is **satisfiable** if there is a probability function $\pi : \mathbb{I}_C \to \mathbb{R}^{\geq 0}$ such that $P_\pi(\varphi_i) \geq p_i$ for all $1 \leq i \leq m$.

Example 2. *For simplicity, consider a propositional language over $\mathcal{P} = \{a, b\}$. Assigning probability lower bounds to formulas, we have PSAT instances; e.g.,* $\Gamma = \{P(a \wedge b) \geq 0.5, P(\neg a \vee \neg b) \geq 0.5\}$. *To see that Γ is (probabilistically) satisfiable, consider the following (classical) interpretations: i_1 assigns T both atoms; and i_2 assigns F both atoms. Now consider a probability function π such that $\pi(i_1) = \pi(i_2) = 0.5$. Since $i_1(a \wedge b) = T$ and $i_2(a \wedge b) = F$, we have that $P_\pi(a \wedge b) = 0.5$. Analogously, $i_1(\neg a \vee \neg b) = F$ and $i_2(\neg a \vee \neg b) = T$ imply $P_\pi(\neg a \vee \neg b) = 0.5$. Note that $\Gamma' = \{P(a \wedge b) \geq 0.5, P(\neg a \vee \neg b) \geq 0.6\}$ would be unsatisfiable; intuitively, both probabilities should sum up to one, as one formula is the negation of the other.*

3. Inconsistency Measures and their Properties

An inconsistency measure assigns a nonnegative real value or infinity to every knowledgebase. In this paper we consider only absolute inconsistency measures, that is, those measures that measure the total amount of inconsistency. For such measures we make two requirements that we explain below.

Definition 1. *A function $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an* **inconsistency measure** *if the following two conditions hold for all $K, K' \in \mathcal{K}$:*

1. *$I(K) = 0$ iff K is consistent.*
2. *If $K \subseteq K'$, then $I(K) \leq I(K')$.*

The first requirement ensures that all and only consistent knowledgebases get measure 0. The second requirement enforces that the addition of a set formulas cannot decrease the inconsistency measure. The above requirements are taken from Hunter and Konieczny (2006) where (1) is called *consistency* and (2) is called *monotony.* A further property, usually required of inconsistency measures, but not satisfied by all of them, requires that the addition of a free formula not increase the inconsistency measure (Hunter & Konieczny, 2006):

Property 1 (Independence)**.** *For all $\varphi \in \mathcal{L}$ and $K \in \mathcal{K}$, if $\varphi \in \mathsf{Free}(K \cup \{\varphi\})$, then $I(K) \geq I(K \cup \{\varphi\})$.*

We wrote the independence property in a general way. However, for an inconsistency measure, on account of the second property, independence is equivalent to writing $I(K) = I(K \cup \{\varphi\})$.

There are many other properties that researchers have suggested for inconsistency measures. We next introduce five such properties, usually called rationality postulates.

MI-separability If $\mathsf{MI}(K \cup K') = \mathsf{MI}(K) \cup \mathsf{MI}(K')$ and $\mathsf{MI}(K) \cap \mathsf{MI}(K') = \varnothing$, then $I(K \cup K') = I(K) + I(K')$.

2. Equivalently, upper bounds or precise probabilities could be used; see for instance (De Bona, Cozman, & Finger, 2014).

Penalty If $\varphi \in \mathsf{Problematic}(K)$, then $I(K) > I(K \setminus \{\varphi\})$.

Super-additivity If $K \cap K' = \varnothing$, then $I(K \cup K') \geq I(K) + I(K')$.

Attenuation If K, K' are minimal inconsistent sets and $|K| < |K'|$, then $I(K) > I(K')$.

Equal Conflict If K, K' are minimal inconsistent sets and $|K| = |K'|$, then $I(K) = I(K')$.

Almost Consistency If $K_1, K_2, \ldots$ is a sequence of minimal inconsistent sets with $\lim\limits_{i\to\infty} |K_i| = \infty$, then $\lim\limits_{i\to\infty} I(K_i) = 0$.

MI-separability comes from Hunter and Konieczny (2010). Penalty and Super-additivity come from Thimm (2009). Attenuation, Equal Conflict, and Almost Consistency come from Mu, Liu, and Jin (2011a).

Now we introduce ten inconsistency measures from the literature: the rationale for each is given below.

Definition 2. *For a knowledgebase K, the inconsistency measures I_B, I_M, I_A, I_P, I_C, $I_\#$, I_H, I_{nc}, I_{hs} and I_η are such that*

- $I_B(K) = 1$ *if* $K \vdash \bot$ *and* $I_B(K) = 0$ *if* $K \not\vdash \bot$
- $I_M(K) = |\mathsf{MI}(K)|$
- $I_A(K) = (|\mathsf{MC}(K)| + |\mathsf{Selfcontradictions}(K)|) - 1$
- $I_P(K) = |\mathsf{Problematic}(K)|$
- $I_C(K) = \mathsf{Contension}(K)$
- $I_\#(K) = \begin{cases} 0 & \textit{if } K \textit{ is consistent} \\ \sum_{X \in \mathsf{MI}(K)} \frac{1}{|X|} & \textit{otherwise} \end{cases}$
- $I_H(K) = \mathsf{min}\{|X| \mid X \subseteq K \textit{ and } \forall M \in \mathsf{MI}(K)(X \cap M \neq \varnothing)\}$
- $I_{nc}(K) = |K| - \max\{n \mid \forall K' \subseteq K : |K'| = n \textit{ implies } K' \not\vdash \bot\}$
- $I_{hs}(K) = \min\{|H| \mid H \textit{ is a hitting set of } K\} - 1$, *where* $\min \varnothing = \infty$
- $I_\eta(K) = 1 - \max\big\{\eta \in [0,1] | \{P(\varphi) \geq \eta \mid \varphi \in K\} \textit{ is satisfiable}\big\}$

We explain the measures as follows: I_B (Hunter & Konieczny, 2008) assigns the same value, 1, to all inconsistent knowledgebases. $I_M(K)$ (Hunter & Konieczny, 2008) counts the number of minimal inconsistent subsets of K. $I_A(K)$ (Grant & Hunter, 2011) counts the sum of the number of maximal consistent subsets together with the number of contradictory formulas but 1 must be subtracted to make $I(K) = 0$ when K is consistent. $I_P(K)$ (Grant & Hunter, 2011) counts the number of formulas in minimal inconsistent subsets of K. $I_C(K)$ (Konieczny et al., 2003; Grant & Hunter, 2011) counts the minimum number of atoms that need to be assigned B amongst the 3VL models of K. $I_\#(K)$ (Hunter & Konieczny, 2008) computes the weighted sum of the minimal inconsistent subsets of K, where the weight is the inverse of the size of the minimal inconsistent subset (and hence smaller minimal inconsistent subsets are regarded as more inconsistent than larger ones). $I_H(K)$ (Grant & Hunter, 2013), originally called the d-hit inconsistency measure, is the size of the smallest set that has a nonempty intersection with every minimal inconsistent subset. $I_{nc}(K)$ (Doder, Rašković, Marković, & Ognjanović, 2010; Thimm, 2016b) finds the maximum size for a subset of K to be surely consistent and subtracts it from the size of the K. $I_{hs}(K)$ (the hitting set measure) (Thimm, 2016b) computes the minimum number of classical interpretations $i \in \mathbb{I}_C$ needed for all formulas in K be evaluated to T by some i; then one is subtracted. This minimum will be infinite if there is a selfcontradiction in K. $I_\eta(K)$ (Knight, 2002) is one minus the maximum probability lower bound one can consistently assign to all formulas in K. Each of these measures satisfies the definition of being an inconsistency measure (i.e. Definition 1) and all but I_{nc} and I_C satisfy the property of independence. To see the failure of independence, consider $K = \{a \wedge \neg a \wedge \neg b, b\}$. Here b is free in K, but $I_{nc}(K) = I_C(K) = 2$ and $I_{nc}(K \setminus \{b\}) = I_C(K \setminus \{b\}) = 1$. For other properties that hold for these measures, see (Thimm, 2016a).

The use of minimal inconsistent subsets, such as for I_M, I_P, and $I_\#$, and the use of maximal consistent subsets such as I_A, have been proposed previously for measures of inconsistency (Hunter & Konieczny, 2004, 2008). The idea of a measure that is sensitive to the number of formulas to produce an inconsistency emanates from Knight (2002) in which the more formulas needed to produce the inconsistency, the less inconsistent the set. As explored in Hunter and Konieczny (2008), this sensitivity is obtained with $I_\#$. Another approach involves looking inside the formulas for the interaction of the atoms, such as I_C, which is a semantic approach based on three-valued logic (Konieczny et al., 2003; Grant & Hunter, 2011), and similar to the ones based on four-valued logic (e.g. Hunter, 2002).

4. Inconsistency and Consistency Graphs

We now introduce several graphical representations of a knowledgebase. We will be using for most of this article a special class of graphs, called bipartite graphs (bigraphs) where the vertices are divided into two groups and edges may connect only vertices that are in different groups. Although we will use labels primarily in Section 7, it is convenient for the presentation to start with labeled bigraphs. The following defines both (unlabeled) bigraphs and labeled bigraphs.

Definition 3. *Bigraphs and labeled bigraphs.*

1. *A* **bigraph** *is a tuple $G = \langle U, V, E \rangle$ where U and V are sets, $U \cap V = \varnothing$ and E is a set of pairs of the form $\{u, v\}$ where $u \in U$ and $v \in V$.*

2. *A* **labeled bigraph** *is a tuple $LG = \langle U, V, E, L \rangle$ where $\langle U, V, E \rangle$ is a bigraph and L is a function $L : U \cup V \to S$, where S is a set of labels.*

We will use u with subscripts for elements of U and v with subscripts for elements of V. For ease of presentation when we draw bigraphs we will assume that $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. We will draw bigraphs using small circles without labels representing the vertices in two rows: the ones on the upper row represent the vertices in U, that we assume to be $u_1, \ldots, u_m$ from left to right, while the ones on the lower row represent the vertices in V that we assume to be $v_1, \ldots, v_n$ from left to right and the edges are lines between the circles. For labeled graphs we use boxes for the vertices with the labels inside. We note that the ordering of the vertices is arbitrary: U and V are sets, not tuples; while a reordering would make the picture look different it would represent the same (labeled) graph.

The following is a brief review of commonly used definitions about (bi)graphs. We write G for a generic graph and assume that we have such a G. An edge $e = \{u, v\}$ is said to **connect** (and be **incident to**) u and v, which are then called **adjacent** vertices. We write $\mathsf{Adj}(u)$ (resp. $\mathsf{Adj}(v)$) for the set of vertices adjacent to u (resp. v). A vertex is said to be **isolated** if there is no edge incident to it. Then $\mathsf{deg}(u) = |\mathsf{Adj}(u)|$ (resp. $\mathsf{deg}(v) = |\mathsf{Adj}(v)|$). The **null bigraph** is $\langle\varnothing,\varnothing,\varnothing\rangle$ and its labeled version is $\langle\varnothing,\varnothing,\varnothing,\varnothing\rangle$.

The concept of subgraph is defined in the usual way, that is, a bigraph $G' = \langle U', V', E'\rangle$ is a **subgraph** of $G = \langle U, V, E\rangle$ if $U' \subseteq U$, $V' \subseteq V$ and $E' \subseteq E$. As usual, a **proper subgraph** is a subgraph not identical to the (bi)graph. For $G = \langle U, V, E\rangle$ let $U' \subseteq U$, $V' \subseteq V$, and $W = U' \cup V'$. The subgraph G' of G **induced by** W is $G' = \langle U', V', E'\rangle$ where $E' = \{\{u, v\} \in E \mid u \in U'$ and $v \in V'\}$.

At this point we move from general graphs to our interest where the graphs are used to represent information about the inconsistency in knowledgebases. We are interested in representing the structure of minimal inconsistent subsets; that is, how the formulas combine to form them. We start by using labeled bigraphs where the set of labels S is $\mathcal{L} \cup \mathcal{K}$. Every $u \in U$ is labeled as a formula, and every $v \in V$, as a set of formulas. Our interest is in using labeled bigraphs to represent the minimal inconsistent subsets of a knowledgebase.

The following definition defines both labelled inconsistency graphs and labelled augmented graphs. These are identical notions except that the former considers only the formulas in $\mathsf{Problematic}(K)$ whereas the latter considers all formulas in K.

Definition 4. *The* **labeled inconsistency graph** *(resp.* **labeled augmented inconsistency graph***) for knowledgebase K where $\mathsf{Problematic}(K) = \{\varphi_1, \ldots, \varphi_m\}$ (resp. $K = \{\varphi_1, \ldots, \varphi_m\}$) and $\mathsf{MI}(K) = \{\Delta_1, \ldots, \Delta_n\}$ is the labeled bigraph $\mathsf{LIG}(K)$(resp. $\mathsf{LIG}^+(K)) = \langle U, V, E, L\rangle$ such that*

- $U = \{u_1, \ldots, u_m\}$
- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{u_i, v_j\} \mid \varphi_i \in \Delta_j\}$ *(every edge corresponds to the set membership relation between a problematic formula $\varphi_i \in K$ and a minimal inconsistent subset $\Delta_j \in \mathsf{MI}(K)$)*
- $L : U \cup V \to \mathcal{L} \cup \mathcal{K}$*, $L(u_i) = \varphi_i$ for all $i, 1 \le i \le m$ and $L(v_j) = \Delta_j$ for all $j, 1 \le j \le n$*

The labeled inconsistency graph for K, $\mathsf{LIG}(K)$, is a subgraph of the labeled augmented inconsistency graph for K, $\mathsf{LIG}^+(K)$, and is a proper subgraph if and only if $\mathsf{Free}(K) \neq \varnothing$. The free formulas have no effect on the minimal inconsistent subsets but without them we cannot reconstruct the original knowledgebase. That is the reason for the augmentation. The labeled inconsistency graph of a consistent knowledgebase is the null labeled bigraph. Isomorphic graphs are considered identical, so that renaming vertices does not change the graph. We consider two labeled graphs as isomorphic if there is a bijection between the sets of vertices that preserves both the edges and the labels. Examples 3 and 4 illustrate labeled augmented inconsistency graphs for two knowledgebases.

Example 3. *Let $K = \{a, \neg a \vee \neg b, b, \neg a \vee c, \neg c \vee d, \neg d\}$. Then $\mathsf{MI}(K) = \{\{a, \neg a \vee \neg b, b\}, \{a, \neg a \vee c, \neg c \vee d, \neg d\}\}$. As $\mathsf{Free}(K) = \varnothing$, $\mathsf{LIG}^+(K) = \mathsf{LIG}(K)$.*

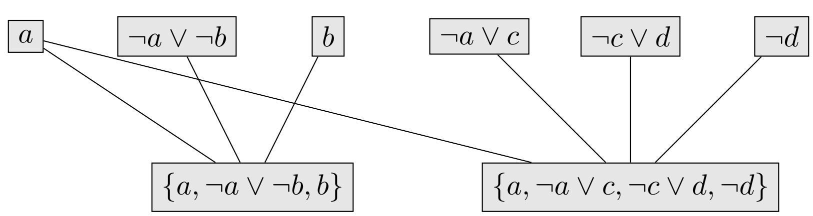


Figure 1: LIG for $K = \{a, \neg a \vee \neg b, b, \neg a \vee c, \neg c \vee d, \neg d\}$

Example 4. *Let* $K = \{\neg a, a \vee b, \neg b, b \vee c, \neg c, c \vee d, \neg d, a \vee d, e\}$*. Here* $\mathsf{MI}(K) = \{\neg b, b \vee c, \neg c\}, \{\neg a, a \vee b, \neg b\}, \{\neg c, c \vee d, \neg d\}, \{\neg a, a \vee d, \neg d\}$*. In this case* $\mathsf{Free}(K) = \{e\} \neq \varnothing$ *hence the augmented graph has an additional vertex in* U *with label* e*. Below we draw the labeled augmented inconsistency graph.*

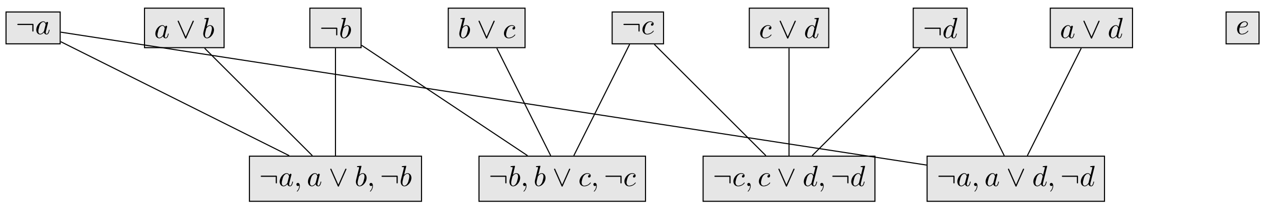


Figure 2: LIG^+ for $K = \{\neg a, a \vee b, \neg b, b \vee c, \neg c, c \vee d, \neg d, a \vee d, e\}$

In the first part of this paper our interest will be the structure of the inconsistency graphs of knowledgebases without labels.

Definition 5. *The inconsistency graph (resp. augmented inconsistency graph) for knowledgebase* K *is the bigraph* $\mathsf{IG}(K) = \langle U, V, E\rangle$ *(resp.* $\mathsf{IG}^+(K) = \langle U, V, E\rangle$*) obtained from* $\mathsf{LIG}(K)$ *(resp.* $\mathsf{LIG}^+(K)$*) by omitting the labeling function* L*.*

Now we show how to obtain the (augmented) inconsistency graphs for the examples given earlier.

Example 5. *Here is* $\mathsf{IG}^+(K) = \mathsf{IG}(K)$ *for the knowledgebase of Example 3.*

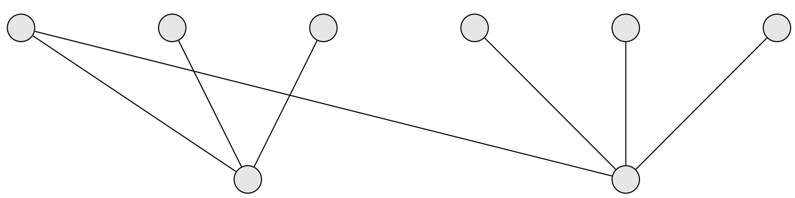


Figure 3: $\mathsf{IG}^+(K) = \mathsf{IG}(K)$ for the knowledgebase of Example 3

Example 6. *Here is* $\mathsf{IG}^+(K)$ *for the knowledgebase of Example 4.*

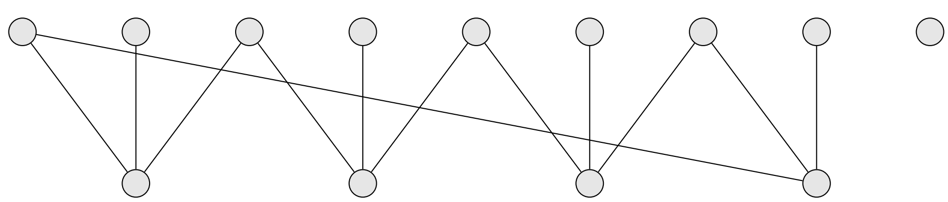
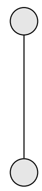


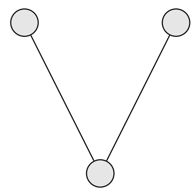
Figure 4: $\mathsf{IG}^+(K)$ for the knowledgebase of Example 4

(Unlabeled) inconsistency graphs are sufficient if one wants to focus on the structure of conflicts of the base. They convey all the information on minimal inconsistent sets and their relationships.

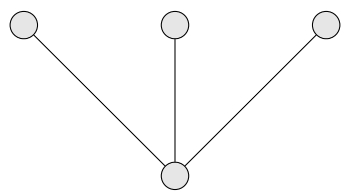
We illustrate further the construction of inconsistency graphs in Figures 5 and 6. None of the illustrated knowledgebases have any free formulas, hence the graphs are both the inconsistency and the augmented inconsistency graphs.



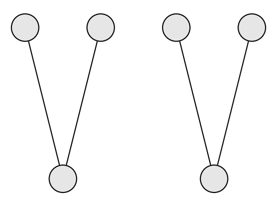
(a) Example: $K = \{a \wedge \neg a\}$ and $\mathsf{MI}(K) = \{\{a \wedge \neg a\}\}$.



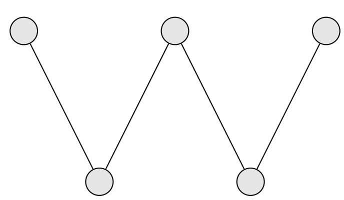
(b) Example: $K = \{a, \neg a\}$ and $\mathsf{MI}(K) = \{\{a, \neg a\}\}$.



(c) Example: $K = \{a, a \rightarrow b, \neg b\}$ and $\mathsf{MI}(K) = \{\{a, a \rightarrow b, \neg b\}\}$.

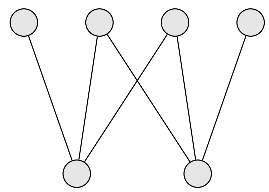


(d) Example: $K = \{a, \neg a, b, \neg b\}$ and $\mathsf{MI}(K) = \{\{a, \neg a\}, \{b, \neg b\}\}$.

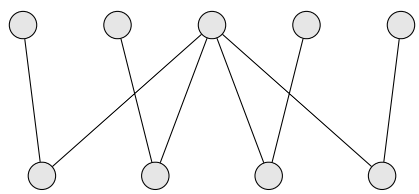


(e) Example: $K = \{a, \neg a \wedge \neg b, b\}$ and $\mathsf{MI}(K) = \{\{a, \neg a \wedge \neg b\}, \{\neg a \wedge \neg b, b\}\}$.

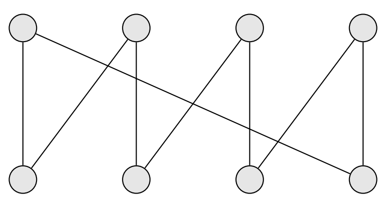
Figure 5: Examples of inconsistency graphs



(a) Example: $K = \{a, b, (\neg a \vee \neg b) \wedge (\neg b \vee \neg c), c\}$ and $\mathsf{MI}(K) = \{\{a, b, (\neg a \vee \neg b) \wedge (\neg b \vee \neg c)\}, \{b, (\neg a \vee \neg b) \wedge (\neg b \vee \neg c), c\}\}$.



(b) Example: $K = \{a, b, \neg a \wedge \neg b \wedge \neg c \wedge \neg d, c, d\}$ and $\mathsf{MI}(K) = \{\{a, \neg a \wedge \neg b \wedge \neg c \wedge \neg d\}, \{b, \neg a \wedge \neg b \wedge \neg c \wedge \neg d\}, \{\neg a \wedge \neg b \wedge \neg c \wedge \neg d, c\}, \{\neg a \wedge \neg b \wedge \neg c \wedge \neg d, d\}\}$.



(c) Example: $K = \{a, \neg a \wedge b, \neg b \wedge c, \neg a \wedge \neg c\}$ and $\mathsf{MI}(K) = \{\{a, \neg a \wedge b\}, \{a, \neg a \wedge \neg c\}, \{\neg a \wedge b, \neg b \wedge c\}, \{\neg b \wedge c, \neg a \wedge \neg c\}\}$.

Figure 6: Additional examples of inconsistency graphs

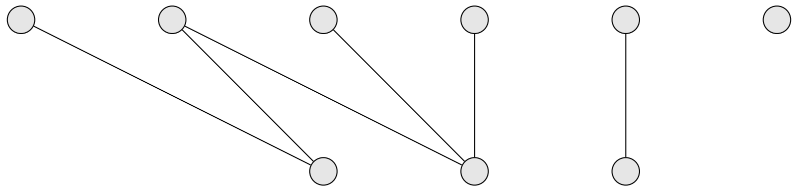


Figure 7: An inconsistency graph used in Example 7.

Our first result about inconsistency graphs is a representation theorem. We show under what conditions a bigraph may represent an augmented inconsistency graph.

Theorem 1. *Let $G = \langle U, V, E \rangle$ be a bigraph. Then $G = \mathsf{IG}^+(K)$ for some knowledgebase K iff the following two conditions hold for G:*

1. *No vertex in V is isolated.*
2. *For all $v, v' \in V$, if $v \neq v'$ then $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.*

Corollary 1. *Let $G = \langle U, V, E \rangle$ be a bigraph. Then $G = \mathsf{IG}(K)$ for some knowledgebase K iff the following two conditions hold for G:*

1. *G contains no isolated vertex.*
2. *For all $v, v' \in V$, if $v \neq v'$ then $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.*

Example 7. *To illustrate the construction employed in the proof of Theorem 1, consider the graph $G = \langle U, V, E \rangle$ in Figure 7. In this graph the vertices for U will be u_1, u_2, u_3, u_4, u_5, and u_6 and for V will be v_1, v_2, and v_3. Hence we will be using the atoms a_1^1, a_2^1, a_2^2, a_3^2, a_4^2, a_5^3, and a_6^0. Then $\mathsf{Adj}(v_1) = \{u_1, u_2\}$, $\mathsf{Adj}(v_2) = \{u_2, u_3, u_4\}$, and $\mathsf{Adj}(v_3) = \{u_5\}$.*

- *For v_1 we obtain $\psi_1^1 = a_1^1 \wedge \neg a_2^1$ and $\psi_2^1 = a_1^1 \rightarrow a_2^1$.*
- *For v_2 we obtain $\psi_2^2 = a_2^2 \wedge \neg a_4^2$, $\psi_3^2 = a_2^2 \rightarrow a_3^2$, and $\psi_4^2 = a_3^2 \rightarrow a_4^2$.*
- *For the last vertex v_3 we obtain $\psi_5^3 = a_5^3 \wedge \neg a_5^3$.*
- *Finally, for the formulas in K we get $\varphi_1 = a_1^1 \wedge \neg a_2^1$, $\varphi_2 = a_1^1 \rightarrow a_2^1 \wedge a_2^2 \wedge \neg a_4^2$, $\varphi_3 = a_2^2 \rightarrow a_3^2$, $\varphi_4 = a_3^2 \rightarrow a_4^2$, $\varphi_5 = a_5^3 \wedge \neg a_5^3$, $\varphi_6 = a_6^0$.*

The graphs that we have constructed so far used minimal inconsistent subsets. In the same way we can use maximal consistent subsets for the (labels of the) vertices in V to obtain 4 types of (labeled) (augmented) consistency graphs. As we will show later there is a close relationship between the various types of inconsistency graphs and their corresponding consistency graphs for a knowledgebase. We use the notations LCG^+, LCG, CG^+, and CG for the four graphs obtained via maximal consistent subsets.

Definition 6. *The* **labeled consistency graph** *(resp.* **labeled augmented consistency graph***) for knowledgebase K where $\mathsf{Problematic}(K) = \{\varphi_1, \ldots, \varphi_m\}$ (resp. $K = \{\varphi_1, \ldots, \varphi_m\}$) and $\mathsf{MC}(K) = \{\Delta_1, \ldots, \Delta_n\}$ is the labeled bigraph $\mathsf{LCG}(K)$(resp. $\mathsf{LCG}^+(K)$) $= \langle U, V, E, L \rangle$ such that*

- $U = \{u_1, \ldots, u_m\}$

- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{u_i, v_j\} \mid \varphi_i \in \Delta_j\}$ *(every edge corresponds to the set membership relation between a problematic formula $\varphi_i \in K$ and a maximal consistent subset $\Delta_j \in \mathsf{MC}(K)$).*
- $L : U \cup V \to \mathcal{L} \cup \mathcal{K}$, $L(u_i) = \varphi_i$ *for all* $i, 1 \leq i \leq m$ *and* $L(v_j) = \Delta_j$ *for all* $j, 1 \leq j \leq n$.

Example 8. *Let* $K = \{a, \neg a \vee \neg b, b, \neg a \wedge c\}$*. Then* $\mathsf{MI}(K) = \{\{a, \neg a \vee \neg b, b\}, \{a, \neg a \wedge c\}\}$, $\mathsf{MC}(K) = \{\{a, \neg a \vee \neg b\}, \{a, b\}, \{\neg a \vee \neg b, b, \neg a \wedge c\}\}$*, and* $\mathsf{LCG}^+(K)$ *is given in Figure 8. As* $\mathsf{Free}(K) = \varnothing$*, we have* $\mathsf{LCG}^+(K) = \mathsf{LCG}(K)$.

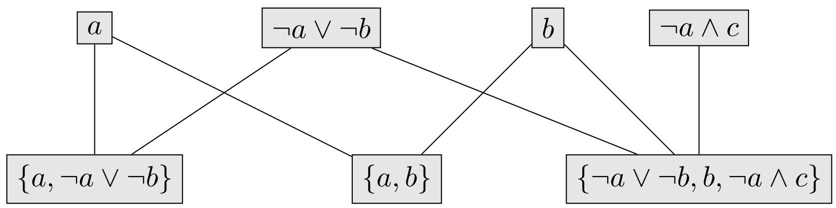


Figure 8: LCG for $K = \{a, \neg a \vee \neg b, b, \neg a \wedge c\}$

Example 9. *Let* $K = \{a, \neg a \vee \neg b, b, \neg a \wedge a\}$*. Then* $\mathsf{MI}(K) = \{\{a, \neg a \vee \neg b, b\}, \{\neg a \wedge a\}\}$, $\mathsf{MC}(K) = \{\{a, \neg a \vee \neg b\}, \{a, b\}, \{\neg a \vee \neg b, b\}\}$*, and* $\mathsf{LCG}^+(K)$ *is given in Figure 9. As* $\mathsf{Free}(K) = \varnothing$*, we have* $\mathsf{LCG}^+(K) = \mathsf{LCG}(K)$.

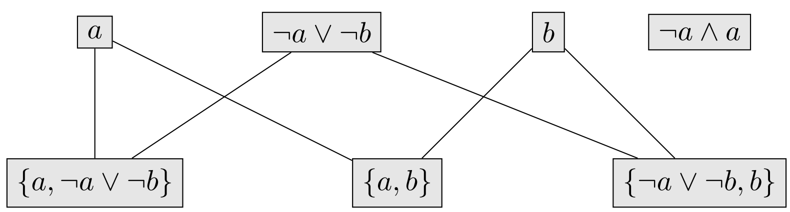


Figure 9: LCG^+ for $K = \{a, \neg a \vee \neg b, b, \neg a \wedge a\}$

Example 10. *Let* $K = \{a, \neg a, b, \neg b, c\}$*. Then* $\mathsf{MI}(K) = \{\{a, \neg a\}, \{b, \neg b\}\}$*, and* $\mathsf{MC}(K) = \{ \{a, b, c\}, \{a, \neg b, c\}, \{\neg a, b, c\}, \{\neg a, \neg b, c\} \}$*. In this case* $\mathsf{Free}(K) = \{c\}$*, so* $\mathsf{LCG}^+(K) \neq \mathsf{LCG}(K)$. $\mathsf{LCG}^+(K)$ *is given Figure 10.*

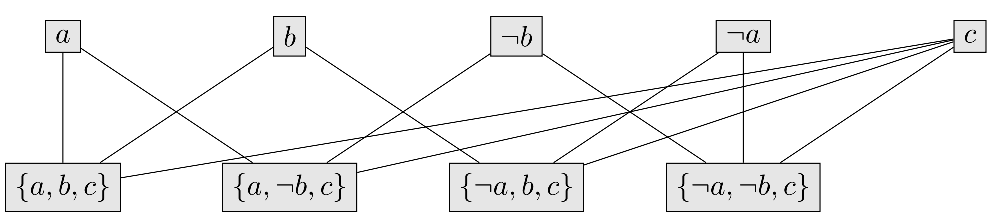


Figure 10: LCG^+ for $K = \{a, \neg a, b, \neg b, c\}$

Definition 7. *The consistency graph (resp. augmented consistency graph) for knowledgebase* K *is the bigraph* $\mathsf{CG}(K) = \langle U, V, E \rangle$ *(resp.* $\mathsf{CG}^+(K) = \langle U, V, E \rangle$*) obtained from* $\mathsf{LCG}(K)$ *(resp.* $\mathsf{LCG}^+(K)$*) by omitting the labeling function* L.

Example 11. *We return to the knowledgebase in Example 10. For this,* $\mathsf{CG}(K)$ *is in Figure 11 left, and* $\mathsf{CG}^+(K)$ *is in Figure 11 right. Note that the difference is that* $\mathsf{CG}^+(K)$ *contains a node for the free formula* c *with edges to all the nodes in* V *because a free formula is a member of all the maximal consistent subsets.*

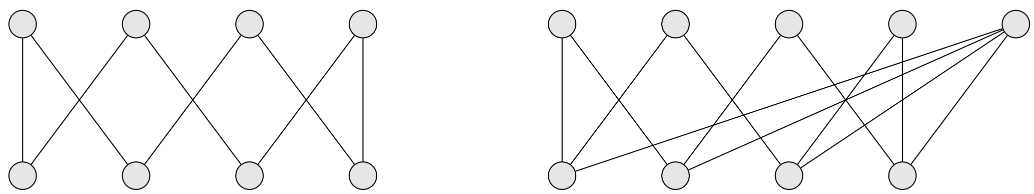


Figure 11: CG^+ and CG for the knowledgebase in Example 10

By analogy with the inconsistency graphs we now obtain representation theorems for the two types of consistency graphs without labels.

Theorem 2. *Let* $G = \langle U, V, E\rangle$ *be a bigraph. Then* $G = \mathsf{CG}^+(K)$ *for some knowledgebase* K *iff the following two conditions hold for* G*:*

1. $V \neq \varnothing$.
2. *For all* $v, v' \in V$, *if* $v \neq v'$ *then* $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.

Corollary 2. *Let* $G = \langle U, V, E\rangle$ *be a bigraph. Then* $G = \mathsf{CG}(K)$ *for some knowledgebase* K *iff the following three conditions hold for* G*:*

1. $V \neq \varnothing$.
2. *For every* $u \in U$, $\mathsf{Adj}(u) \neq V$.
3. *for all* $v, v' \in V$, *if* $v \neq v'$ *then* $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.

Again we show how the construction of the augmented consistency graph is performed by doing an example.

Example 12. *To illustrate the construction employed in the proof of Theorem 2, we use the same graph we used in Example 7. In this graph the vertices for* U *will be* u_1, u_2, u_3, u_4, u_5, *and* u_6 *and for* V *will be* v_1, v_2, *and* v_3. *Hence we will be using the atoms* a_1^1, a_2^1, a_2^2, a_3^2, a_4^2, a_5^3, *and* a_6^0. *Then* $\mathsf{Adj}(v_1) = \{u_1, u_2\}$, $\mathsf{Adj}(v_2) = \{u_2, u_3, u_4\}$, *and* $\mathsf{Adj}(v_3) = \{u_5\}$. *We obtain the following formulas for the vertices in* U*:*

$$\begin{aligned}
\varphi_1 &= a_1^1 \wedge \neg a_3^2 \wedge \neg a_4^2 \wedge \neg a_5^3 \\
\varphi_2 &= a_2^1 \wedge \neg a_5^3 \\
\varphi_3 &= a_3^2 \wedge \neg a_1^1 \wedge \neg a_5^3 \\
\varphi_4 &= a_4^2 \wedge \neg a_1^1 \wedge \neg a_5^3 \\
\varphi_5 &= a_5^3 \wedge \neg a_1^1 \wedge \neg a_2^1 \wedge \neg a_3^2 \wedge \neg a_4^2 \\
\varphi_6 &= a_6^0 \wedge \neg a_6^0
\end{aligned}$$

Although ignoring free formulas while representing the maximal consistent subsets of a knowledgebase may not appear reasonable at first, it can be argued that they only change the size of all such subsets. The next result shows the connection between using and ignoring free formulas. First we observe that by the definitions the following hold:

1. For every set $M \in \mathsf{MC}(K)$, $\mathsf{Free}(K) \subseteq M$ (that is, every maximal consistent subset contains all the free formulas).

2. For every $M \subseteq K$, $M = (\mathsf{Free}(K) \cap M) \cup (\mathsf{Problematic}(K) \cap M)$ is a disjoint union.

Proposition 1. *For every knowledgebase K and $M \subseteq K$, $M \in \mathsf{MC}(K)$ iff $\mathsf{Free}(K) \subseteq M$ and $M \cap \mathsf{Problematic}(K) \in \mathsf{MC}(\mathsf{Problematic}(K))$.*

5. Relationships between Inconsistency Measures and Graphs

In this section we investigate the relationship of inconsistency and consistency graphs to inconsistency measures. First, we show how several inconsistency measures from the literature can be represented as functions from the inconsistency or the consistency graph, and how the latter is a function of the former as well. Then, we study when functions on inconsistency graphs yield inconsistency measures. Finally, we briefly discuss how inconsistency graphs (or even the whole knowledgebase) could be recovered from some contrived inconsistency measures.

5.1 Computing Inconsistency Measures from the Inconsistency Graph

Employing only the inconsistency graph for a knowledgebase allows us to focus on the structure of the conflicts (interrelationships between problematic formulas).

In order to characterize the inconsistency measures that are functions of the inconsistency graph, as those in Proposition 2, we denote by $\mathcal{G}$ ($\mathcal{G}^+$) the set of all (augmented) inconsistency graphs.

Definition 8. *An inconsistency measure $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an* **augmented IG (aIG) measure** *if there is a function $f : \mathcal{G}^+ \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = f(\mathsf{IG}^+(K))$ for all $K \in \mathcal{K}$. I is an* **IG measure** *if there is a function $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = f(\mathsf{IG}(K))$ for all $K \in \mathcal{K}$.*

As we show next, numerous known inconsistency measures in the literature use this information only.

Proposition 2. *I_B, I_M, I_P, $I_\#$ and I_H are IG measures.*

We can show this proposition by using the following ways of calculating the measures where $\mathsf{IG}(K) = \langle U, V, E\rangle$ is the inconsistency graph for a knowledgebase K. Then,

1. $I_B(K) = \begin{cases} 0 & \text{if } V = \varnothing \\ 1 & \text{otherwise} \end{cases}$

2. $I_M(K) = |V|$

3. $I_P(K) = |U|$

4. $I_\#(K) = \begin{cases} 0 & \text{if } V = \varnothing \\ \sum_{v \in V} \frac{1}{|\mathsf{deg}(v)|} & \text{otherwise} \end{cases}$

5. $I_H(K) = \mathsf{min}\{|X| \mid X \subseteq U \text{ and every } v \in V \text{ is adjacent to some } u \in X\}$.

Since inconsistency graphs are recovered from augmented inconsistency graphs by simply discarding the isolated vertices (which correspond to the free formulas), every IG inconsistency measure is also an aIG measure. Even though we will show that the converse does not hold, in general most aIG inconsistency measures in the literature are indeed IG; thus we focus on the latter. This is due to the fact that IG measures are exactly the aIG measures satisfying the independence property (i.e. Property 1), which holds for most measures in the literature (Thimm, 2018). Intuitively, the independence property guarantees that free formulas do not affect the inconsistency measure.

Proposition 3. *An inconsistency measure is an IG measure iff it is an aIG measure and satisfies the independence property.*

Corollary 3. *I_{nc} is not an IG measure.*

Another way to look at the result above is through the characterization of $I_{nc}(K)$ for any inconsistent K as $I_{nc}(K) = |K| - \min\{|M| \mid M \in \mathsf{MI}(K)\} + 1$. Besides the sizes of the minimal inconsistent subsets in K, in order to compute $I_{nc}(K)$, one also needs to know the size of K. By ignoring free formulas, inconsistency graphs do not encode $|K|$, but augmented inconsistency graphs do.

Proposition 4. *I_{nc} is an aIG measure.*

An aIG inconsistency measure depends only on how the formulas in a knowledgebase can be combined to form minimal inconsistent subsets and on the quantity of free formulas. Consequently, inconsistency measures that are sensitive to the formulas themselves are not aIG.

In particular, measures that are not based on the syntax of the knowledgebase and on minimal inconsistent subsets can not be IG or aIG. This is for instance the case for variable-based measures such as I_C.

Proposition 5. *I_C is not an aIG measure.*

5.2 Computing Inconsistency Measures from the Consistency Graph

We wish to show that we can recover the (augmented) consistency graph from the (augmented) inconsistency graph and vice versa. First we prove two lemmas that characterize the maximal consistent subsets in an augmented inconsistency graph and the minimal inconsistent subsets in an augmented consistency graph.

Lemma 1. *Let $\mathsf{IG}^+(K) = \langle U, V, E \rangle$, $S \subseteq K$, and write U_S for the subset of U corresponding to the elements of S. Then S is a maximal consistent subset of K iff U_S is a maximal subset of U such that there is no $v \in V$ with $\mathsf{Adj}(v) \subseteq U_S$.*

As a direct consequence, the consistency graph can be recovered from the inconsistency graph, as $\mathsf{CG}(K) = \mathsf{CG}^+(\mathsf{Problematic}(K))$ and $\mathsf{IG}^+(\mathsf{Problematic}(K)) = \mathsf{IG}(K)$.

Lemma 2. *Let $\mathsf{CG}^+(K) = \langle U, V, E \rangle$, $S \subseteq K$, and write U_S for the subset of U corresponding to the elements of S. Then S is a minimal inconsistent subset of K iff U_S is a minimal subset of U such that there is no $v \in V$ with $U_S \subseteq \mathsf{Adj}(v)$.*

Again, if one is interested in recovering only the inconsistency graph, the consistency graph suffices, since $\mathsf{IG}(K) = \mathsf{IG}^+(\mathsf{Problematic}(K))$ and $\mathsf{CG}^+(\mathsf{Problematic}(K)) = \mathsf{CG}(K)$.

Using these results, we can prove that the (augmented) inconsistency graph and the (augmented) consistency graph are actually two sides of the same coin. This result is based on the well-known relationship between minimally inconsistency subsets and maximally consistent subsets (see for example Reiter, 1987). We denote by $\mathcal{G}_c$ ($\mathcal{G}_c^+$) the set of all (augmented) consistency graphs.

Theorem 3. *There is a bijection $h : \mathcal{G}^+ \to \mathcal{G}_c^+$ such that, for any $K \in \mathcal{K}$, $G = \mathsf{IG}^+(K)$ iff $h(G) = \mathsf{CG}^+(K)$.*

Corollary 4. *There is a bijection $h : \mathcal{G} \to \mathcal{G}_c$ such that, for any $K \in \mathcal{K}$, $G = \mathsf{IG}(K)$ iff $h(G) = \mathsf{CG}(K)$.*

Next we illustrate Theorem 3 on an example.

Example 13. *We use the graph of Example 7 for G. In order to construct* $h(G)$ *we determine the following:* $\mathsf{Adj}(v_1) = \{u_1, u_2\}$, $\mathsf{Adj}(v_2) = \{u_2, u_3, u_4\}$, $\mathsf{Adj}(v_3) = \{u_5\}$, $X_1 = \{u_1, u_3, u_4, u_6\}$, $X_2 = \{u_2, u_3, u_6\}$, $X_3 = \{u_2, u_4, u_6\}$. *We draw the graph* $h(G)$ *in Figure 12.*

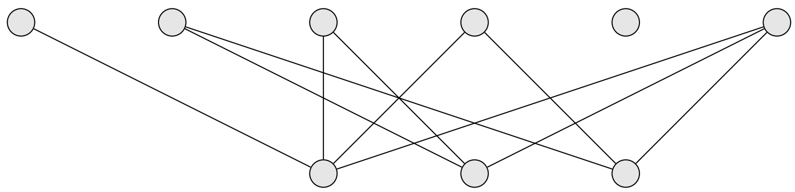


Figure 12: $h(G)$ for the graph of Example 7

In order to make the formulas easier to read we rewrite Example 7 using different letters instead of subscripts and superscripts: $K = \{\varphi_1, \ldots, \varphi_6\}$ *where*

$$\varphi_1 = a \wedge \neg b, \varphi_2 = a \rightarrow b \wedge c \wedge \neg e, \varphi_3 = c \rightarrow d, \varphi_4 = d \rightarrow e, \varphi_5 = f \wedge \neg f, \varphi_6 = g.$$

Then $\mathsf{MI}(K) = \{\{\varphi_1, \varphi_2\}, \{\varphi_2, \varphi_3, \varphi_4\}, \{\varphi_5\}\}$. *So, for the structure of the graph G we have* $U = \{u_1, \ldots, u_6\}$ *(each* φ_i *associated with* u_i *and* $V = \{v_1, v_2, v_3\}$ *associated with the elements of* $\mathsf{MI}(K)$. *For* $h(G)$ *we get* $\mathsf{Adj}(v_1) = \{u_1, u_3, u_4, u_6\}$, $\mathsf{Adj}(v_2) = \{u_2, u_3, u_6\}$, $\mathsf{Adj}(v_3) = \{u_2, u_4, u_6\}$, *that is,* $\mathsf{MC}(K) = \{\{\varphi_1, \varphi_3, \varphi_4, \varphi_6\}, \{\varphi_2, \varphi_3, \varphi_6\}, \{\varphi_2, \varphi_4, \varphi_6\}\}$. *It is easy to check that* $h'(h(G)) = G$.

Going in the opposite direction again we use the graph of Example 7 for G. In order to construct $h'(G)$ *we determine the following:* $\mathsf{Adj}(v_1) = \{u_1, u_2\}$, $\mathsf{Adj}(v_2) = \{u_2, u_3, u_4\}$, $\mathsf{Adj}(v_3) = \{u_5\}$, $X_1 = \{u_1, u_3\}$, $X_2 = \{u_1, u_4\}$, $X_3 = \{u_1, u_5\}$, $X_4 = \{u_2, u_5\}$, $X_5 = \{u_3, u_5\}$, $X_6 = \{u_4, u_5\}$, $X_7 = \{u_6\}$. *We draw the graph* $h'(G)$ *in Figure 13.*

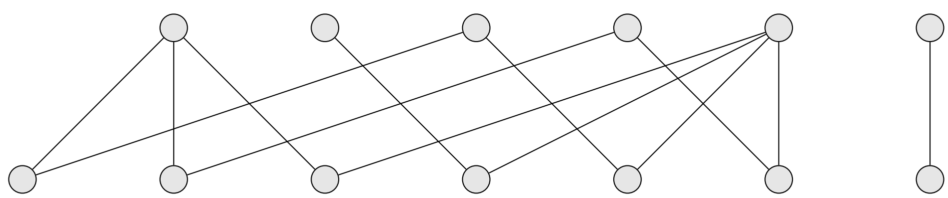


Figure 13: $h'(G)$ for the graph of Example 7

In particular, suppose that K is the knowledgebase obtained in Example 12 but again using different letters instead of subscripts and superscripts: $K = \{\varphi_1, \ldots, \varphi_6\}$ *where*

$$\varphi_1 = a \wedge \neg b \wedge \neg c \wedge \neg d, \varphi_2 = e \wedge \neg d, \varphi_3 = \neg a \wedge b \wedge \neg d,$$
$$\varphi_4 = \neg a \wedge c \wedge \neg d, \varphi_5 = \neg a \wedge \neg e \wedge \neg b \wedge \neg c \wedge d, \varphi_6 = f \wedge \neg f.$$

Then $\mathsf{MC}(K) = \{\{\varphi_1, \varphi_2\}, \{\varphi_2, \varphi_3, \varphi_4\}, \{\varphi_5\}\}$. *So, for the structure of the graph G we have* $U = \{u_1, \ldots, u_6\}$ *(each* φ_i *associated with* u_i *and each* v_j *associated with an element of* $\mathsf{MC}(K)$*). For* $h'(G)$ *we get* $\mathsf{Adj}(v_1) = \{u_1, u_3\}$, $\mathsf{Adj}(v_2) = \{u_1, u_4\}$, $\mathsf{Adj}(v_3) = \{u_1, u_5\}$, $\mathsf{Adj}(v_4) = \{u_2, u_5\}$, $\mathsf{Adj}(v_5) = \{u_3, u_5\}$, $\mathsf{Adj}(v_6) = \{u_4, u_5\}$, $\mathsf{Adj}(v_7) = \{u_6\}$. *That is,* $\mathsf{MI}(K) = \{\{\varphi_1, \varphi_3\}, \{\varphi_1, \varphi_4\}, \{\varphi_1, \varphi_5\}, \{\varphi_2, \varphi_5\}, \{\varphi_3, \varphi_5\}, \{\varphi_4, \varphi_5\}, \{\varphi_6\}\}$. *It is easy to check that* $h(h'(G)) = G$.

As a consequence of this link between inconsistency and consistency graphs, we can show that measures based on maximal consistent subsets are IG measures.

Corollary 5. *1. An inconsistency measure $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an aIG measure iff there is a function $g : \mathcal{G}_c^+ \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = g(\mathsf{CG}^+(K))$ for all $K \in \mathcal{K}$.*

2. An inconsistency measure $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an IG measure iff there is a function $g : \mathcal{G}_c \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = g(\mathsf{CG}(K))$ for all $K \in \mathcal{K}$.

We next use Corollary 5 to show that some inconsistency measures are IG measures. In fact, the following results show that some measures that are not overtly syntactic are actually in the same category as the measures usually taken to be syntactic.

Proposition 6. *I_A and I_{hs} are IG inconsistency measures.*

We end this subsection by showing that I_η is also an IG inconsistency measure, that is more surprising than the previous ones, since it is not directly related to minimal inconsistent subsets, so the proof will be less obvious. Recall that I_η is defined in terms of the probabilistic satisfiability problem which involves finding a probability function over the set of classical interpretations (for the language of K) so that certain inequalities hold. We will translate this problem to one where the probability function is over $\mathsf{MC}(K)$. For this purpose it is useful to define a more general concept of (uniform) satisfiability as follows. Let Z be a mathematical object with elements such as a set or graph, X a set based on Z, $C(Z, X)$ a condition on elements of Z involving X (that is, $C(z, x)$ is true or false), $p \in [0, 1]$, π a probability function over X, and $P_\pi(z) = \sum_{x \in X} \{\pi(x) \mid C(z, x) \text{ is true }\}$. We say that the set of inequalities $\Gamma = \{P(z) \geq p \mid z \in Z\}$ is $\langle Z, X, C\rangle$-satisfiable for P if there is a probability function π on X such that Γ with P_π for P holds. In particular, the satisfiability that was defined for I_η using this notation is $\langle K, \mathbb{I}_C, i(\varphi) = t\rangle$-satisfiability where $\varphi \in K$ and $i \in \mathbb{I}_C$ (we write $\mathbb{I}_C$ for the set of classical interpretations of the atoms of K). We will also use the notation $T(i) = \{\varphi \in K \mid i(\varphi) = T\}$, that is, $T(i)$ is the set of formulas of K that are true for the interpretation i. In the proof we will show in 2 steps that for P, $\langle K, \mathbb{I}, i(\varphi) = T\rangle$-satisfiablity is equivalent to $\langle G, V, u \in \mathsf{Adj}(v)\rangle$-satisfiability where G is the consistency graph for K.

Proposition 7. *I_η is an IG inconsistency measure.*

Given this duality between consistency and inconsistency graphs, we will focus on the latter in the remainder of the paper since their relation to *inconsistency* measuring is more intuitive. Nonetheless, there are some existing proposals for inconsistency measures that are based on maximal consistent subsets of a knowledgebase (for example Ammoura, Salhia, Oukachab, & Raddaoui, 2017)) and these can equivalently be defined in terms of an augmented consistency graph.

5.3 Devising Inconsistency Measures from the Inconsistency Graph

We will use the inconsistency graph to study the structure of the inconsistencies in a knowledgebase. In particular, we will define several new IG inconsistency measures $I(K) = f(\mathsf{IG}(K))$, via functions f on the inconsistency graph. If $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ is a function on inconsistency graphs, we denote by $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ the function on knowledgebases defined as $I_f(K) = f(\mathsf{IG}(K))$ for every $K \in \mathcal{K}$. Not every $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ yields a function $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ that is an inconsistency measure, for I_f must satisfy the rationality postulates of consistency and monotony, given in Definition 1. After defining the concept of subgraph for bigraphs, we gave the usual definition for an induced subgraph. That is, for $G = \langle U, V, E\rangle$, the induced subgraph is defined using a set $W \subseteq U \cup V$. For our purpose now it will be useful to have another definition where the subgraph is induced by just a subset of U.

Definition 9. *Let $G = \langle U, V, E\rangle$ be a bigraph and $W \subseteq U$. Let $V' = \{v \in V | \mathsf{adj}(v) \subseteq W\}$. Then, let $U' = \{u \in W | \exists v \in V'$ such that $\{u, v\} \in E\}$. Finally, let $E' = \{\{u, v\} \in E | u \in U'$ and $v \in V'\}$. Then we say that $G' = \langle U', V', E'\rangle$ was* **U-induced** *by W.*

We use this definition to obtain a correspondence between subsets of a knowledgebase and subgraphs of the inconsistency graph of the knowledgebase.

Proposition 8. *Let $G = \langle U, V, E\rangle$ be the inconsistency graph of a knowledgebase K where, as usual, $K = \{\varphi_1, \ldots, \varphi_m\}$ and $\mathsf{MI}(K) = \{\Delta_1, \ldots, \Delta_n\}$. Let $K' \subseteq K$. Then $G' = \langle U', V', E'\rangle$ is the inconsistency graph of K' iff G' is the bigraph U-induced from G by W where $W \subseteq U$ corresponds to the elements of K'.*

Proposition 8 gives the exact process of obtaining all the inconsistency graphs for all the subsets of a knowledgebase. This allows us to specify a necessary and sufficient condition that a function f on inconsistency graphs must satisfy in order for I_f to be an inconsistency measure.

Proposition 9. *Let $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$. $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an inconsistency measure iff the following two conditions hold:*

1. *$f(G) = 0$ iff $G = \langle \varnothing, \varnothing, \varnothing\rangle$;*
2. *If $G' = \langle U', V', E'\rangle$ was U-induced by W ($W \subseteq U$) from $G = \langle U, V, E\rangle$ then $f(G') \leq f(G)$.*

Corollary 6. *Let $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ be such that*

1. *$f(G) = 0$ iff $G = \langle \varnothing, \varnothing, \varnothing\rangle$,*
2. *If $G' \subseteq G$ then $f(G') \leq f(G)$.*

Then I_f is an inconsistency measure.

Besides the inconsistency measures from Proposition 2, we can conceive of a number of IG measures I_f based on functions on inconsistency graphs.

Proposition 10. *The following functions $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ defined below yield inconsistency measures $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$. We put in parentheses the meaning for the corresponding knowledgebase.*

- *$f_1(G) = |U| + |V|$ (the number of problematic formulas plus the number of minimal inconsistent subsets)*
- *$f_2(G) = |E|$ (the sum of the sizes of the minimal inconsistent subsets — as specified in Definition 3, every edge in E denotes the membership of a formula in a minimally inconsistent subset.)*
- *$f_3(G) = |U| + |V| + |E|$ ($f_1(G) + f_2(G)$)*
- $f_4(G) = \begin{cases} 0 & \text{if } U = \varnothing \\ \sum\limits_{v \in V} \dfrac{\sum_{u \in \mathsf{Adj}(v)} \deg(u)}{\deg(v)^2} & \text{otherwise} \end{cases}$

  (0 if K is consistent, otherwise the sum of the reciprocals of the sizes of the minimal inconsistent subsets weighted by the average number of minimal inconsistent subsets containing their elements)

- $f_5(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ 1 + |\{u \in U \mid \mathsf{deg}(u) \geq 2\}| & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise one plus the number of formulas that are in at least two minimal inconsistent subsets)

- $f_6(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ 1 + \sum \frac{1}{|\mathsf{Adj}(v) \cap \mathsf{Adj}(v')|} (\forall v, v' \notin V, v \neq v', \mathsf{Adj}(v) \cap \mathsf{Adj}(v') \neq \varnothing) & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise one plus the sum of the reciprocals of the sizes of the intersections of each pair of minimal inconsistent subsets)

- $f_7(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ \max\{\mathsf{deg}(u) \mid u \in U\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the maximum number of minimal inconsistent subsets containing the same formula)

- $f_8(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ |\{v \in V \mid \mathsf{deg}(v) = 1\}| + \max\{\mathsf{deg}(u) \mid u \in U\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the number of self-contradictions plus the maximum number of minimal inconsistent subsets containing the same formula)

- $f_9(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ \max\{\mathsf{deg}(v) \mid v \in V\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the maximum number of formulas in a minimal inconsistent subset)

We next show which of the rationality postulates we introduced earlier the inconsistency measures obtained from these functions satisfy.

Proposition 11. *Consider the inconsistency measures introduced in Proposition 10. I_{f_i} satisfies Independence for $1 \leq i \leq 9$, MI-separability for $i = 2$, Penalty for $1 \leq i \leq 4$, Super-additivity for $1 \leq i \leq 6$, Attenuation for $i = 4$, Equal Conflict for $1 \leq i \leq 9$. and Almost Consistency for $i = 4$.*

Previously we obtained several inconsistency measures using functions on the inconsistency graph by finding conditions to yield inconsistency measures. However, functions on inconsistency graphs violating these conditions may also be valuable, for there is other useful information in such graphs concerning the inconsistency besides the inconsistency degree itself. For example, we may be interested in the average size of the minimal inconsistent subsets. For this purpose we can define a function

$$f_{AV}(K) = \begin{cases} 0 & \text{if } U = \varnothing \\ \frac{1}{|V|} \sum_{v \in V} \mathsf{deg}(v) & \text{otherwise} \end{cases}$$

This function over inconsistency graphs does not lead to an inconsistency measure because the monotony postulate is not satisfied. For example, unless all inconsistencies are due to selfcontradictions, the addition of a selfcontradiction to an inconsistent knowledgebase K decreases $f_{AV}(\mathsf{IG}(K))$. Nevertheless, $f_{AV}((K)) = I_{f_2}(K)/I_M(K)$.

Example 14. *Consider the table below showing 3 knowledgebases and the calculations of f_{AV} for them. On the left hand side of the equals sign, the first number is the reciprocal of the number of vertices V (i.e. the number of minimal inconsistent sets) and the numbers in brackets are the degrees of the vertices in V (i.e. the size of the minimal inconsistent sets).*

K	$f_{AV}(K)$
$\{a, \neg a, b, \neg b\}$	$1/2 \times (2+2) = 2$
$\{a, b, c, \neg a \vee \neg b \vee \neg c, \varphi, \neg\varphi \vee \neg\psi, \psi\}$	$1/2 \times (4+3) = 7/2$
$\{a \wedge b, a \wedge \neg b, \neg a \wedge b, \neg a \wedge \neg b\}$	$1/6 \times (2+2+2+2+2+2) = 2$

Table 2: Examples of computations of f_{AV}

In the same way that we devised new measures of inconsistency from the inconsistency graph (as presented in Proposition 10), we can devise new measures of inconsistency from the consistency graph. Because, of the equivalence of inconsistency and consistency graphs (as investigated in Section 5.2), and the observation that it appears easier to motivate inconsistency measures using inconsistency graphs, we will not present new inconsistency measures based on consistency graphs in this paper. However, for some applications it may be easier and more intuitive to define inconsistency measures based on consistency graphs. Consider for instance the I_A measure which is defined in terms of maximal consistent subsets, or the I_η measure that, as we show via Proposition 8, can be calculated in terms of maximal consistent subsets.

5.4 From Inconsistency Measures to the Inconsistency Graph

We have shown in the previous subsection that under certain conditions we get an inconsistency measure from a function on bigraphs. In this subsection we investigate the reverse problem, that is, finding a function from inconsistency measures to inconsistency graphs. The functions we have constructed on inconsistency graphs are not 1-1, hence we cannot simply take their inverse and must proceed in a different way. Formally, we would like to get an inconsistency measure I and a function $g : \mathbb{R}^{\geq 0}_{\infty} \to \mathcal{G}$ such that for all knowledgebases K, $\mathsf{IG}(K) = g(I(K))$. In order to construct this functions, we rely on an important fact:

Proposition 12. *Both $\mathcal{K}$ and $\mathcal{G}$ are countable.*

Now from Proposition 12 we conclude that there is an enumeration $e : \mathcal{K} \to \mathbb{N}$, (which is a bijection) and hence has an inverse $e^{-1} : \mathbb{N} \to \mathcal{K}$. We would like to define an inconsistency measure from e, I_e, as follows:

$$I_e(K) = \begin{cases} 0 & \text{if } K \text{ is consistent} \\ 1 + e(K) & \text{otherwise} \end{cases} \tag{1}$$

We add one to the number corresponding to an inconsistent knowledgebase in order to avoid the possibility of $I_e(K) = 0$ in those cases. Although I_e is not 1-1 and hence has no inverse, all inconsistent knowledgebases K can be recovered because $e^{-1}(I_e(K) - 1) = K$. As all consistent knowledgebases have the same inconsistency graph, there is a function $g_e : \mathbb{N} \to \mathcal{G}$ such that $g_e(I_e(K)) = \mathsf{IG}(K)$ for all $K \in \mathcal{K}$:

$$g_e(x) = \begin{cases} \langle \varnothing, \varnothing, \varnothing \rangle & \text{if } x = 0 \\ \mathsf{IG}(e^{-1}(x-1)) & \text{otherwise} \end{cases} \tag{2}$$

For I_e to be an inconsistency measure, monotony has to be satisfied as well. That is, we have to show that there is a suitable enumeration e.

Proposition 13. *There is a bijection $e : \mathcal{K} \to \mathbb{N}$ such that, if $K \subseteq K'$, then $e(K) \leq e(K')$.*

Employing the enumeration $e : \mathcal{K} \to \mathbb{N}$ presented in the proof of Proposition 13, the function $I_e : \mathcal{K} \to \mathbb{N}$ defined in Equation (1) is indeed an inconsistency measure. Also, as we pointed out earlier, the function $g_e : \mathbb{N} \to \mathcal{G}$ defined in Equation (2) is such that $\mathsf{IG}(K) = g_e(I_e(K))$ for all $K \in \mathcal{K}$. In fact, from the natural number $I_e(K)$ we can recover even the knowledgebase K if it is inconsistent, and therefore its augmented inconsistency graph $\mathsf{IG}^+(K)$. By employing the bijection h between augmented consistency and augmented inconsistency graphs from Theorem 3, the augmented consistency graph $\mathsf{CG}^+(K)$ can be recovered as well.

It is worth remarking that the independence property does not hold for I_e. If this property is enforced, no inconsistency measure can be used to recover an inconsistent knowledgebase or its augmented inconsistency graph since the free formulas would be ignored. However, as the inconsistency graph takes into account only the problematic formulas, it could be recovered via an inconsistency measure satisfying the independence property. Let the function $I'_e : \mathcal{K} \to \mathbb{N}$ be defined as $I'_e(K) = I_e(\mathsf{Problematic}(K))$ for all $K \in \mathcal{K}$. Applying g_e to $I'_e(K)$ yields $\mathsf{IG}(\mathsf{Problematic}(K))$, and $\mathsf{IG}(\mathsf{Problematic}(K)) = \mathsf{IG}(K)$ for all $K \in \mathcal{K}$. Then, the consistency graph can also be recovered using I'_e by applying the bijection h from Corollary 4.

The functions I_e and I'_e, despite satisfying the basic properties required for inconsistency measures, seem to be quite arbitrary in the sense that they lack an underlying intuition for the inconsistency measurements. This poses the question of which further desiderata could be imposed on inconsistency measures to preclude such cases. Discussions on how reasonable are the inconsistency measuring postulates in the literature have claimed that they are over-constraining to some extent (Ammoura, Raddaoui, Salhi, & Oukacha, 2015; Besnard, 2014; Thimm, 2016a). In contrast, the absence of strong rationality postulates leave a vacuum for rather meaningless proposals to arise, such as I_e and I'_e. To address this dilemma, more or less demanding properties, as those proposed in De Bona and Hunter (2017), could be employed to describe and classify inconsistency measures, instead of being required in their definition. Augmented inconsistency graphs, or simply inconsistency graphs, can be useful for the formulation of properties of this kind.

In this section, we have shown how by focusing on the inconsistency graph, we can calculate a number of the existing inconsistency measures. In other words, the inconsistency graph has sufficient information that can be used to calculate these measures. In addition, we have identified further interesting measures, each of which can be calculated from the inconsistency graph, such as I_{f_4} and I_{f_5} in Proposition 10. Both measures take the overlap of minimal inconsistent subsets into account to give finer grained inconsistency measures. I_{f_4} is a measure that regards smaller inconsistent sets as worse than larger inconsistent sets but then weights this by the average size of the inconsistent sets containing the elements, and I_{f_5} is a measure that counts the number of formulas that are in at least two minimal inconsistent sets, thereby flagging the formulas that are more than a "one-off" problem. These measures are indicative of a range of further measures that can be defined based on the inconsistency graph. We also showed that, without requiring more than the postulates of monotony and consistency, inconsistency measures can encode the whole knowledgebase.

6. A Hierarchy for Syntactic Measures

So far we have investigated inconsistency measures that can be calculated from various graphs constructed from knowledgebases. We had a particular interest in using the inconsistency graph. But some inconsistency measures that can be computed from the inconsistency graph (the ones we called IG measures) do not actually use all the information provided by the inconsistency graph. This suggests that we look for simpler representations that still convey enough information to calculate them. In this section we will build a hierarchy of such representations that will then provide a corresponding hierarchy for inconsistency measures based on how much information is needed to compute them.

We call such a representation an abstraction as it abstracts some information from a knowledgebase in a uniform way. We write A for the set of objects for a particular abstraction and call A an abstraction space. For instance so far we have used the set of bigraphs as an abstraction space. We also need a mapping that takes each knowledgebase to an object in the abstraction space. We do not formally define what types of operations are allowed for a mapping but note that determining if a set of formulas is consistent or inconsistent is allowed. In our case the mapping constructs whichever bigraph we are considering, such as the inconsistency graph.

Definition 10. *An* **abstraction class** C *(or simply a* **class***) is a pair* $C = \langle A, m_C \rangle$ *where* $m_C : \mathcal{K} \to A$.
An inconsistency measure I *is in class* C *if there is a function* $f_C : A \to \mathbb{R}^{\geq 0}_{\infty}$ *such that* $I(K) = f_C(m_C(K))$ *for all knowledgebases* K*. We then call* I *a* C *measure.*

We will sometimes write a class as C with a subscript but often it will be convenient to use just the subscript of C for the class name. Thus we write $C_{IG} = IG = \langle \mathcal{G}, IG \rangle$, where $\mathcal{G}$ is the set of bigraphs and $\mathsf{IG}(K)$ is the inconsistency graph of K, is a class and the IG inconsistency measures are exactly the ones in C_{IG}. In order to be precise in the definition of f_C, we should use $Range(m_C)$ instead of A because it is possible that not all elements of A are in $Range(m_C)$. For example, consider the class C_{IG}. $\mathcal{G}$ is the set of all bigraphs but as we showed in Corollary 1 not every bigraph is an inconsistency graph. So in that case we really require the domain of f_{IG} to be those bigraphs that are inconsistency graphs, namely the ones that satisfy the two conditions of Corollary 1, that is, $Range(IG)$. However, in order to make the presentation less cumbersome we will use A instead of $Range(m_C)$ for all classes.

Another example is $C_B = \langle \{0, 1\}, I_B \rangle$, the **binary** class where I_B was given in Definition 2. I_B is a measure in this class. In this case $f_{C_B} = \iota_{\{0,1\}}$, the identity function on $\{0, 1\}$. Due to the consistency postulate for inconsistency measures, no non-empty class can have a smaller abstraction space. In general, for any inconsistency measure I we can define the class $C_I = \langle \mathbb{R}^{\geq 0}_{\infty}, I \rangle$. Another trivial case occurs if an inconsistency measure I' is obtained as a function of some I, that is, $I'(K) = g(I(K))$ in which case an I' measure is automatically an I measure. An example is the function g that doubles the value of its argument (i.e. $g(I(k)) = 2 \times I(k)$). We are interested in classes that encompass genuinely different inconsistency measures given in the literature. We should also note that not every class yields an inconsistency measure. For example, let $C = \langle \mathbb{N}, m_C \rangle$ where $m_C(K)$ is the number of formulas in K. There is no way to get an inconsistency measure if that is the only information stored about K. We will be interested only in classes that yield inconsistency measures. We call such a class **proper** and will discuss only proper classes.

Next we show how to compare classes with respect to their inconsistency measures. Consider the class $MI = \langle \mathbb{N}, MI \rangle$ which is obtained by abstracting from the knowledgebase its number of minimal inconsistent subsets. But we can also calculate the number of minimal inconsistent subsets from the inconsistency graph by counting the number of vertices in V. However, given the number of inconsistent subsets we cannot construct the inconsistency graph. Thus, intuitively, the class IG is more general than the class MI. The most general class is the one where we retain the entire knowledgebase. This is what happens when we construct the labeled augmented inconsistency graphs. We call this class the **universal** class and denote it as $U = \langle \mathcal{LG}, \mathsf{LIG}^+ \rangle$ where $\mathcal{LG}$ is the set of labeled bigraphs (that can be obtained from knowledgebases) and $\mathsf{LIG}^+(K)$ is the labeled augmented inconsistency graph of K. Moving away from graph representation, we can simply write this class as $U = \langle \mathcal{K}, \iota_{\mathcal{K}} \rangle$ where $\iota_{\mathcal{K}}$ is the identity function on $\mathcal{K}$. Our interest now is in comparing the generality of classes. The following definition captures how we can represent classes as a hierarchy. So we define the $\succeq$, $\succ$, and $\sim$ relations to specify the hierarchy.

Definition 11. *A class $C = \langle A, m_C\rangle$ is* **at least as (more or equally) general** *as the class $C' = \langle A', m_{C'}\rangle$, denoted $C \succeq C'$, if the relation $h_{C,C'} = \{\langle m_C(K), m_{C'}(K)\rangle \in A \times A' \mid K \in \mathcal{K}\}$ is a function. As usual, we write $C \not\succeq C'$ when $C \succeq C'$ does not hold. We say that C and C' are* **equally general** *written $C \sim C'$ if both $C \succeq C'$ and $C' \succeq C$ hold. Furthermore, C is* **more general** *than C', denoted $C \succ C'$, when $C \succeq C'$ but $C \not\sim C'$. Finally, we call C and C'* **incomparable** *(for generality) in case neither $C \succeq C'$ nor $C' \succeq C$ holds When $h_{C,C'}$ is a function we will use function notation, that is, write $m_{C'}(K) = h_{C,C'}(m_C(K))$.*

We are not using the phrase "more general" in the sense of "more abstract", but rather in the sense of "more discriminative". We will show that whenever $C \succeq C'$ holds, any inconsistency measure in class C' must also be in class C.

Example 15. *Consider the class $C_{IG} = IG = \langle \mathcal{G}, IG\rangle$, where $\mathcal{G}$ is the set of bigraphs and $\mathsf{IG}(K)$ is the inconsistency graph of K, and the class $C_B = \langle\{0,1\}, I_B\rangle$ where I_B was given in Definition 2. So for each $G = \langle U, V, E\rangle \in \mathcal{G}$, if $U = \varnothing$, then $h_{IG,B} = 0$, otherwise $h_{IG,B} = 1$.*

We start our investigation of the generality of classes by showing two fundamental properties of the relation $\succeq$.

Proposition 14. *The relation $\succeq$ is reflexive and transitive.*

Next we present several results that we will use to find generality relations between classes. We start with a characterization for equally general classes.

Proposition 15. *$C = \langle A, m_C\rangle$ and $C' = \langle A', m_{C'}\rangle$ are equally general iff $h_{C,C'}$ and $h_{C',C}$ are inverse functions.*

Corollary 7. *If $C \succeq C'$ and $h_{C,C'}$ is not one-to-one then $C \succ C'$.*

As we next show, the generality relation between classes extends to their inconsistency measures.

Proposition 16. *If $C \succeq C'$ then every C' inconsistency measure is also a C inconsistency measure.*

Corollary 8. *If $C \succeq C'$ and there is a C inconsistency measure that is not a C' inconsistency measure then $C \succ C'$.*

We can also characterize the case where two classes have incomparable generalities.

Proposition 17. *$C = \langle A, m_C\rangle \not\succeq C' = \langle A', m_{C'}\rangle$ iff there exist knowledgebases K and K' such that $m_C(K) = m_C(K')$ but $m_{C'}(K) \neq m_{C'}(K')$.*

Corollary 9. *The generalities of $C = \langle A, m_C\rangle$ and $C' = \langle A', m_{C'}\rangle$ are incomparable iff*

1. *There exist K and K' such that $m_C(K) = m_C(K')$ but $m_{C'}(K) \neq m_{C'}(K')$ and*
2. *There exist K'' and K''' such that $m_{C'}(K'') = m_{C'}(K''')$ but $m_C(K'') \neq m_C(K''')$.*

We now formulate several classes that are intuitively less expressive than IG. In all of these cases we first show how to obtain the value from the knowledgebase and in parentheses we indicate how to obtain it from the inconsistency graph. We also give one or more examples of inconsistency measures in that class. These inconsistency measures were defined in Definition 2 and Proposition 10. Table 3 gives all the formal definitions.

Definition 12. *Classes obtained from IG:*

- PC (Problematic-Count) *counts the number of problematic formulas (the size of U).*
  Example: I_P is the number of problematic formulas.
- CC (Conflict-Count) *counts the number of minimal inconsistent subsets (the size of V).*
  Example: I_M is the number of minimally inconsistent subsets.
- VC (Vertex-Count) *combines the PC and CC values into a pair of numbers.*
  Example: I_{f_1} is the sum of I_P and I_M.
- PD (Problematic-Degree) *counts for each positive integer n the number of formulas (if not 0) that are in n minimal inconsistent subsets (the number of vertices in U that have degree n) to form a set of ordered pairs of positive integers.*
  Examples: I_{f_5} is one plus the number of formulas that are in at least two minimal inconsistent subsets and I_{f_7} is the maximum number of minimal inconsistent subsets containing the same formula.
- CD (Conflict-Degree) *counts for each positive integer n the number of minimal inconsistent subsets (if not 0) that contain n formulas (the number of vertices in V that have degree n) to form a set of ordered pairs of positive integers.*
  Examples: $I_{\#}$ is the sum of the reciprocals of the sizes of the minimal inconsistent subsets and I_{f_9} is the maximum number of formulas in a minimal inconsistent subset.
- VD (Vertex-Degree) *combines the PD and CD sets into an ordered pair.*
  Examples: I_{f_4} is the sum of the reciprocals of the sizes of the minimal inconsistent subsets weighted by the average number of minimal inconsistent subsets containing their elements and I_{f_8} is the number of self-contradictions plus the maximum number of minimal inconsistent subsets containing the same formula.
- EC (Edge-Count) *counts the sum of the sizes of the minimal inconsistent sets*
  Example: I_{f_2} is the sum of the sizes of the minimal inconsistent subsets.

Example 16. *Consider the knowledgebase $K = \{a, \neg a \wedge \neg b, b, c\}$ whose inconsistency graph G is given in Figure 5(e). Then $h_{IG,PC}(G) = 3$ (the size of U), $h_{IG,CC}(G) = 2$ (the size of V), $h_{IG,VC}(G) = \langle 3, 2\rangle$ (placing the previous 2 numbers into a pair), $h_{IG,PD}(G) = \{\langle 2, 1\rangle, \langle 1, 2\rangle\}$ (2 vertices in U have degree 1 and 1 vertex has degree 2), $h_{IG,CD}(G) = \{\langle 2, 2\rangle\}$ (the 2 vertices in V both have degree 2), $h_{IG,VD}(G) = \langle\{\langle 2, 1\rangle, \langle 1, 2\rangle\}, \{\langle 2, 2\rangle\}\rangle$ (placing the previous 2 answers into a pair), and $h_{IG,EC}(G) = 4$ (there are 4 edges).*

The generality relations among these classes are given by the next result and illustrated graphically in Figure 14.

Theorem 4. *The following generality relations hold between classes where B refers to class C_B (i.e. the binary class):*

1. *IG^+ is more general than IG: $IG^+ \succ IG$.*
2. *IG is more general than VD: $IG \succ VD$.*
3. *VD is more general than each of PD, VC, and CD: $VD \succ PD, VC, CD$.*

Name	C	A	$m_C(K)$	$h_{IG,C}(G)$
problematic-count	PC	$\mathbb{N}$	$\lvert \mathsf{Problematic}(K) \rvert$	$\lvert U \rvert$
conflict-count	CC	$\mathbb{N}$	$\lvert \mathsf{MI}(K) \rvert$	$\lvert V \rvert$
vertex-count	VC	$\mathbb{N}^2$	$\langle \lvert \mathsf{Problematic}(K) \rvert, \lvert \mathsf{MI}(K) \rvert \rangle$	$\langle \lvert U \rvert, \lvert V \rvert \rangle$
problematic-degree	PD	$2^{\mathbb{N}^2}$	see caption	see caption
conflict-degree	CD	$2^{\mathbb{N}^2}$	see caption	see caption
vertex-degree	VD	$2^{\mathbb{N}^2} \times 2^{\mathbb{N}^2}$	$\langle m_{PD}(K), m_{CD}(K) \rangle$	$\langle h_{IG,PD}(G), h_{IG,CD}(G) \rangle$
edge-count	EC	$\mathbb{N}$	$\sum_{\Delta \in \mathsf{MI}(K)} \lvert \Delta \rvert$	$\lvert E \rvert$

Table 3: Classes based on abstracting the inconsistency graph $\mathsf{IG}(K) = \langle U, V, E \rangle$
For each class we give the full name as well as the class name, the abstraction space, the mapping m_C from K, and the function $h_{IG,C}$ from the inconsistency graph. The missing cases are below:
For PD,
$$m_C(K) = \{\langle x, y \rangle \mid x > 0 \text{ and } x = |\{\varphi \in \mathsf{Problematic}(K) \text{ s.t. } |\{M \in \mathsf{MI}(K) \mid \varphi \in M\}| = y\}|\}$$
$$h_{IG,C}(G) = \{\langle x, y \rangle \mid x > 0 \text{ and } x = |\{u \in U \mid \mathsf{deg}(u) = y\}|\}$$
For CD,
$$m_C(K) = \{\langle x, y \rangle \mid x > 0 \text{ and } x = |\{M \in \mathsf{MI}(K) \text{ s.t.i } |M| = y\}|\}$$
$$h_{IG,C}(G) = \{\langle x, y \rangle \mid x > 0 \text{ and } x = |\{v \in V \mid \mathsf{deg}(v) = y\}|\}$$

4. *PD is more general than both PC and EC: $PD \succ PC, EC$.*

5. *VC is more general than both PC and CC: $VC \succ PC, CC$.*

6. *CD is more general than both EC and CC: $CD \succ EC, CC$.*

7. *Each of PC, EC, and CC is more general than B: $PC, EC, CC \succ B$.*

8. *The generalities of PD, VC, and CD are pairwise incomparable.*

9. *The generalities of PC, EC, and CC are pairwise incomparable.*

To summarize, Figure 14 illustrates the generality relation between the proposed classes. We can construct a similar hierarchy for the consistency graphs. Given the equivalence of (augmented) consistency graphs and (augmented) inconsistency graphs, the top part and the bottom element of the hierarchies would be the same. However, in between the top and bottom, new classes would need to be defined for consistency graphs. We leave this to future work.

Using our hierarchy, we are able to position many of the existing proposals for inconsistency measures. As we have already shown, we can situate I_B (Hunter & Konieczny, 2008), $I_M(K)$ (Hunter & Konieczny, 2008), $I_A(K)$ (Grant & Hunter, 2011) , $I_P(K)$ (Grant & Hunter, 2011), $I_\#(K)$ (Hunter & Konieczny, 2008), $I_{nc}(K)$ (Doder et al., 2010; Thimm, 2016b), $I_{hs}(K)$ (Thimm, 2016b), $I_H(K)$ (Grant & Hunter, 2013), and $I_\eta(K)$ (Knight, 2002). There are a number of further measures that build on these measures, and these can also be captured in our hierarchy (e.g. Mu et al., 2011a, 2011b; Jabbour et al., 2014; Jabbour, Ma, Raddaoui, Sais, & Salhi, 2016)). What we cannot capture in the hierarchy (except in the most general class) are the semantic-based measures such as $I_C(K)$ (Konieczny et al., 2003; Grant & Hunter, 2011) which counts the minimum number of atoms that need to be assigned B amongst the 3VL models of K or measures based on fuzzy answer set semantics (Madrid & Ojeda-Aciego, 2011). There are also syntactic measures based on

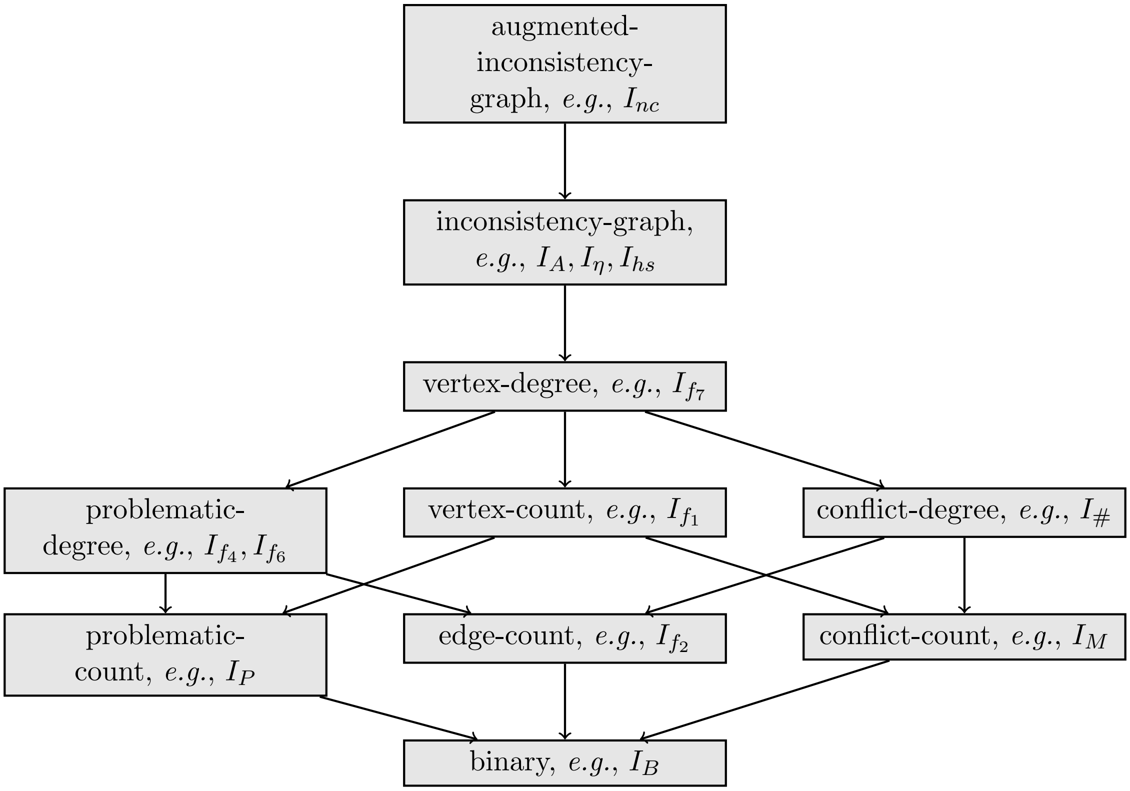


Figure 14: The generalization hierarchy for the classes that contain syntactic inconsistency measures. Each arc from class C to C' denotes $C \succ C'$. Each class is followed by at least one inconsistency measure in the class that is not in a less general class.

other kinds of logics such as description logics (Ma et al., 2007; Qi & Hunter, 2007; Zhou et al., 2009) that currently do not fit into the hierarchy but if we were to expand the framework to other logics, it would be straightforward to also include those.

Much of the current version of the hierarchy could have been developed without recourse to graphs, perhaps using just minimal inconsistent and maximal consistent subsets directly, but this would not have captured all the measures that we wanted to consider, and it would not have allowed us to easily identify some of the interesting new proposals for inconsistency measures. Furthermore, by not basing the framework on graphs, we would miss the interesting potential to consider other graph theoretic notions like connected components, matchings, cycles, kernels, etc. that could give rise to further (yet unconsidered but) meaningful notions of inconsistency measures, where the graph structure really would be particularly useful. We will be investigating these graph-theoretic options in future work.

Finally, we note that there are some related works that have hypergraph representation of minimal inconsistency subsets for inconsistency measures (Jabbour et al., 2014, 2016). The hypergraph representation captures the overlap of the minimal inconsistent subsets, and can be captured using our inconsistency graphs. The main point of their work is to define some specific new inconsistency measures whereas the point of our work is to compare and classify inconsistency measures.

7. A Hierarchy of Non-Syntactic Measures

In the previous sections we showed that what are usually called syntactic inconsistency measures are in the IG (or IG^+) class. But then we found that some such measures do not need all the information from the inconsistency graph. This allowed us to define a hierarchy of abstraction classes, so that when an inconsistency measure I is in class C but not in class C', where C is more general than C' and I' is in C', I requires more information to compute than I'. But not all inconsistency measures are syntactic. In this section we extend the hierarchy we previously constructed from B (the binary class) to IG by constructing a hierarchy on top of IG that goes up to the universal class. While this hierarchy is not intended to capture all possible non-syntactical approaches to inconsistency measuring, it provides an example of how our framework can be extended to cope with some non-syntactic measures. We wish to point out that we are using the word "syntactic" as it has been used in the literature, that is, at the knowledgebase level and based on the minimal inconsistent subsets. Syntax-based measures are concerned with how the inconsistency is spread among the elements (formulas) of the knowledgebase, without looking inside the formulas. The new measures that we define in this section depend on both the structure of the minimal inconsistent subsets and the formulas themselves.

The question we consider is how to make inconsistency graphs more informative so that we can build these new classes from which we can define non-IG measures. The inconsistency graphs we constructed do not reveal any information about the formulas themselves. Consider that one of the syntactic measures we dealt with, I_P, counts the number of problematic formulas. But we may want to probe further and also consider the atoms in the formulas. For example, it seems reasonable to count the number of atoms in the problematic formulas as an inconsistency measure. This cannot be done with inconsistency graphs. Recall now that we started with labeled inconsistency graphs. Then, for a knowledgebase K, $\mathsf{LIG}^+(K)$ contains K: that is, $K = \cup_{i=1}^{m} L(u_i)$; V is not even needed to recover K. But in order to count the number of atoms we don't really need all the formulas, just what atoms each formula contains. Our idea in this section is similar to what we did for the hierarchy based on inconsistency graphs where we reduced the amount of information. The difference is that now we start with the labeled inconsistency graph and reduce the information in the labels. Then, in order to build the hierarchy on top of IG we must retain the entire unlabeled graph.

Consider now the measure we presented earlier that is not a syntactic measure, I_C. Recall that $I_C(K)$ is the minimal number of atoms that must be assigned the value B to obtain a 3VL model for K. Let us contrast I_C with the one we just suggested, the one that counts the number of distinct atoms in $\mathsf{Problematic}(K)$ that we call I_1. Let $K_1 = \{a \wedge \neg a \wedge b \wedge \neg c\}$. The inconsistency graph is very simple with $|U| = |V| = |E| = 1$. Here $I_1(K_1) = 3$ (for a, b, and c) but $I_C(K_1) = 1$ as I_C counts only the atoms that are "really" involved in an inconsistency. After all, b and c just happen to appear in the problematic formula without causing any problems themselves. Next, let $K_2 = \{a \wedge \neg a \wedge b \wedge \neg c, \neg b \wedge d\}$. The only difference is the addition of the second formula which happens to be free in K_2. So $\mathsf{IG}(K_1) = \mathsf{IG}(K_2)$; however, the augmented inconsistency graph has an extra vertex in U. The formula $\neg b \wedge d$ is called an iceberg inconsistency (De Bona & Hunter, 2017) as it is free in this set but if the first formula were broken into two formulas, as $a \wedge \neg a$ and $b \wedge \neg c$, where the two formulas together are logically equivalent to the original formula, $\neg b \wedge d$ would no longer be free. $I_1(K_2) = I_1(K_1) = 3$ as the atom d in the free formula is not counted. But $I_C(K_2) = 2 \neq I_C(K_1) = 1$ because b must also be given the value B for a 3VL model. In this way I_C is sensitive to iceberg inconsistencies that are not considered for I_1. So it seems that even with labels (that have limited information, not actual formulas) the inconsistency graph is not the appropriate structure for computing I_C. About all we can say is that I_1 is an upper bound for

I_C, that is, for every knowledgebase K, $I_C(K) \leq I_1(K)$, as assigning the B value to all atoms in problematic formulas always yields a 3VL model.

To simplify matters we will consider only inconsistency measures that satisfy the independence property. Hence we will omit the free formulas and not deal with augmented inconsistency graphs.

We start by defining four new types of labeled inconsistency graphs. In each case U, V, and E are the same as for the labeled inconsistency graphs defined in Definition 4. So these graphs deal only with the formulas in $\mathsf{Problematic}(K)$. The difference is in the labeling. Hence in our definitions it suffices to write only the labeling functions, whose codomains are assumed to be equal to the ranges. As in some cases we will need **multisets**, we will use the notation where the elements of a multiset are placed in brackets, as opposed to the braces used for sets. We also need the concept of **the length of a formula**, denoted $\mathsf{Length}(\varphi)$ that we define as the total number of occurrences of symbols in the formula φ.

Definition 13. *New types of labeled inconsistency graphs:*

1. *For a* **multiset labeled inconsistency graph** $\mathsf{MLIG}(K) = \langle U, V, E, L^M \rangle$ *where* $L^M(u_i) =$ *the multiset of atoms that occur in* $\varphi_i \in \mathsf{Problematic}(K)$ *for all* i, $1 \leq i \leq m$ *and* $L^M(v_j) =$ *the multiset of atoms that occur in the formulas of* $\Delta_j \in \mathsf{MI}(K)$ *for all* j, $1 \leq j \leq n$.

2. *For a* **set labeled inconsistency graph** $\mathsf{SLIG}(K) = \langle U, V, E, L^S \rangle$ *where* $L^S(u_i) =$ *the set of atoms that occur in* $\varphi_i \in \mathsf{Problematic}(K)$ *for all* i, $1 \leq i \leq m$ *and* $L^S(v_j) =$ *the set of atoms that occur in the formulas of* $\Delta_j \in \mathsf{MI}(K)$ *for all* j, $1 \leq j \leq n$.

3. *For a* **number labeled inconsistency graph** $\mathsf{NLIG}(K) = \langle U, V, E, L^N \rangle$ *where* $L^N(u_i) = |L^M(u_i)|$ *for all* i, $1 \leq i \leq m$ *and* $L^N(v_j) = |L^M(v_j)|$ *for all* j, $1 \leq j \leq n$.

4. *For a* **length labeled inconsistency graph** $\mathsf{LLIG}(K) = \langle U, V, E, L^L \rangle$ *where* $L^L(u_i) = \mathsf{Length}(\varphi_i)$ *for all* i, $1 \leq i \leq m$ *and* $L^L(v_j) = \sum_{\varphi_i \in \Delta_j} \mathsf{Length}(\varphi_i)$ *for all* j, $1 \leq j \leq n$.

Next we illustrate the new labeled graphs.

Example 17. *In Example 3 we constructed* $\mathsf{LIG}(K)$ *for* $K = \{a, \neg a \vee \neg b, b, \neg a \vee c, \neg c \vee d, \neg d\}$, $\mathsf{MI}(K) = \{\{a, \neg a \vee \neg b, b\}, \{a, \neg a \vee c, \neg c \vee d, \neg d\}\}$, *and* $\mathsf{Problematic}(K) = K$. *Here we draw the 4 new labeled graphs just defined for* K.

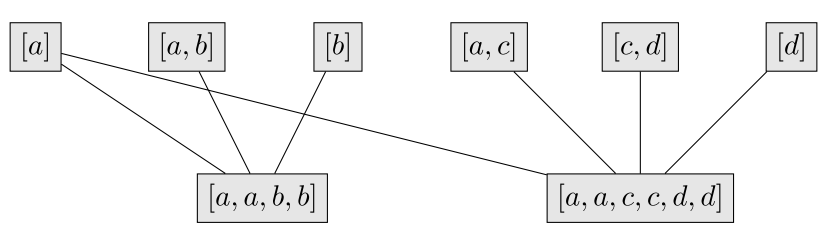


Figure 15: $\mathsf{MLIG}(K)$

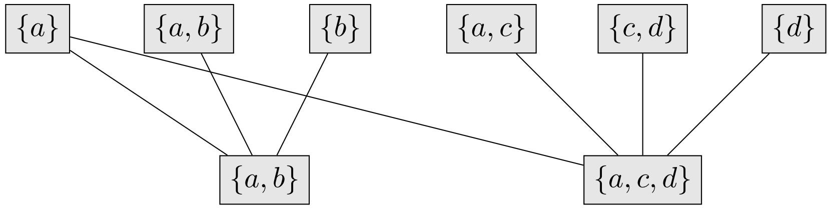


Figure 16: $\mathsf{SLIG}(K)$

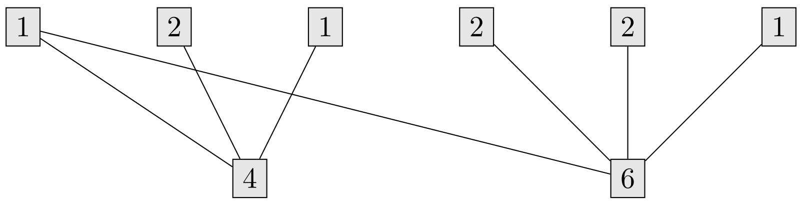


Figure 17: $\mathsf{NLIG}(K)$

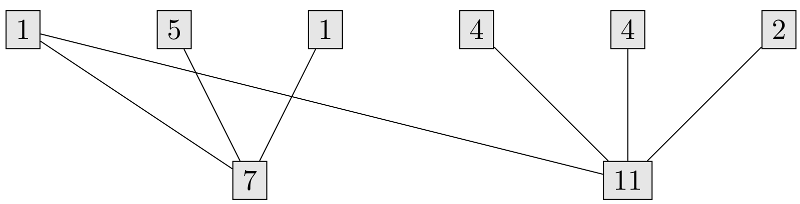


Figure 18: $\mathsf{LLIG}(K)$

As in the previous sections we define a class for each of these types of labeled graphs, that is, the classes $MLIG$, $SLIG$, $NLIG$, and $LLIG$, where the corresponding functions $m_C : \mathcal{K} \to A$ are given in Definition 13 (homonymous to the classes), and the abstraction spaces A can be defined via the range of those functions.

Next we define 9 new inconsistency measures based on the new labeled graphs. It is clear from the definitions that all of them satisfy the definition of inconsistency measure as well as the independence property.

Definition 14. 1. *$I_1(K) =$ the number of distinct atoms in the formulas of* $\mathsf{Problematic}(K)$.

2. *$I_2(K) =$ the number of occurrences of atoms in the formulas of* $\mathsf{Problematic}(K)$.

3. *$I_3(K) =$ the sum of the lengths of the formulas of* $\mathsf{Problematic}(K)$.

4. *$I_4(K) =$ the maximum number of distinct atoms in any formula of* $\mathsf{Problematic}(K)$.

5. *$I_5(K) =$ the maximum number of occurrences of atoms in any formula of* $\mathsf{Problematic}(K)$.

6. *$I_6(K) =$ the maximum length of any formula of* $\mathsf{Problematic}(K)$.

7. *$I_7(K) =$ the maximum number of distinct atoms in any minimal inconsistent subset of K.*

8. $I_8(K)$ = *the maximum number of occurrences of atoms in any minimal inconsistent subset of* K.

9. $I_9(K) = 1+$ *the number of formulas of* $\mathsf{Problematic}(K)$ *in which an atom occurs more than once, if* $\mathsf{Problematic}(K) = \varnothing$ *then* $I_9(K) = 0$.

To illustrate the new inconsistency measures we compute them for the knowledgebase of Example 17.

Example 18. *For the knowledgebase* K *of Example 17 (i.e. for* $K = \{a, \neg a \vee \neg b, b, \neg a \vee c, \neg c \vee d, \neg d\}$, $\mathsf{MI}(K) = \{\{a, \neg a \vee \neg b, b\}, \{a, \neg a \vee c, \neg c \vee d, \neg d\}\}$, *and* $\mathsf{Problematic}(K) = K$*).* $I_1(K) = 4$, $I_2(K) = 9$, $I_3(K) = 17$, $I_4(K) = 2$, $I_5(K) = 2$, $I_6(K) = 5$, $I_7(K) = 3$, $I_8(K) = 6$ *and* $I_9(K) = 1$.

These new measures provide further details about the inconsistencies. Each offers a dimension that could be considered in an inconsistency resolution. For instance, in an incremental process for reducing inconsistency in a software requirements specification, there could be an emphasis on reducing the number of distinct atoms in the formulas of $\mathsf{Problematic}(K)$ (i.e. I_1) or the maximum length of any formula of $\mathsf{Problematic}(K)$ (i.e. I_5). Choosing the exact measures may depend on the application and on the participants involved in the inconsistency resolution process, but this definition provides a useful range of options that could be used in practice.

Next we show to which class each new inconsistency measure belongs by writing its definition using one of the new labeled inconsistency graphs of Definition 13.

Proposition 18. *The classes for the 9 inconsistency measures are as follows:*

- I_1, I_4, I_7 *are in* $SLIG$
- I_2, I_5, I_8 *are in* $NLIG$
- I_3, I_6 *are in* $LLIG$
- I_9 *is in* $MLIG$

We now show the generality relation among the new classes we have defined and connect them with the previous hierarchy.

Theorem 5. *The following generality relations hold among classes:*

1. LIG^+ *is more general than* LIG *and* IG^+*:* $LIG^+ \succ LIG, IG^+$.

2. LIG *is more general than* $MLIG$ *and* $LLIG$*:* $LIG \succ MLIG, LLIG$.

3. $MLIG$ *is more general than* $SLIG$ *and* $NLIG$*:* $MLIG \succ SLIG, NLIG$.

4. $SLIG$, $NLIG$, *and* $LLIG$ *are more general than* IG*:* $SLIG, NLIG, LLIG \succ IG$.

5. *The generalities of* LIG *and* IG^+ *are incomparable.*

6. *The generalities of* $MLIG$, $LLIG$ *and* IG^+ *are pairwise incomparable.*

7. *The generalities of* $SLIG$, $NLIG$, $LLIG$ *and* IG^+ *are pairwise incomparable.*

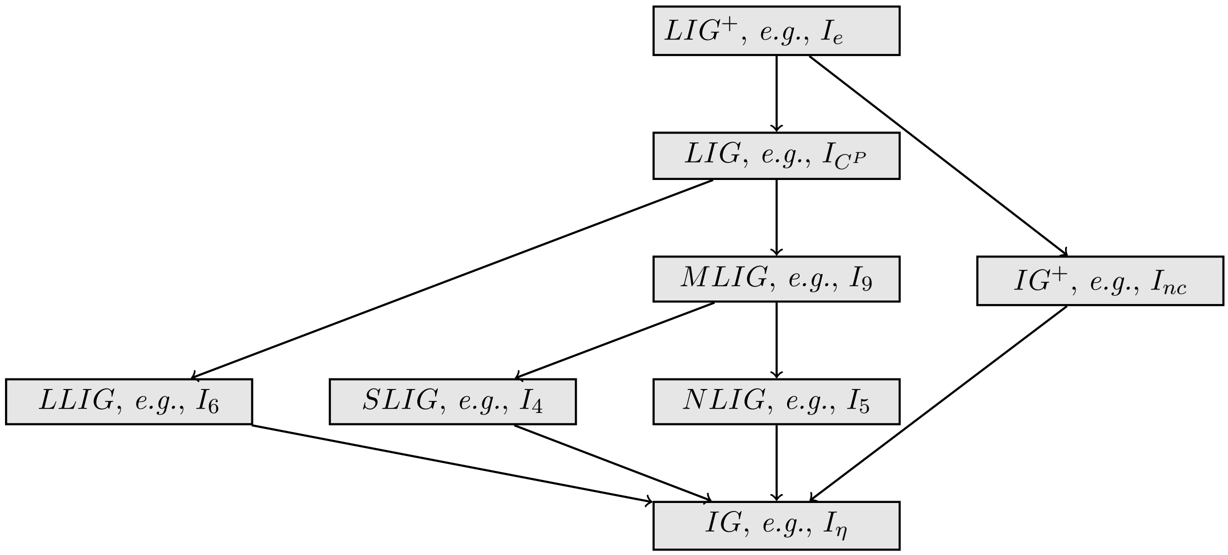


Figure 19: The generalization hierarchy for the classes using the new labeled inconsistency graphs. Each arc from class C to C' denotes $C \succ C'$. Each class is followed by one inconsistency measure in the class that is not in less general classes.

The hierarchy based on the new labeled inconsistency graphs and the above theorem is illustrated in Figure 19. Note that I_e, the inconsistency measure based on the enumeration e of the knowledgebases, needs the whole knowledgebase to be computed, including the free formulas. To illustrate the LIG class, we can define the measure $I_{C^P}(K) = I_C(\mathsf{Problematic}(K))$, that counts the minimum number of atoms in the problematic formulas that must be assigned a B value in order for a 3VL model to be obtained. To compute I_{C^P} one must know the connectives involved in each problematic formula, not only its atoms, thus I_{C^P} is in LIG but is not in any less general class in Figure 19.

Finally, let us sum up the whole hierarchy build from results of this paper in Figure 20.

8. Conclusion and Future Work

We have proposed in this paper a framework to classify syntactic inconsistency measures. We started by introducing the inconsistency graph, which captures the structure of the minimal inconsistent subsets in the knowledgebase, and showed how several inconsistency measures from the literature can be computed from it. Abstracting the inconsistency graph, we introduced a hierarchy that can order inconsistency measures according to the information needed for their calculation.

The introduction of the (augmented) inconsistency graph sheds light on the organization of the inconsistency measures proposed in the literature. Even though we have intentionally avoided defining syntactic inconsistency measures in this paper, the definition of IG measures seems to formally capture this intuitive idea. Evidence for this is the fact that the inconsistency measures usually considered syntactic in the literature are indeed IG, but we don't know any measure said to be semantic that is so. Within the IG measures, our framework of abstraction classes and the generality relationship provides a principled way to classify the existing and possible future proposals of syntactic inconsistency measures.

From a theoretical point of view, the framework helps to make sense of the space of possible syntactic measures. We organize existing measures, providing relations between them through our

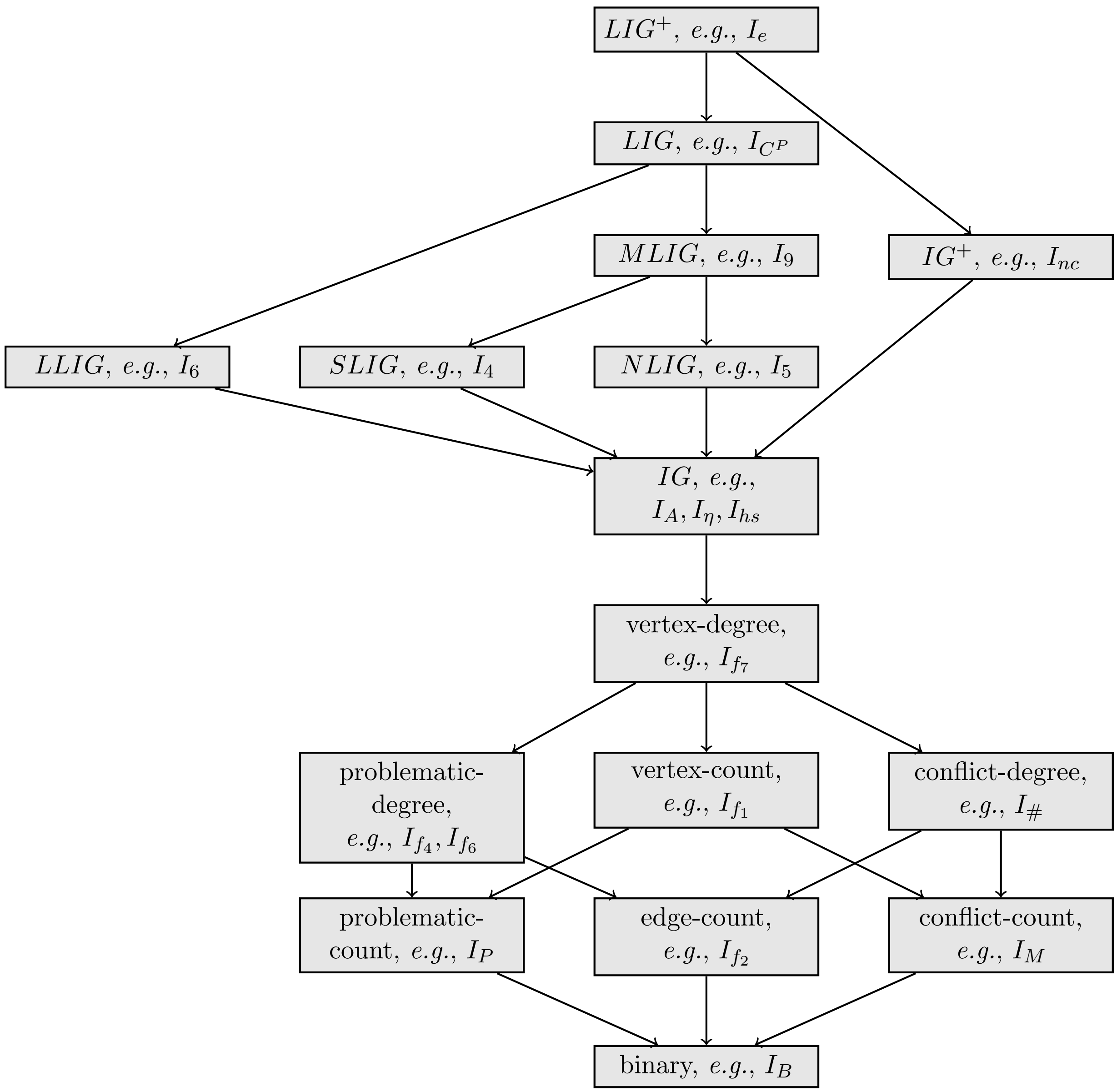


Figure 20: The abstraction hierarchy for inconsistency graphs

hierarchical structure, and we can fit new measures into the hierarchy which may involve identifying new categories in the hierarchy. New measures can also be readily identified in the hierarchy.

From a practical point of view, this hierarchy can help us understand what measures mean in applications, and how the measures relate to each other. Also, the hierarchy can help us develop algorithms by for instance calculating different measures using the same subcomputations.

As future work note that the abstraction classes we introduced are based on simple computations such as the number of vertices and edges. It would be good to refine the classification by introducing additional abstraction classes that represent more complex interactions between formulas and how inconsistencies are formed from them.

Finally, the framework is restricted to syntactic measures, plus some new non-syntactic measures. So it does not include consideration of semantic measures. However, syntactic measures are currently the most important in applications studies. Nonetheless, we intend to investigate how semantic measures can be organized either by extending the current hierarchy or by developing a new framework.

Acknowledgements

We wish to thank the editor and the reviewers whose comments and suggestions were very helpful in refining and improving our paper.
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Appendix A: Proofs

Theorem 1. *Let $G = \langle U, V, E\rangle$ be a bigraph. Then $G = \mathsf{IG}^+(K)$ for some knowledgebase K iff the following two conditions hold for G:*

1. *No vertex in V is isolated.*
2. *For all $v, v' \in V$, if $v \neq v'$ then $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.*

Proof. ($\rightarrow$) Let $G = \mathsf{IG}^+(K)$ for some knowledgebase K. By definition, each vertex in V represents a minimal inconsistent subset; hence it must be incident to the vertices in U that comprise the formulas in the subset. This gives condition 1). Condition 2) follows from the fact that a minimal inconsistent subset cannot be a subset of another minimal inconsistent subset.

($\leftarrow$) Let $G = \langle U, V, E\rangle$ be a bigraph that satisfies the two conditions. We show how to construct a knowledgebase K such that $G = \mathsf{IG}^+(K)$. First we do special cases. If $G = \langle \varnothing, \varnothing, \varnothing\rangle$ then let $K = \varnothing$. Next, if $V = E = \varnothing$ and $|U| = n > 0$, let $K = \{a_1, \ldots, a_n\}$. Now we can assume that $V \neq \varnothing$. By condition 1) this means that $E \neq \varnothing$. The idea of constructing K is as follows. For each $v \in V$ and each vertex adjacent to v we construct a formula so that the set of formulas in $\mathsf{Adj}(v)$ form a minimal inconsistent set. For $\deg(v) = n$ the formulas will have the form: $b_1 \wedge \neg b_n, b_1 \rightarrow b_2, \ldots, b_{n-1} \rightarrow b_n$ where the atoms $b_1, \ldots, b_n$ do not appear in any formula associated with a vertex different from $v \in V$. When a vertex in U has edges to several vertices in V, we take the conjunction of the appropriate formulas for each vertex in V, each of which contains different atoms. This does not change the minimal inconsistent sets because of condition 2).

Formally, we proceed as follows. Let $G = \langle U, V, E\rangle$ where $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. K will have atoms of the form a_i^j, for all $\{u_i, v_j\} \in E$ as well as atoms of the form a_i^0 for all isolated vertices (of U). The number of atoms will be the sum of the number of edges and the number of isolated vertices. We will construct a formula φ_i for each i, $1 \leq i \leq m$ that will stand for the formulas in U as follows. For each $v_j \in V$, let $\mathsf{Adj}(v_j) = \{u_{j1}, \ldots, u_{jt_j}\}$

where $j1 < \ldots < jt_j$. We associate with each v_j, t_j formulas as follows: $\psi^j_{j1} = a^j_{j1} \wedge \neg a^j_{jt_j}, \psi^j_{j2} = a^j_{j1} \rightarrow a^j_{j2}, \ldots, \psi^j_{jtj} = a^j_{j(t_j-1)} \rightarrow a^j_{jt_j}$. We take $\varphi_i = \bigwedge_{\{j|\{u_i,v_j\}\in E\}} \psi^j_i$. For every isolated vertex $u_i \in U$, we take $\varphi_i = a^0_i$. Then $U = \{\varphi_1, \ldots, \varphi_m\}$. The construction guarantees that all $v_j \in V$ correspond to a minimal inconsistent set Δ_j that contains the formulas associated with all $u_i \in U$ that are adjacent to v_j, that is, $\Delta_j = \{\varphi_{j1}, \ldots, \varphi_{jt_j}\}$.

□

Corollary 1. *Let $G = \langle U, V, E\rangle$ be a bigraph. Then $G = \mathsf{IG}(K)$ for some knowledgebase K iff the following two conditions hold for G:*

1. *G contains no isolated vertex.*
2. *For all $v, v' \in V$, if $v \neq v'$ then $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.*

Proof. The only difference between $\mathsf{IG}^+(K)$ and $\mathsf{IG}(K)$ is that the free formulas of K are excluded from consideration for $IG(K)$. Hence the proof is the same as for the theorem but now the isolated vertices for the free formulas (in U) cannot be present in the graph. Hence no vertex in U or V can be isolated. □

Theorem 2. *Let $G = \langle U, V, E\rangle$ be a bigraph. Then $G = \mathsf{CG}^+(K)$ for some knowledgebase K iff the following two conditions hold for G:*

1. *$V \neq \varnothing$.*
2. *For all $v, v' \in V$, if $v \neq v'$ then $\mathsf{Adj}(v) \not\subseteq \mathsf{Adj}(v')$.*

Proof. ($\rightarrow$) Let $G = CG^+(K)$ for some knowledgebase K. The empty set, $\varnothing$, is a consistent subset of every K. Hence there must be at least one maximal consistent subset. This proves condition 1). Condition 2) follows from the fact that a maximal consistent subset cannot be a subset of another maximal consistent subset.

($\leftarrow$) Let $G = \langle U, V, E\rangle$ be a bigraph that satisfies the two conditions. We show how to construct a knowledgebase K such that $G = \mathsf{CG}^+(K)$. First we do the special case where $|V| = 1$. In this case, for $|U| = m$ we choose $K = \{a_1, \ldots, a_m\}$. Now we can assume that $|V| > 1$. By condition 2) this means that $U \neq \varnothing$ and $E \neq \varnothing$. The idea of constructing K is as follows. We associate an atom a^j_i with every edge connecting u_i and v_j. Then we obtain a formula φ_i for each u_i that allows the formulas of all the vertices for every $\mathsf{Adj}(v_j)$ to be consistent but also including sufficient negated atoms so that those consistent sets are maximal.

Formally, we proceed as follows. Let $G = \langle U, V, E\rangle$ where $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. For each $\{u_i, v_j\} \in E$ we will have an atom a^j_i. Also, for each isolated vertex u_i we will have the atom a^0_i. The formula for an isolated vertex u_i, $\varphi_i = a^0_i \wedge \neg a^0_i$. For the rest of the proof we will ignore the isolated vertices. For every u_i let $A_i = \{a^j_i | a^j_i$ is an atom$\}$. Also, let $B_i = \{a^j_i | a^j_i$ is an atom such that $v_j \notin \mathsf{Adj}(u_i)\}$. Then we choose $\varphi_i = \bigwedge_{a^j_i \in A_i} a^j_i \wedge \bigwedge_{a^j_i \in B_i} \neg a^j_i$. We choose $\Delta_j = \{\varphi_i | u_i \in \mathsf{Adj}(v_j)\}$. It follows from the construction that each Δ_j must be consistent. Then, because of the negated atoms, if a formula associated with a vertex not in $\mathsf{Adj}(v_j)$ is added, the set becomes inconsistent. Hence those sets are maximal consistent. □

Corollary 2. *Let $G = \langle U, V, E\rangle$ be a bigraph. Then $G = \mathsf{CG}(K)$ for some knowledgebase K iff the following three conditions hold for G:*

1. *$V \neq \varnothing$.*

2. *For every* $u \in U$, $\mathsf{Adj}(u) \neq V$.
3. *for all* $v, v' \in V$, *if* $v \neq v'$ *then* $\mathsf{Adj}(v) \nsubseteq \mathsf{Adj}(v')$.

Proof. The only difference between $\mathsf{CG}^{+}(K)$ and $\mathsf{CG}(K)$ is that the free formulas of K are excluded from consideration for $\mathsf{CG}(K)$. In the augmented consistency graph the vertex for a free formula is adjacent to all the vertices in V. Hence the proof is the same as for the theorem but now there cannot be any vertex in U that is adjacent to all vertices in V. □

Proposition 1. *For every knowledgebase* K *and* $M \subseteq K$, $M \in \mathsf{MC}(K)$ *iff* $\mathsf{Free}(K) \subseteq M$ *and* $M \cap \mathsf{Problematic}(K) \in \mathsf{MC}(\mathsf{Problematic}(K))$.

Proof. We use the contrapositive in both directions.

($\rightarrow$) Suppose that $\mathsf{Free}(K) \nsubseteq M$. Then there must be a $\varphi \in \mathsf{Free}(K) \setminus M$, and $M \cup \{\varphi\} \subseteq K$ is consistent and $M \notin \mathsf{MC}(K)$. Now suppose that $M \cap \mathsf{Problematic}(K) \notin \mathsf{MC}(\mathsf{Problematic(K)})$. There are two possible reasons. First, suppose that $M \cap \mathsf{Problematic}(K)$ is not consistent. As M is a superset of an inconsistent set, it cannot be consistent. Now suppose that $M \cap \mathsf{Problematic}(K)$ is consistent but not maximal. Then there must be $\varphi \in \mathsf{Problematic}(K) \setminus M$ such that $(M \cap \mathsf{Problematic}(K)) \cup \{\varphi\}$ is consistent. But then $M \cup \{\varphi\} = \big((M \cap \mathsf{Problematic}(K)) \cup \{\varphi\}\big) \cup \big((M \cap \mathsf{Free}(K)) \cup \{\varphi\}\big)$ may contain only additional free formulas and hence is also consistent. Hence $M \notin \mathsf{MC}(K)$.

($\leftarrow$) Suppose that $M \notin \mathsf{MC}(K)$. There are two possible reasons for this. First, suppose that M is not consistent. Then $M \cap \mathsf{Problematic}(K)$ must already contain an inconsistency. Now suppose that M is consistent but not maximal. Then there is a $\varphi \in K$ such that $M \cup \{\varphi\} \neq M$ is consistent. If $\varphi \in \mathsf{Free}(K)$, then $\mathsf{Free}(K) \nsubseteq M$. Otherwise, it follows that $(M \cup \{\varphi\}) \cap \mathsf{Problematic}(K)$, a (consistent) subset of $M \cup \{\varphi\}$, strictly contains $M \cap \mathsf{Problematic}(K)$, hence $M \cap \mathsf{Problematic}(K) \notin \mathsf{MC}(\mathsf{Problematic}(K))$. □

Proposition 2. I_B, I_M, I_P, $I_{\#}$ *and* I_H *are IG measures.*

Proof. Let $\mathsf{IG}(K) = \langle U, V, E\rangle$ be the inconsistency graph for a knowledgebase K. Then,

1. $I_B(K) = \begin{cases} 0 & \text{if } V = \varnothing \\ 1 & \text{otherwise} \end{cases}$
2. $I_M(K) = |V|$
3. $I_P(K) = |U|$
4. $I_{\#}(K) = \begin{cases} 0 & \text{if } V = \varnothing \\ \sum_{v \in V} \frac{1}{|\,\mathsf{deg}(v)|} & \text{otherwise} \end{cases}$
5. $I_H(K) = \mathsf{min}\{|X| \mid X \subseteq U \text{ and every } v \in V \text{ is adjacent to some } u \in X\}$.

We justify the above claims as follows: (1) K is consistent iff $V = \varnothing$; (2) By the definition of $\mathsf{IG}(K)$, $|V|$ is the number of minimal inconsistent subsets of K; (3) By the definition of $\mathsf{IG}(K)$, $|U|$ is the number of problematic formulas in K; (4) The degree of a vertex $v \in V$ is the size of the corresponding minimal inconsistent subset of K; and (5) The formula computes the size of the smallest hitting set for all the minimal inconsistent subsets of K. □

Proposition 3. *An inconsistency measure is an IG measure iff it is an aIG and satisfies the independence property.*

Proof. ($\rightarrow$) If I is an IG measure then it must be an aIG as explained above. Now suppose that $\varphi \in \mathsf{Free}(K \cup \{\varphi\})$. Then $\mathsf{IG}(K) = \mathsf{IG}(K \cup \{\varphi\})$, hence $I(K) = I(K \cup \{\varphi\})$.

($\leftarrow$) Suppose that I is an aIG measure and satisfies the independence property. By independence, $I(K) = I(\mathsf{Problematic}(K))$. As I is an aIG, $I(\mathsf{Problematic}(K)) = f(\mathsf{IG}^+(\mathsf{Problematic}(K)))$ for some function f. As $\mathsf{IG}^+(\mathsf{Problematic}(K)) = IG(K)$, $I(K) = f(IG(K))$, and I is an IG measure. □

Corollary 3. *I_{nc} is not an IG measure.*

Proof. Consider the knowledgebases $K_1 = \{a \wedge \neg a\}$ and $K_2 = K_1 \cup \{b\}$ and note that $I_{nc}(K_1) = 1 < 2 = I_{nc}(K_2)$, even though b is free in K_2. Hence, I_{nc} violates the independence property, and the result follows from Proposition 3. □

Proposition 4. *I_{nc} is an aIG measure.*

Proof. Using $\mathsf{IG}^+(K) = \langle U, V, E\rangle$, $I_{nc}(K)$ can be written as:

$$I_{nc}(K) = \begin{cases} 0 & \text{if } V = \varnothing \\ |U| - \min\{\mathsf{deg}(v) \mid v \in V\} + 1 & \text{otherwise} \end{cases}$$

□

Proposition 5. *I_C is not an aIG measure.*

Proof. Consider the knowledgebases $K_1 = \{a \wedge \neg a\}$ and $K_2 = \{a \wedge \neg a \wedge b \wedge \neg b\}$. Note that $I_C(K_1) = 1 < 2 = I_C(K_2)$. However, K_1 and K_2 have identical augmented inconsistency graphs $\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_2) = \langle \{u_1\}, \{v_1\}, \{\{u_1, v_1\}\}\rangle$. □

Lemma 1. *Let $\mathsf{IG}^+(K) = \langle U, V, E\rangle$, $S \subseteq K$, and write U_S for the subset of U corresponding to the elements of S. Then S is a maximal consistent subset of K iff U_S is a maximal subset of U such that there is no $v \in V$ with $\mathsf{Adj}(v) \subseteq U_S$.*

Proof. If $V = \varnothing$ then the only maximal consistent subset is U and the result follows. Assume now that $V \neq \varnothing$.

($\rightarrow$) Let $S \subseteq K$ be consistent. Then there cannot be any $v \in V$ such that $\mathsf{Adj}(v) \subseteq U_S$. As S is maximal consistent, U_S must be a maximal subset with this property.

($\leftarrow$) Let $S \subseteq K$ be such that there is no $v \in V$ with $\mathsf{Adj}(v) \subseteq U_S$. Then S must be consistent. As U_S is a maximal subset of U with this property, it must be maximal consistent. □

Lemma 2. *Let $\mathsf{CG}^+(K) = \langle U, V, E\rangle$, $S \subseteq K$, and write U_S for the subset of U corresponding to the elements of S. Then S is a minimal inconsistent subset of K iff U_S is a minimal subset of U such that there is no $v \in V$ with $U_S \subseteq \mathsf{Adj}(v)$.*

Proof. ($\rightarrow$) Let $S \subseteq K$ be inconsistent. Then there cannot be a $v \in V$ such that $U_S \subseteq \mathsf{Adj}(v)$ because each $\mathsf{Adj}(v)$ is consistent. As S is minimal inconsistent, U_S must also be a minimal subset of U with this property.

($\leftarrow$) Let $S \subseteq K$ be such that there is no $v \in V$ with $U_S \subseteq \mathsf{Adj}(v)$. Then S must be inconsistent. Minimality for S follows because U_S is a minimal subset of U with this propeerty. □

Theorem 3. *There is a bijection $h : \mathcal{G}^+ \rightarrow \mathcal{G}_c^+$ such that, for any $K \in \mathcal{K}$, $G = \mathsf{IG}^+(K)$ iff $h(G) = \mathsf{CG}^+(K)$.*

Proof. We use Lemma 1 to construct the function h. Let $\mathsf{IG}^+(K) = G = \langle U, V, E\rangle \in \mathcal{G}^+$. We define $h(G) = \langle U, h(V), h(E)\rangle$ where $h(V)$ contains a vertex v_X for each $X \subseteq U$ such that X is a maximal subset of U with the property that there is no $v \in V$ with $\mathsf{Adj}(v) \subseteq X$. Then $h(E)$ contains edges only between each such v_X and the elements of the corresponding X. By Lemma 1, $h(G) = \mathsf{CG}^+(K)$. Next, let $\mathsf{CG}^+(K) = G = \langle U, V, E\rangle \in \mathcal{G}_c^+$. We define a function $h' : \mathcal{G}_c^+ \to \mathcal{G}^+$ as follows: $h'(G) = \langle U, h'(V), h'(E)\rangle$ where $h'(V)$ contains a vertex v_X for each $X \subseteq U$ such that X is a minimal subset of U with the property that there is no $v \in V$ with $X \subseteq \mathsf{Adj}(v)$. Then $h'(E)$ contains edges only between each such v_X and the elements of the corresponding X. By Lemma 2, $h'(G) = \mathsf{IG}^+(K)$. But actually, by the construction, $h' = h^{-1}$, that is, h' is the inverse of h; hence h is a bijection. □

Corollary 4. *There is a bijection $h : \mathcal{G} \to \mathcal{G}_c$ such that, for any $K \in \mathcal{K}$, $G = \mathsf{IG}(K)$ iff $h(G) = \mathsf{CG}(K)$.*

Proof. Just note that $\mathsf{IG}(K) = \mathsf{IG}^+(\mathsf{Problematic}(K))$ and $\mathsf{CG}(K) = \mathsf{CG}^+(\mathsf{Problematic}(K))$ for any $K \in \mathcal{K}$ and the result follows from Theorem 3. □

Corollary 5. *1. An inconsistency measure $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an aIG measure iff there is a function $g : \mathcal{G}_c^+ \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = g(\mathsf{CG}^+(K))$ for all $K \in \mathcal{K}$.*

2. An inconsistency measure $I : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an IG measure iff there is a function $g : \mathcal{G}_c \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I(K) = g(\mathsf{CG}(K))$ for all $K \in \mathcal{K}$.

Proof. This follows from Corollary 4 as we can use $f = g \circ h$ in Definition 4. □

Proposition 6. *I_A and I_{hs} are IG inconsistency measures.*

Proof. $\mathbf{I_A}$. Recall that $I_A(K) = (|\mathsf{MC}(K)| + |\mathsf{Selfcontradictions}(K)|) - 1$ for each $K \in \mathcal{K}$. In $\mathsf{CG}(K)$ each self-contradiction is an isolated vertex in U. We write $\mathsf{Iso}(U)$ for the isolated vertices of U. Then for $\mathsf{CG}(K) = \langle U, V, E\rangle$ define $g(\mathsf{CG}(K)) = |V| + |\mathsf{Iso}(U)| - 1$ and apply Corollary 5.

$\mathbf{I_{hs}}$. I_{hs} is based on the concept of a hitting set H which is a set of classical interpretations such that each formula is true in at least one element of H. In particular, this measure calculates the smallest size that H might have. The maximal sets of formulas that can be made true by an interpretation are exactly the maximal consistent sets. But these are exactly the ones represented by the elements of V in $\mathsf{CG}(K)$. Hence using $\mathsf{CG}(K)$ we calculate $I_{hs}(K) = min_{|W|}\{W \subseteq V \text{ and } U \subseteq \bigcup_{v \in W} \mathsf{Adj}(v)\} - 1$ and apply Corollary 5. □

Proposition 7. *I_η is an IG inconsistency measure.*

Proof. Let $G = \langle U, V, E\rangle$ be the consistency graph of K. As I_η is known to satisfy the independence property (see Thimm, 2016a) it suffices to deal only with the problematic formulas of K which are exactly the ones represented in U. We take care of two special cases first. If $U = \varnothing$ (K is consistent) then let $I_\eta(K) = 0$; if U contains at least one isolated node (representing a selfcontradiction) then let $I_\eta(K) = 1$.

For the rest of the proof we assume that U is not empty and does not contain an isolated vertex. The key portion of the proof is to show that $\langle K, \mathbb{I}_C, i(\varphi) = t\rangle$-satisfiability for P is equivalent to $\langle K, \mathsf{MC}(K), \in\rangle$-satisfiability. For this purpose we will construct two functions between $\mathbb{I}_C$ and $\mathsf{MC}(K)$, namely $f : \mathbb{I}_C \to \mathsf{MC}(K)$ such that $T(i) \subseteq f(i)$ for all $i \in \mathbb{I}_C$ and $g : \mathsf{MC}(K) \to \mathbb{I}_C$ such that $M \subseteq T(g(M))$ for all $M \in \mathsf{MC}(K)$.

Suppose that $\Gamma = \{P(\varphi) \geq p \mid \varphi \in K\}$ is satisfiable. Then there is a probability function $\pi : \mathbb{I}_C \to [0, 1]$ such that $P_\pi(\varphi) = \sum_{i \in \mathbb{I}_C}\{\pi(i) \mid i(\varphi) = T\} \geq p$ for all $\varphi \in K$. Let $i \in \mathbb{I}_C$.

Since $T(i)$ is a consistent subset of K there must be $M \in \mathsf{MC}(K)$ such that $T(i) \subseteq M$. For the function f pick such an M for each i so that $T(i) \subseteq M$. Now define $\delta : \mathsf{MC}(K) \to [0,1]$ as follows: $\delta(M) = \sum_{i \in \mathbb{I}_C} \{\pi(i) \mid f(i) = M\}$. This definition assures that δ is a probability function over $\mathsf{MC}(K)$ and also from Γ we obtain that $\forall \varphi \in K, \sum_{M \in \mathsf{MC}(K)} \{\delta(M) \mid \varphi \in M\} \geq p$ (by the condition that $T(i) \subseteq f(i)$ for all $i \in \mathbb{I}_C$). Hence Γ is $\langle K, \mathsf{MC}(K), \in \rangle$-satisfiable.

Going in the opposite direction suppose that Γ is $\langle K, \mathsf{MC}(K), \in \rangle$-satisfiable. Then there is a probability function over $\mathsf{MC}(K)$ such that for all $\varphi \in K$, $\sum_{M \in \mathsf{MC}(K)} \{\delta(M) \mid \varphi \in M\} \geq p$. By the consistency of M there must be $i \in \mathbb{I}_C$ such that $M \subseteq T(i)$. Pick such an i for each M to define g that is $g : \mathsf{MC}(K) \to \mathbb{I}_C$ such that $M \subseteq T(g(M))$. Then we define $\pi : \mathbb{I}_C \to [0,1]$ as $\pi(i) = \sum_{M \in \mathsf{MC}(K)} \{\delta(M) \mid g(M) = i\}$. This definition assures that π is a probability function over $\mathbb{I}$ that satisfies Γ and hence that Γ is $\langle K, \mathbb{I}_C, i(\varphi) = T \rangle$-satisfiable.

Finally, we observe that $\langle K, \mathsf{MC}(K), \in \rangle$-satisfiability is the same as $\langle G, V, u \in \mathsf{Adj}(v) \rangle$-satisfiability where G is the consistency graph of K. Hence the formula for $I_\eta(K)$ is as follows.
Let $G = \mathsf{IG}(K)$. If $U = \varnothing$ then $I_\eta(K) = 0$. If $\exists u \in U$ such that $\deg(u) = 0$, then $I_\eta(K) = 1$. Otherwise
$I_\eta(K) = 1 - (max\{\eta \in [0,1] \mid \Gamma \text{ is } \langle G, V, u \in \mathsf{Adj}(v) \rangle\text{-satisfiable for } \eta\})$. So by Corollary 5 I_η is an IG inconsistency measure. □

Proposition 8. *Let $G = \langle U, V, E \rangle$ be the inconsistency graph of a knowledgebase K where, as usual, $K = \{\varphi_1, \ldots, \varphi_m\}$ and $\mathsf{MI}(K) = \{\Delta_1, \ldots, \Delta_n\}$. Let $K' \subseteq K$. Then $G' = \langle U', V', E' \rangle$ is the inconsistency graph of K' iff G' is the bigraph U-induced from G by W where $W \subseteq U$ corresponds to the elements of K'.*

Proof. The minimal inconsistent sets of K' are exactly the minimal inconsistent sets of K all of whose elements are in K'. In the construction of Definition 9 we start by putting into V' exactly the elements of V that correspond to the minimal inconsistent sets of K'. Then we obtain U' by deleting from K' the formulas that have become free so that U' corresponds to $\mathsf{Problematic}(K')$. Finally, E' connects the appropriate edges for the vertices in U' and V' from E. □

Proposition 9. *Let $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$. $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$ is an inconsistency measure iff the following two conditions hold:*

1. *$f(G) = 0$ iff $G = \langle \varnothing, \varnothing, \varnothing \rangle$;*
2. *If $G' = \langle U', V', E' \rangle$ was U-induced by W ($W \subseteq U$) from $G = \langle U, V, E \rangle$ then $f(G') \leq f(G)$.*

Proof. The two conditions correspond exactly to the conditions of consistency and monotony of Definition 1 by the fact stated earlier that $\langle \varnothing, \varnothing, \varnothing \rangle$ is the inconsistency graph for all consistent knowledgebases only and then using Proposition 8.

□

Corollary 6. *Let $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ be such that*

1. *$f(G) = 0$ iff $G = \langle \varnothing, \varnothing, \varnothing \rangle$,*
2. *If $G' \subseteq G$ then $f(G') \leq f(G)$.*

Then I_f is an inconsistency measure.

Proof. From these two conditions, the two conditions of Theorem 9 follow. □

Proposition 10. *The following functions $f : \mathcal{G} \to \mathbb{R}^{\geq 0}_{\infty}$ defined below yield inconsistency measures $I_f : \mathcal{K} \to \mathbb{R}^{\geq 0}_{\infty}$. We put in parentheses the meaning for the corresponding knowledgebase.*

- $f_1(G) = |U| + |V|$ *(the number of problematic formulas plus the number of minimal inconsistent subsets)*

- $f_2(G) = |E|$ *(the sum of the sizes of the minimal inconsistent subsets — as specified in Definition 3, every edge in E denotes the membership of a formula in a minimally inconsistent subset.)*

- $f_3(G) = |U| + |V| + |E|$ $(f_1(G) + f_2(G))$

- $f_4(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ \sum\limits_{v \in V} \frac{\sum\limits_{u \in \mathsf{Adj}(v)} \deg(u)}{\deg(v)^2} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the sum of the reciprocals of the sizes of the minimal inconsistent subsets weighted by the average number of minimal inconsistent subsets containing their elements)

- $f_5(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ 1 + |\{u \in U \mid \mathsf{deg}(u) \geq 2\}| & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise one plus the number of formulas that are in at least two minimal inconsistent subsets)

- $f_6(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ 1 + \sum \frac{1}{|\mathsf{Adj}(v) \cap \mathsf{Adj}(v')|} (\forall v, v' \notin V, v \neq v', \mathsf{Adj}(v) \cap \mathsf{Adj}(v') \neq \varnothing) & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise one plus the sum of the reciprocals of the sizes of the intersections of each pair of minimal inconsistent subsets)

- $f_7(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ \max\{\mathsf{deg}(u) \mid u \in U\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the maximum number of minimal inconsistent subsets containing the same formula)

- $f_8(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ |\{v \in V \mid \mathsf{deg}(v) = 1\}| + \max\{\mathsf{deg}(u) \mid u \in U\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the number of self-contradictions plus the maximum number of minimal inconsistent subsets containing the same formula)

- $f_9(G) = \begin{cases} 0 & \textit{if } U = \varnothing \\ \max\{\mathsf{deg}(v) \mid v \in V\} & \textit{otherwise} \end{cases}$
  (0 if K is consistent, otherwise the maximum number of formulas in a minimal inconsistent subset)

Proof. This follows from Corollary 6. □

Proposition 11. *Consider the inconsistency measures introduced in Proposition 10. I_{f_i} satisfies Independence for $1 \leq i \leq 9$, MI-separability for $i = 2$, Penalty for $1 \leq i \leq 4$, Super-additivity for $1 \leq i \leq 6$, Attenuation for $i = 4$, Equal Conflict for $1 \leq i \leq 9$. and Almost Consistency for $i = 4$.*

Proof. (Independence) To see that all measures satisfy independence, note that they are all defined via the inconsistency graph, which does not take free formulas into account.

(MI-separability) If $\mathsf{MI}(K) \cap \mathsf{MI}(K') = \varnothing$ and $\mathsf{MI}(K \cup K') = \mathsf{MI}(K) \cup \mathsf{MI}(K')$, then $\mathsf{MI}(K), \mathsf{MI}(K')$ form a partition of $\mathsf{MI}(K \cup K')$. Therefore, the sum of the sizes of the elements of $\mathsf{MI}(K \cup K')$

(i.e., $I_{f_2}(K \cup K')$) equals the sum of sizes of the elements of $\mathsf{MI}(K)$ (i.e., $I_{f_2}(K)$) plus the sum of the sizes of the elements of $\mathsf{MI}(K')$ (i.e., $I_{f_2}(K')$), and I_{f_2} satisfies MI-separability. Now consider $K_1 = \{\neg a, a\}$, $K_2 = \{a, \neg a \wedge \neg a\}$, $K_3 = \{a \wedge \neg a\}$ and $K_4 = \{b \wedge \neg b\}$, which are all different minimal inconsistent sets. Note furthermore that $\mathsf{MI}(K_1 \cup K_2) = \mathsf{MI}(K_1) \cup \mathsf{MI}(K_2)$ and $\mathsf{MI}(K_3 \cup K_4) = \mathsf{MI}(K_3) \cup \mathsf{MI}(K_4)$. Nevertheless, $I_{f_1}(K_1 \cup K_2) = 5 \neq 3 + 3 = I_{f_1}(K_1) + I_{f_1}(K_2)$, $I_{f_3}(K_1 \cup K_2) = 9 \neq 5 + 5 = I_{f_3}(K_1) + I_{f_3}(K_2)$, $I_{f_4}(K_1 \cup K_2) = 1.5 \neq 0.5 + 0.5 = I_{f_4}(K_1) + I_{f_4}(K_2)$, $I_{f_5}(K_3 \cup K_4) = 1 \neq 1 + 1 = I_{f_5}(K_3) + I_{f_5}(K_4)$, $I_{f_6}(K_3 \cup K_4) = 1 \neq 1 + 1 = I_{f_6}(K_3) + I_{f_6}(K_4)$, $I_{f_7}(K_3 \cup K_4) = 1 \neq 1 + 1 = I_{f_7}(K_3) + I_{f_7}(K_4)$, $I_{f_8}(K_3 \cup K_4) = 3 \neq 2 + 2 = I_{f_8}(K_3) + I_{f_8}(K_4)$, and $I_{f_9}(K_3 \cup K_4) = 1 \neq 1 + 1 = I_{f_9}(K_3) + I_{f_9}(K_4)$.

(Penalty) Removing a problematic formula φ from a knowledgebase K implies discarding one vertex in U in the inconsistency graph $\mathsf{IG}(K) = \langle U, V, E\rangle$, which implies discarding also at least one edge in E and at least one vertex in V. Hence, $I_{f_i}(K \setminus \{\varphi\}) < I_{f_i}(K)$ for $1 \leq i \leq 3$. Furthermore, $I_{f_4}(K \setminus \{\varphi\})$ sums strictly fewer terms $\sum\limits_{u \in \mathsf{Adj}(v)} \mathsf{deg}(u)/\mathsf{deg}(v)^2$ than $I_{f_4}(K)$, due to the smaller $|V|$, and the numerator in each term cannot increase, for the degree of each u can only decrease when discarding a formula. As each such term is positive, $I_{f_4}(K \setminus \{\varphi\}) < I_{f_4}(K)$. To see that penalty fails for the other measures, consider $K = \{a, \neg a, b, \neg b\}$. Note that $I_{f_i}(K) = 1 = I_{f_i}(K \setminus \{a\})$ for $5 \leq i \leq 8$, and $I_{f_9}(K) = 2 = I_{f_9}(K \setminus \{a\})$.

(Super-additivity) Consider knowledgebases K_1 and K_2 such that $K_1 \cap K_2 = \varnothing$, $\mathsf{IG}(K_1) = \langle U_1, V_1, E_1\rangle$, $\mathsf{IG}(K_2) = \langle U_2, V_2, E_2\rangle$, $K = K_1 \cup K_2$ and $\mathsf{IG}(K) = \langle U, V, E\rangle$. As $|U| \geq |U_1| + |U_2|$, $|V| \geq |V_1| + |V_2|$ and $|E| \geq |E_1| + |E_2|$, $I_{f_i}(K) \geq I_{f_i}(K_1) + I_{f_i}(K_2)$ for $1 \leq i \leq 3$. Vertices $u \in U_1 \cup U_2$ with $\mathsf{deg}(u) \geq 2$ are also in U, so $I_{f_5}(K) \geq I_{f_5}(K_1) + I_{f_5}(K_2)$. Vertices $v \in V_1 \cup V_2$ are also in V, and the degree of the vertices in $u \in \mathsf{Adj}(v)$ in $\mathsf{IG}(K)$ cannot be smaller than in $\mathsf{IG}(K_1)$ or $\mathsf{IG}(K_2)$. Thus, $I_{f_4}(K) \geq I_{f_4}(K_1) + I_{f_4}(K_2)$. Note that intersections of pairs, in K_1 or K_2, of minimal inconsistent subsets will be present in K as well, therefore $I_{f_6}(K) \geq I_{f_6}(K_1) + I_{f_6}(K_2)$. Now consider $K_1 = \{a, \neg a\}, K_2 = \{b, \neg b\}$ and $K = K_1 \cup K_2$. Note that $I_{f_i}(K) = 1 < 2 = I_{f_i}(K_1) + I_{f_i}(K_2)$ for $7 \leq i \leq 8$ and $I_{f_9}(K) = 2 < 4 = I_{f_9}(K_1) + I_{f_9}(K_2)$.

(Attenuation) Consider a minimal inconsistent set K with $|K| = m > 0$, and its inconsistency graph $\mathsf{IG}(K) = \langle U, V, E\rangle$. Note that $\mathsf{deg}(u) = 1$ for all $u \in U$ and that V has a single vertex v, with $\mathsf{deg}(v) = m$. Thus, $I_{f_4}(K) = m/m^2 = 1/m$ and, the greater the m, the smaller the inconsistency measurement, satisfying attenuation. On the other hand, the measures $I_{f_1}(K) = 1+m$, $I_{f_2}(K) = m$, $I_{f_3}(K) = 1 + 2m$ and $I_{f_9}(K) = m$ actually increase when m increases, violating attenuation. For $5 \leq i \leq 7$, $I_{f_i}(K) = 1$ is constant, also violating attenuation. Finally, if $m \geq 2$, K is not a self-contradiction, and $I_{f_8}(K) = 1$ is also constant, and attenuation does not hold.

(Equal Conflict) This follows from the fact that all minimal inconsistent sets of a specific size have the same inconsistency graph.

(Almost Consistency) If K is a minimal inconsistent set then $I_{f_4}(K) = \frac{|K|}{|K|^2} = \frac{1}{|K|}$ whose limit is 0 as K increases in size. All the other measures have value at least 1 no matter the size of the minimal inconsistent set. $\square$

Proposition 12. *Both $\mathcal{K}$ and $\mathcal{G}$ are countable.*

Proof. As we are considering a countable set of atoms $\mathcal{P}$, the set $\mathcal{L}$ of all formulas is also countable. Since each element of $\mathcal{K}$ is a finite subset of $\mathcal{L}$, $\mathcal{K}$ is countable as well. As each knowledgebase has a single inconsistency graph, $|\mathcal{G}| \leq |\mathcal{K}|$; hence $\mathcal{G}$ is also countable. $\square$

Proposition 13. *There is a bijection $e : \mathcal{K} \to \mathbb{N}$ such that, if $K \subseteq K'$, then $e(K) \leq e(K')$.*

Proof. Let $e_{\mathcal{L}} : \mathcal{L} \to \mathbb{N}$ be an enumeration of $\mathcal{L}$. Define $e : \mathcal{K} \to \mathbb{N}$ as $e(K) = \sum\{2^{e_\ell(\varphi)} \mid \varphi \in K\}$ for all $K \in \mathcal{K}$. Representing $e(K)$ in binary notation, one can see that e is a bijection. Furthermore, it is clear by the construction of e that $K \subseteq K'$ implies $e(K) \leq e(K')$, as $e(K \cup \{\varphi\}) = e(K) + 2^{e_\ell(\varphi)}$ for any $\varphi \in \mathcal{L} \setminus K$. $\square$

Proposition 14. *The relation $\succeq$ is reflexive and transitive.*

Proof. In this proof we use K for an arbitrary knowledgebase. To show that $\succeq$ is reflexive we must show that for every class $C = \langle A, m_c \rangle$, $C \succeq C$. In this case $h_{C,C} : \iota_A$ (the identity function on A).

For transitivity assume that $C_1 \succeq C_2$ and $C_2 \succeq C_3$ where $C_i = \langle A_i, m_{C_i} \rangle$ for $i = 1, 2, 3$. Hence h_{C_1,C_2} and h_{C_2,C_3} are functions such that $h_{C_1,C_2} : A_1 \to A_2$ and $h_{C_2,C_3} : A_2 \to A_3$. So $m_{C_2}(K) = h_{C_1,C_2}(m_{C_1}(K))$ and $m_{C_3}(K) = h_{C_2,C_3}(m_{C_2}(K))$. Therefore,

$$m_{C_3}(K) = h_{C_2,C_3}(m_{C_2}(K)) = h_{C_2,C_3}(h_{C_1,C_2}(m_{C_1}(K))) = (h_{C_2,C_3} \circ h_{C_1,C_2})(m_{C_1}(K)).$$

So $h_{C_1,C_3} = h_{C_2,C_3} \circ h_{C_1,C_2}$ is a function (composition of functions) and hence satisfies $m_{C_3}(K) = h_{C_1,C_3}(m_{C_1}(K))$ meaning that $C_1 \succeq C_3$. $\square$

Proposition 15. *$C = \langle A, m_C \rangle$ and $C' = \langle A', m_{C'} \rangle$ are equally general iff $h_{C,C'}$ and $h_{C',C}$ are inverse functions.*

Proof. In the proof K is an arbitrary knowledgebase.
($\to$) Let C and C' be equally general. By definition there exist $h_{C,C'} : A \to A'$ such that $m_{C'}(K) = h_{C,C'}(m_C(K))$ and $h_{C',C} : A' \to A$ such that $m_C(K) = h_{C',C}(m_{C'}(K))$. Hence,
$(h_{C',C} \circ h_{C,C'})(m_C(K)) = h_{C',C}(h_{C,C'}(m_C(K))) = h_{C',C}(m_{C'}(K)) = m_C(K)$ and similarly
$(h_{C,C'} \circ h_{C',C})(m_{C'}(K)) = h_{C,C'}(h_{C',C}(m_{C'}(K))) = h_{C,C'}(m_C(K)) = m_{C'}(K)$. This shows that $h_{C,C'}$ and $h_{C',C}$ are inverse functions.
($\leftarrow$) Let $h_{C,C'}$ and $h_{C'C}$ be inverse functions. $C \succeq C'$ and $C' \succeq C$ follow from the definitions. $\square$

Corollary 7. *If $C \succeq C'$ and $h_{C,C'}$ is not one-to-one then $C \succ C'$.*

Proof. Immediate from Proposition 15. $\square$

Proposition 16. *If $C \succeq C'$ then every C' inconsistency measure is also a C inconsistency measure.*

Proof. In the proof K is an arbitrary knowledgebase. Let $C = \langle A, m_C \rangle \succeq C' = \langle A', m'_C \rangle$. Then there is a function $h_{C,C'} : A \to A'$ such that $m_{C'}(K) = h_{C,C'}(m_C(K))$. Let I' be a C' inconsistency measure. Hence there is a function $f_{C'} : A' \to \mathbb{R}^{\geq 0}_{\infty}$ such that $I'(K) = f_{C'}(m_{C'}(K))) = f_{C'}(h_{C,C'}(m_C(K))$. Choose $f_C : A \to \mathbb{R}^{\geq 0}_{\infty}$ such that $f_C = f_{C'} \circ h_{C,C'}$ and so $I(K) = f_C(m_C(K))$. Therefore I' is a C inconsistency measure. $\square$

Corollary 8. *If $C \succeq C'$ and there is a C inconsistency measure that is not a C' inconsistency measure then $C \succ C'$.*

Proof. Immediate from Proposition 16. $\square$

Proposition 17. *$C = \langle A, m_C \rangle \not\succeq C' = \langle A', m_{C'} \rangle$ iff there exist knowledgebases K and K' such that $m_C(K) = m_C(K')$ but $m_{C'}(K) \neq m_{C'}(K')$.*

Proof. By the definition of $\succeq$, $C \succeq C'$ means the existence of a function $h_{C,C'} : A \to A'$ such that for all K, $m_{C'}(K) = h_{C,C'}(m_C(K))$. This means that for any K and K', if $m_C(K) = m_C(K')$ then $m_{C'}(K) = m_{C'}(K')$. Hence such a function cannot exist iff there exist K and K' such that $m_C(K) = m_C(K')$ but $m_{C'}(K) \neq m_{C'}(K')$. $\square$

Corollary 9. *The generalities of $C = \langle A, m_C \rangle$ and $C' = \langle A', m_{C'} \rangle$ are incomparable iff*

1. *There exist K and K' such that $m_C(K) = m_C(K')$ but $m_{C'}(K) \neq m_{C'}(K')$ and*

2. *There exist K'' and K''' such that $m_{C'}(K'') = m_{C'}(K''')$ but $m_C(K'') \neq m_C(K''')$.*

Proof. Using Proposition 17 1. is equivalent to $C \not\succeq C'$ and 2. is equivalent to $C' \not\succeq C$. □

Theorem 4. *The following generality relations hold between classes where B refers to class C_B (i.e. the binary class):*

1. *IG^+ is more general than IG: $IG^+ \succ IG$.*

2. *IG is more general than VD: $IG \succ VD$.*

3. *VD is more general than each of PD, VC, and CD: $VD \succ PD, VC, CD$.*

4. *PD is more general than both PC and EC: $PD \succ PC, EC$.*

5. *VC is more general than both PC and CC: $VC \succ PC, CC$.*

6. *CD is more general than both EC and CC: $CD \succ EC, CC$.*

7. *Each of PC, EC, and CC is more general than B: $PC, EC, CC \succ B$.*

8. *The generalities of PD, VC, and CD are pairwise incomparable.*

9. *The generalities of PC, EC, and CC are pairwise incomparable.*

Proof. First we will prove cases where $C \succ C'$ and then where they are incomparable. In the first case we will start by showing that $h_{C,C'}$ is a function to establish that $C \succeq C'$. Then, in all such proofs, except for one, we will show that $h_{C,C'}$ is not one-to-one, that is, find knowledgebases K and K' such that $m_{C'(K)} = m_{C'(K')}$ but $m_{C(K)} \neq m_{C(K')}$ and use Corollary 7. For the second case we will use Corollary 9 which requires finding 4 knowledgebases on which m_C and $m_{C'}$ differ.

In this process we will use 12 knowledgebases that we present here:

$K_1 = \{a, \neg a\}$

$K_2 = \{a \wedge \neg a\}$

$K_3 = \{a, \neg a, b, \neg b\}$

$K_4 = \{a, \neg a \vee \neg b, b\}$

$K_5 = \{a, \neg a \wedge \neg b, b\}$

$K_6 = \{a \wedge \neg a, b \wedge \neg b\}$

$K_7 = \{a \wedge \neg a, b \wedge \neg b, c \wedge \neg c\}$

$K_8 = \{a, \neg a \vee b, \neg b, b \vee c, \neg c, c \vee d, \neg d\}$

$K_9 = \{a, \neg a \vee b, \neg b, \neg b \wedge e, c, \neg c \vee d, \neg d\}$

$K_{10} = \{a, \neg a, \neg a \vee b, \neg b \vee c, \neg c, c \vee d, \neg d\}$

$K_{11} = \{a, \neg a \wedge b, \neg b \wedge c, \neg c\}$

$K_{12} = \{a, b, c, (\neg a \vee \neg b) \wedge (\neg b \vee \neg c) \wedge (\neg a \vee \neg c)\}$

Next we give the definition of 5 functions that we will use to define $h_{C,C'}$. Z is a set of pairs: $Z = \{\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle\}$.

- $\mathsf{Proj}_1(\langle X, Y\rangle) = X$
- $\mathsf{Proj}_2(\langle X, Y\rangle) = Y$
- $\mathsf{MakePair}(X, Y) = \langle X, Y\rangle$
- $\mathsf{SumFirst}(Z) = \sum_{i=1}^{n} x_i$
- $\mathsf{SumProduct}(Z) = \sum_{i=1}^{n} x_i \times y_i$

Now we are ready to proceed with all the parts of the proof. To make the proof easier to read we will write $C(K)$ instead of $m_C(K)$.

1. $IG^+ \succ IG$. Here $h_{IG^+,IG}$ is the function that deletes the isolated vertices from the augmented inconsistency graph. The result follows from Proposition 4, and Corollaries 3 and 8.

2. $IG \succ VD$. The function $h_{IG,VD}$ is given in Table 3. Consider now K_8 and K_9. $VD(K_8) = VD(K_9) = \langle\{\langle 5,1\rangle, \langle 2,2\rangle\}, \{\langle 3,3\rangle\}\rangle$. However $IG(K_8) \neq IG(K_9)$ because $IG(K_9)$ has 2 V-vertices, v_1 and v_2 with edges to the same 2 U-vertices, u_1 and u_2 (corresponding to two minimal inconsistent subsets sharing two formulas) but that is not the case for $IG(K_8)$.

3. There are three parts.

   (a) $VD \succ PD$. Here $h_{VD,PD} = \mathsf{Proj}_1$. Consider K_4 and K_7. Then $PD(K_4) = PD(K_7) = \{\langle 3,1\rangle\}$ but $VD(K_4) = \langle\{\langle 3,1\rangle\}, \{\langle 1,3\rangle\}\rangle \neq VD(K_7) = \langle\{\langle 3,1\rangle\}, \{\langle 3,1\rangle\}\rangle$.

   (b) $VD \succ VC$. Here $h_{VD,VC} = \mathsf{MakePair}(\mathsf{SumFirst} \circ \mathsf{Proj}_1, \mathsf{SumFirst} \circ \mathsf{Proj}_2)$. Consider K_9 and K_{10}. Then $VC(K_9) = VC(K_{10}) = \langle 7,3\rangle$ but $VD(K_9) = \langle\{\langle 5,1\rangle, \langle 2,2\rangle\}, \{\langle 3,3\rangle\}\rangle \neq VD(K_{10}) = \langle\{\langle 5,1\rangle, \langle 2,2\rangle\}, \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle\}\rangle$.

   (c) $VD \succ CD$. Here $h_{VD,CD} = \mathsf{Proj}_2$. Consider now K_3 and K_5. Then $CD(K_3) = CD(K_5) = \{\langle 2,2\rangle\}$ but $PD(K_3) = \{\langle 4,1\rangle\} \neq PD(K_5) = \{\langle 2,1\rangle, \langle 1,2\rangle\}$ and so $VD(K_3) \neq VD(K_5)$.

4. There are two parts.

   (a) $PD \succ PC$. Here $h_{PD,PC} = \mathsf{SumFirst}$. Consider now K_4 and K_5. Then $PC(K_4) = PC(K_5) = 3$ but $PD(K_4) = \{\langle 3,1\rangle\} \neq PD(K_5) = \{\langle 2,1\rangle, \langle 1,2\rangle\}$.

   (b) $PD \succ EC$. Here $H_{PD,EC} = \mathsf{SumProduct}$. Consider K_3 and K_5. Then, $EC(K_3) = EC(K_5) = 4$ but as we showed in 3(c) $PD(K_3) \neq PD(K_5)$.

5. There are two parts.

   (a) $VC \succ PC$. Here $h_{VC,PC} = \mathsf{Proj}_1$. Consider K_4 and K_5. Then $PC(K_4) = PC(K_5) = 3$ but $VC(K_4) = \langle 3,1\rangle \neq VC(K_5) = \langle 3,2\rangle$.

   (b) $VC \succ CC$. Here $h_{VC,CC} = \mathsf{Proj}_2$. Consider K_1 and K_2. Then $CC(K_1) = CC(K_2) = 1$ but $VC(K_1) = \langle 2,1\rangle \neq VC(K_2) = \langle 1,1\rangle$.

6. There are two parts.

   (a) $CD \succ EC$. Here $h_{CD,EC} = \mathsf{SumProduct}$. Consider K_1 and K_6. Then $EC(K_1) = EC(K_6) = 2$ but $CD(K_1) = \{\langle 1,2\rangle\} \neq CD(K_6) = \{\langle 2,1\rangle\}$.

   (b) $CD \succ CC$. Here $h_{CD,CC} = \mathsf{SumFirst}$. Consider K_1 and K_2. Then $CC(K_1) = CC(K_2) = 1$ but $CD(K_1) = \{\langle 1,2\rangle\} \neq CD(K_2) = \{\langle 1,1\rangle\}$.

7. $PC, EC, CC \succ B$ Instead of showing that $PC, EC, CC \succeq B$ we show a more general result. We show that for every proper class C, $C = \langle A, m_C\rangle \succeq B = \langle\{0,1\}, I_B\rangle$ (every proper class is at least as general as the binary class). For this purpose we need to find $h_{C,B} : A \to \{0,1\}$ such that $I_B(K) = h_{C,B}(m_C(K))$. As C is a proper class there must be an inconsistency measure, say I' in C and so we can calculate $I'(K)$. We define $h_{C,B} : A \to \{0,1\}$ by setting $h_{C,B}(A) = 0$ if $I'(K) = 0$, $h_{C,B}(A) = 1$ otherwise.
   For the second part, note that $B(K_i) = 1$ for all K_i, $1 \leq i \leq 10$. But for K_1 and K_3, $PC(K_1) = 2 \neq PC(K_3) = 4$, $EC(K_1) = 2 \neq EC(K_3) = 4$, and $CC(K_1) = 1 \neq CC(K_3) = 2$.

8. There are three parts.

   (a) The generalities of PD and VC are incomparable.
   $PD(K_1) = PD(K_6) = \{\langle 2,1\rangle\}$
   but $VC(K_1) = \langle 2,1\rangle \neq VC(K_6) = \langle 2,2\rangle$
   and $VC(K_{11}) = VC(K_{12}) = \langle 4,3\rangle$
   but $PD(K_{11}) = \{\langle 2,1\rangle, \langle 2,2\rangle\} \neq PD(K_{12}) = \{\langle 3,2\rangle, \langle 1,3\rangle\}$.

   (b) The generalities of PD and CD are incomparable.
   $PD(K_8) = PD(K_{10}) = \{\langle 5,1\rangle, \langle 2,2\rangle\}$
   but $CD(K_8) = \{\langle 3,3\rangle\} \neq CD(K_{10}) = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle\}$
   and $CD(K_3) = CD(K_5) = \{\langle 2,2\rangle\}$
   but $PD(K_3) = \{\langle 4,1\rangle\} \neq PD(K_5) = \{\langle 2,1\rangle, \langle 1,2\rangle\}$.

   (c) The generalities of VC and CD are incomparable.
   $VC(K_8) = VC(K_{10} = \langle 7,3\rangle$
   but as shown in (b) $CD(K_8) \neq CD(K_{10})$
   and as shown in (b) $CD(K_3) = CD(K_5)$
   but $VC(K_3) = \langle 4,2\rangle \neq VC(K_5) = \langle 3,2\rangle$.

9. There are three parts.

   (a) The generalities of PC and EC are incomparable.
   $PC(K_4) = PC(K_5) = 3$ but $EC(K_4) = 3 \neq EC(K_5) = 4$ and
   $EC(K_3) = EC(K_5) = 4$ but $PC(K_3) = 4 \neq PC(K_5) = 3$.

   (b) The generalities of PC and CC are incomparable.
   $PC(K_4) = PC(K_5) = 3$ but $CC(K_4) = 1 \neq CC(K_5) = 2$ and
   $CC(K_1) = CC(K_2) = 1$ but $PC(K_1) = 2 \neq PC(K_2) = 1$.

(c) The generalities of EC and CC are incomparable.
$EC(K_1) = EC(K_6) = 2$ but $CC(K_1) = 1 \neq CC(K_6) = 2$ and
$CC(K_1) = CC(K_2) = 1$ but $EC(K_1) = 2 \neq EC(K_2) = 1$.

□

Proposition 18. *The classes for the 9 inconsistency measures are as follows:*

- $I_1 \in SLIG$
- $I_2 \in NLIG$
- $I_3 \in LLIG$
- $I_4 \in SLIG$
- $I_5 \in NLIG$
- $I_6 \in LLIG$
- $I_7 \in SLIG$
- $I_8 \in NLIG$
- $I_9 \in MLIG$

Proof. We write the definition of each inconsistency measure using the particular labeled inconsistency graph. In all of these definitions $I(K) = 0$ whenever $U = \varnothing$ so to make the notation easier we deal only with the case where $\mathsf{Problematic}(K) \neq \varnothing$.

$I_1(K) = |\cup_{i=1}^{m} L^S(u_i)|$
$I_2(K) = \sum_{i=1}^{m} L^N(u_i)$
$I_3(K) = \sum_{i=1}^{m} L^L(u_i)$
$I_4(K) = \max_{1 \leq i \leq m}\{|L^S(u_i)|\}$
$I_5(K) = \max_{1 \leq i \leq m}\{L^N(u_i)\}$
$I_6(K) = \max_{1 \leq i \leq m}\{L^L(u_i)\}$
$I_7(K) = \max_{1 \leq j \leq n}\{|L^S(v_j)|\}$
$I_8(K) = \max_{1 \leq j \leq n}\{L^N(v_j)\}$
$I_9(K) = 1 + |\{u \mid u \in U, L^M(u) \text{ is not a set}^3\}|$ □

Theorem 5. *The following generality relations hold among classes:*

1. *LIG^+ is more general than LIG and IG^+: $LIG^+ \succ LIG, IG^+$.*
2. *LIG is more general than $MLIG$ and $LLIG$: $LIG \succ MLIG, LLIG$.*
3. *$MLIG$ is more general than $SLIG$ and $NLIG$: $MLIG \succ SLIG, NLIG$.*
4. *$SLIG$, $NLIG$, and $LLIG$ are more general than IG: $SLIG, NLIG, LLIG \succ IG$.*
5. *The generalities of LIG and IG^+ are incomparable.*

3. i.e. an atom is occuring twice.

6. *The generalities of $MLIG$, $LLIG$ and IG^+ are pairwise incomparable.*

7. *The generalities of $SLIG$, $NLIG$, $LLIG$ and IG^+ are pairwise incomparable.*

Proof. The process is similar to what we did in Theorem 4. First we will prove cases where $C \succ C'$ and then where they are incomparable. In the first case we will start by showing that $h_{C,C'}$ is a function to establish that $C \succeq C'$. Then, we will show that $h_{C,C'}$ is not one-to-one, that is, find knowledgebases K and K' such that $m_{C'(K)} = m_{C'(K')}$ but $m_{C(K)} \neq m_{C(K')}$ and use Corollary 7. For the second case we will use Corollary 9 which requires finding 4 knowledgebases on which m_C and $m_{C'}$ differ.

In this process we will use 7 knowledgebases (note the relevance of the parentheses) that we present here:

$K_1 = \{a, \neg a\}$

$K_2 = \{a, \neg a, b\}$

$K_3 = \{b, \neg b\}$

$K_4 = \{a \wedge a, \neg a\}$

$K_5 = \{a \vee a, \neg a\}$

$K_6 = \{(a), (\neg a)\}$

$K_7 = \{a \wedge b, (\neg a)\}$

As in the previous proof again we will write $C(K)$ instead of $m_C(K)$. In order to make the proof easier to follow here we compute for each class C in the theorem $C(K)$. However, these are all graphs and we don't want to take up so much space drawing all these different inconsistency graphs. Instead we will just write what is in the labels.

For $LIG^+(K_1)$ and $LIG(K_1)$: $L(u_1) = a$, $L(u_2) = \neg a$, $L(v_1) = \{a, \neg a\}$.
$L^M(u_1) = [a]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a]$.
$L^S(u_1) = \{a\}$, $L^S(u_2) = \{a\}$, $L^S(v_1) = \{a\}$.
$L^N(u_1) = 1$, $L^N(u_2) = 1$, $L^N(v_1) = 2$.
$L^L(u_1) = 1$, $L^L(u_2) = 2$, $L^L(v_1) = 3$.

For $LIG^+(K_2)$: $L(u_1) = a$, $L(u_2) = \neg a$, $L(u_3) = b$.
For $LIG(K_2)$: $L(u_1) = a$, $L(u_2) = \neg a$.
The rest are the same for $LIG^+(K_2)$ and $LIG(K_2)$.
$L(v_1) = \{a, \neg a\}$.
$L^M(u_1) = [a]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a]$.
$L^S(u_1) = \{a\}$, $U^S(u_2) = \{a\}$, $L^S(v_1) = \{a\}$.
$L^N(u_1) = 1$, $U^N(u_2) = 1$, $L^N(v_1) = 2$.
$L^L(u_1) = 1$, $L^L(u_2) = 2$, $L^L(v_1) = 3$.

For $LIG^+(K_3)$ and $LIG(K_3)$: $L(u_1) = b$, $L(u_2) = \neg b$, $L(v_1) = \{b, \neg b\}$.
$L^M(u_1) = [b]$, $L^M(u_2) = [b]$, $L^M(v_1) = [b, b]$.
$L^S(u_1) = \{b\}$, $L^S(u_2) = \{b\}$, $L^S(v_1) = \{b\}$.
$L^N(u_1) = 1$, $L^N(u_2) = 1$, $L^N(v_1) = 2$.
$L^L(u_1) = 1$, $L^L(u_2) = 2$, $L^L(v_1) = 3$.

For $LIG^+(K_4)$ and $LIG(K_4)$: $L(u_1) = a \wedge a$, $L(u_2) = \neg a$, $L(v_1) = \{a \wedge a, \neg a\}$.
$L^M(u_1) = [a, a]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a, a]$.
$L^S(u_1) = \{a\}$, $L^S(u_2) = \{a\}$, $L^S(v_1) = \{a\}$.
$L^N(u_1) = 2$, $L^N(u_2) = 1$, $L^N(v_1) = 3$.
$L^L(u_1) = 3$, $L^L(u_2) = 2$, $L^L(v_1) = 5$.

For $LIG^+(K_5)$ and $LIG(K_5)$: $L(u_1) = a \vee a$, $L(u_2) = \neg a$, $L(v_1) = \{a \vee a, \neg a\}$.
$L^M(u_1) = [a, a]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a, a]$.
$L^S(u_1) = \{a\}$, $L^S(u_2) = \{a\}$, $L^S(v_1) = \{a\}$.
$L^N(u_1) = 2$, $L^N(u_2) = 1$, $L^N(v_1) = 3$.
$L^L(u_1) = 3$, $L^L(u_2) = 2$, $L^L(v_1) = 5$.

For $LIG^+(K_6)$ and $LIG(K_6)$: $L(u_1) = (a)$, $L(u_2) = (\neg a)$, $L(v_1) = \{(a), (\neg a)\}$.
$L^M(u_1) = [a]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a]$.
$L^S(u_1) = \{a\}$, $L^S(u_2) = \{a\}$, $L^S(v_1) = \{a\}$.
$L^N(u_1) = 1$, $L^N(u_2) = 1$, $L^N(v_1) = 2$.
$L^L(u_1) = 3$, $L^L(u_2) = 4$, $L^L(v_1) = 7$.

For $LIG^+(K_7)$ and $LIG(K_7)$: $L(u_1) = a \wedge b$, $L(u_2) = (\neg a)$, $L(v_1) = \{a \wedge b, (\neg a)\}$.
$L^M(u_1) = [a, b]$, $L^M(u_2) = [a]$, $L^M(v_1) = [a, a, b]$.
$L^S(u_1) = \{a, b\}$, $L^S(u_2) = \{a\}$, $L^S(v_1) = \{a, b\}$.
$L^N(u_1) = 2$, $L^N(u_2) = 1$, $L^N(v_1) = 3$.
$L^L(u_1) = 3$, $L^L(u_2) = 4$, $L^L(v_1) = 7$.

Next we give the definition of three functions that we will use to define $h_{C,C'}$: $\mathsf{Multiset}$ where $\mathsf{Multiset}(\varphi)$ is the multiset of atoms of φ and $\mathsf{Multiset}(\Delta)$ is the multiset of atoms of all the formulas in Δ; Length where $\mathsf{Length}(\varphi)$ is the length of φ and $\mathsf{Length}(\Delta)$ is the sum of the lengths of all the formulas in Δ; and $\mathsf{MultiToSet}$ where $\mathsf{MultiToSet}(S)$ is the set obtained from the multiset S.

Now we are ready to proceed with all the parts of the proof.

1. There are two parts.

   (a) $LIG^+ \succ LIG$. Here $h_{LIG^+,LIG}$ is the function that deletes the isolated vertices. Then $\mathsf{LIG}(K_1) = \mathsf{LIG}(K_2)$ but $\mathsf{LIG}^+(K_1) \neq \mathsf{LIG}^+(K_2)$.

   (b) $LIG^+ \succ IG^+$. Here h_{LIG^+,IG^+} is the function that omits the labels. Then $\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_3)$ but $\mathsf{LIG}^+(K_1) \neq \mathsf{LIG}^+(K_3)$.

2. There are two parts.

   (a) $LIG \succ MLIG$. Here $h_{LIG,MLIG}(L(u_i)) = \mathsf{Multiset}(L(u_i))$ for all $i, 1 \leq i \leq m$ and $h_{LIG,MLIG}(L(v_j)) = \mathsf{Multiset}(L(v_j))$ for all $j, 1 \leq j \leq n$.
   Then $\mathsf{MLIG}(K_4) = \mathsf{MLIG}(K_5)$ but $\mathsf{LIG}(K_4) \neq \mathsf{LIG}(K_5)$.

   (b) $LIG \succ LLIG$. Here $h_{LIG,LLIG}(L(u_i)) = \mathsf{Length}(L(u_i))$ for all $i, 1 \leq i \leq m$ and $h_{LIG,LLIG}(L(v_j)) = \mathsf{Length}(L(v_j))$ for all $j, 1 \leq j \leq n$.
   Then $\mathsf{LLIG}(K_1) = \mathsf{LLIG}(K_3)$ but $\mathsf{LIG}(K_1) \neq \mathsf{LIG}(K_3)$.

3. There are two parts.

(a) $MLIG \succ SLIG$. Here $h_{MLIG,SLIG}(L(u_i)) = \mathsf{MultiToSet}(L(u_i))$ for all $i, 1 \leq i \leq m$ and $h_{MLIG,SLIG}(L(v_j)) = \mathsf{MultiToSet}(L(v_j))$ for all $j, 1 \leq j \leq n$.
Then $\mathsf{SLIG}(K_1) = \mathsf{SLIG}(K_4)$ but $\mathsf{MLIG}(K_1) \neq \mathsf{MLIG}(K_4)$.

(b) $MLIG \succ NLIG$. Here $h_{MLIG,NLIG}(L^M(u_i)) = |L^M(u_i)|$ for all $i, 1 \leq i \leq m$ and $h_{MLIG,NLIG}(L^M(v_j)) = |L^M(v_j)|$ for all $j, 1 \leq j \leq n$.
Then $\mathsf{NLIG}(K_1) = \mathsf{NLIG}(K_3)$ but $\mathsf{MLIG}(K_1) \neq \mathsf{MLIG}(K_3)$.

4. There are three parts. For each part $h_{C,C'}$ deletes the labels.

(a) $\mathsf{IG}(K_1) = \mathsf{IG}(K_3)$ but $\mathsf{SLIG}(K_1) \neq \mathsf{SLIG}(K_3)$.

(b) $\mathsf{IG}(K_1) = \mathsf{IG}(K_4)$ but $\mathsf{NLIG}(K_1) \neq \mathsf{NLIG}(K_4)$.

(c) $\mathsf{IG}(K_1) = \mathsf{IG}(K_4)$ but $\mathsf{LLIG}(K_1) \neq \mathsf{LLIG}(K_4)$.

5. The generalities of LIG and IG^+ are incomparable.
$\mathsf{LIG}(K_1) = \mathsf{LIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_3)$ but $\mathsf{LIG}(K_1) \neq \mathsf{LIG}(K_3)$.

6. There are three parts.

(a) The generalities of $MLIG$ and $LLIG$ are incomparable.
$\mathsf{LLIG}(K_1) = \mathsf{LLIG}(K_3)$ but $\mathsf{MLIG}(K_1) \neq \mathsf{MLIG}(K_3)$ and
$\mathsf{MLIG}(K_1) = \mathsf{MLIG}(K_6)$ but $\mathsf{LLIG}(K_1) \neq \mathsf{LLIG}(K_6)$ and

(b) The generalities of $MLIG$ and IG^+ are incomparable.
$\mathsf{MLIG}(K_1) = \mathsf{MLIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_3)$ but $\mathsf{MLIG}(K_1) \neq \mathsf{MLIG}(K_3)$.

(c) The generalities of $LLIG$ and IG^+ are incomparable.
$\mathsf{LLIG}(K_1) = \mathsf{LLIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_4)$ but $\mathsf{MLIG}(K_1) \neq \mathsf{MLIG}(K_4)$.

7. There are six parts.

(a) The generalities of $SLIG$ and $NLIG$ are incomparable.
$\mathsf{SLIG}(K_1) = \mathsf{SLIG}(K_4)$ but $\mathsf{NLIG}(K_1) \neq \mathsf{NLIG}(K_4)$ and
$\mathsf{NLIG}(K_1) = \mathsf{NLIG}(K_3)$ but $\mathsf{SLIG}(K_1) \neq \mathsf{SLIG}(K_3)$.

(b) The generalities of $SLIG$ and $LLIG$ are incomparable.
$\mathsf{SLIG}(K_1) = \mathsf{SLIG}(K_4)$ but $\mathsf{LLIG}(K_1) \neq \mathsf{LLIG}(K_4)$ and
$\mathsf{LLIG}(K_1) = \mathsf{LLIG}(K_3)$ but $\mathsf{SLIG}(K_1) \neq \mathsf{SLIG}(K_3)$.

(c) The generalities of $SLIG$ and IG^+ are incomparable.
$\mathsf{SLIG}(K_1) = \mathsf{SLIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_3)$ but $\mathsf{SLIG}(K_1) \neq \mathsf{SLIG}(K_3)$.

(d) The generalities of $NLIG$ and $LLIG$ are incomparable.
$\mathsf{NLIG}(K_1) = \mathsf{NLIG}(K_6)$ but $\mathsf{LLIG}(K_1) \neq \mathsf{LLIG}(K_6)$ and
$\mathsf{LLIG}(K_6) = \mathsf{LLIG}(K_7)$ but $\mathsf{SLIG}(K_6) \neq \mathsf{SLIG}(K_7)$.

(e) The generalities of $NLIG$ and IG^+ are incomparable.
$\mathsf{NLIG}(K_1) = \mathsf{NLIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_4)$ but $\mathsf{NLIG}(K_1) \neq \mathsf{NLIG}(K_4)$.

(f) The generalities of $LLIG$ and IG^+ are incomparable.
$\mathsf{LLIG}(K_1) = \mathsf{LLIG}(K_2)$ but $\mathsf{IG}^+(K_1) \neq \mathsf{IG}^+(K_2)$ and
$\mathsf{IG}^+(K_1) = \mathsf{IG}^+(K_4)$ but $\mathsf{LLIG}(K_1) \neq \mathsf{LLIG}(K_4)$.

□

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