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Preference Orders on Families of Sets—When Can Impossibility Results Be Avoided?

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Abstract

Lifting a preference order on elements of some universe to a preference order on *subsets* of this universe is often guided by postulated properties the lifted order should have. Well-known impossibility results pose severe limits on when such liftings exist if *all* non-empty subsets of the universe are to be ordered. The extent to which these negative results carry over to other families of sets is not known. In this paper, we consider families of sets that induce connected subgraphs in graphs. For such families, common in applications, we study whether lifted orders satisfying the well-studied axioms of dominance and (strict) independence exist for *every* or, in another setting, for *some* underlying order on elements (*strong* and *weak orderability*). We characterize families that are strongly and weakly orderable under dominance and strict independence, and obtain a tight bound on the class of families that are strongly orderable under dominance and independence.

1. Introduction

When agents, individually or as a group, make decisions to select one of several options, they refer to their preference orders (or rankings) of the available choices. In a single-agent setting, the agent simply selects an option that she prefers the most. In a group setting, the agents submit their preference orders, or votes, to a voting rule, which determines the option to select.

In many situations, having a preference order on individual objects is not enough for decision making, and the ability to compare *sets* of alternatives is needed. For instance, when an agent is to select a set of objects subject to some constraints, as in the *knapsack* problem, that agent must have a preference order on the family of feasible sets. Similarly, in the problem of fair allocation of indivisible goods (Bouveret, Chevaleyre, & Maudet, 2016), knowing how agents rank sets of goods is necessary to ensure that the goods are distributed fairly. Some problems involving strategic behaviors in voting (Barberà, 1977; Fishburn, 1977; Bossert, 1989; Brandt & Brill, 2011; Brandt, Saile, & Stricker, 2018) and problems arising when determining optimal matchings and assignments (Roth & Sotomayor, 1990) also require the knowledge of agents' preferences on collections of objects. Finally, preferences on sets of outcomes are needed in decision making, when there is uncertainty about the consequences of an action (Larbi, Konieczny, & Marquis, 2010).

However, often the number of possible sets of alternatives makes explicit enumerations of preference orders or rankings infeasible. To circumvent this problem, researchers proposed that agents' true preference order on sets be approximated by an order that can be derived from their preference order on individual objects. If such an order is given as a utility function on objects, the utility function (preference) on *sets* of objects can be derived assuming, say, some form of additivity. This is a common setting, used for instance in the knapsack problem and fair division.

An alternative and more abstract framework, known as the *ordinal* setting, assumes that preferences on objects in some set X are represented by an order relation on X. The objective is to *lift* this order to an order on a family of non-empty subsets of X. The problem of lifting an order relation on X to an order on the family of *all* non-empty subsets of X has been extensively studied. The paper by Barberà, Bossert, and Pattanaik (2004) provides an excellent extensive overview of this research area. The results can roughly be divided into two groups, those concerned with properties of specific ways to lift an order on objects to an order on sets of objects, and those following the "axiomatic" approach, where one postulates desirable properties a lifted order should have and seeks conditions that would guarantee the existence of such a lifting (Barberà, 1977; Barberà, Barret, & Pattanaik, 1977; Moretti & Tsoukiàs, 2012). Among these properties *dominance* and *independence* are the most studied ones. Informally speaking, dominance ensures that adding an element which is better (worse) than all elements in a set, makes the augmented set better (worse) than the original one. Independence, on the other hand, states that adding a new element a to sets A and B where A is already known to be preferred over B, must not make $B \cup \{a\}$ be preferred over $A \cup \{a\}$ (or, in the strict variant, $A \cup \{a\}$ should remain preferred over $B \cup \{a\}$). A further basic property, called *extension* rule, states that the singletons $\{a\}$ and $\{b\}$ have to be ordered the same way as elements a and b are ordered in the underlying order. In what follows, we shall call properties like dominance, independence, and extensions simply *axioms*.

The most striking results in the group of axiomatic approaches are known as *impossibility* theorems. They say that some natural desiderata are inherently incompatible and cannot be achieved together (Kannai & Peleg, 1984; Barberà et al., 2004; Geist & Endriss, 2011). For instance, given an ordered set X with $|X| \geq 6$, orders on $\mathcal{P}(X) \setminus \{\emptyset\}$ satisfying dominance and independence are not possible (Kannai & Peleg, 1984). Since these impossibility results usually seek liftings to the family of *all* non-empty subsets of a set, they put very strong constraints on the lifted order, constraints that cannot be satisfied together. However, one is often only interested in comparing sets from much smaller families of sets. This will be the setting we study here.

Indeed, if the set of indivisible goods are offices and labs in a new research building and agents are research groups, it is natural to only consider allocations that form topologically contiguous areas. For instance, if the building consists of a single long hall of rooms, legal allocations are only those that split this hall into segments. In such situations, only preferences that research groups may have on contiguous segments of rooms need to be taken into account (Bouveret, Cechlárová, Elkind, Igarashi, & Peters, 2017). For another example, we might consider a problem of farmland fragmentation, where individual farms consist of many small non-contiguous plots of land as the result of divisions of farms among heirs, and acquiring ownership through marriage (King & Burton, 1982). Land consolidation was proposed as a method to improve economic performance. The objective of land consolidation is to reallocate the plots so that they form large contiguous land areas. In both cases, the topology of the set of goods can be modeled by a graph and valid sets of goods are those that induce in this graph a connected subgraph.

The question we are concerned here is whether the impossibility results still hold when the goal is to lift an order on a set X (of goods) to specific collections of subsets of X, namely those determined by the condition of connectivity in a given graph on X. More precisely, we seek characterizations of graphs (topologies) when the impossibility results still hold and those, for which lifting to orders satisfying prescribed postulates is possible. To this end, we introduce the key notions of strongly and weakly orderable graphs. A graph is *strongly* (resp. *weakly*) orderable with respect to a set $\mathcal{A}$ of axioms if every (resp. at least one) linear order on X can be lifted to an order on the collection of all sets inducing in the graph connected subgraphs, such that the lifted order satisfies axioms $\mathcal{A}$. The motivation for studying both variants of orderability is illustrated by a two-step system for delivering a ranking on a collection $\mathcal{C}$ of all sets that induce a connected subgraph in a given graph. In the first step, the user specifies the graph defining the corresponding collection $\mathcal{C}$, and the check is made whether $\mathcal{C}$ is weakly orderable or strongly orderable with respect to the desired axioms $\mathcal{A}$. If $\mathcal{C}$ is not even *weakly* orderable with respect to $\mathcal{A}$, the system reports that no matter which preference order on individual elements is provided by the user, the obtained ranking will violate the axioms (the user hence might then be asked to select a weaker set of axioms or adjust the graph); on the other hand, if $\mathcal{C}$ turns out to be *strongly* orderable with respect to $\mathcal{A}$, the user is asked for a ranking on the individual elements with the guarantee that no matter how the ranking is chosen, a lifted ranking satisfying $\mathcal{A}$ will be produced and delivered.

We summarize the main contributions of our paper next.

1. We show that the disjoint union of orderable graphs yields an orderable graph as well. This enables us to fully describe strong and weak orderability by characterizing the two concepts for connected graphs.

2. We fully characterize orderable connected graphs with respect to the axioms of dominance and strict independence. For these two axioms, the class of strongly orderable graphs is that of trees and the class of weakly orderable graphs is that of connected bipartite graphs. This also holds if, in addition, the axiom of (strong) extension is required.

3. We show that weakening strict independence to independence has minimal effect on strong orderability. In combination with strong extension, we show that the only additional connected strongly orderable graph that arises is the complete graph K_3. Furthermore, we give a full characterization of strong orderability with respect to dominance and independence for *two-connected* graphs. Here we observe that, except for some smaller special cases, two-connected graphs are strongly orderable with respect to dominance and independence if and only if they are cycles or if they do not contain a cycle of length five or more. This result holds also if we additionally require

the extension axiom. Finally, we give a nearly complete picture for strong orderability with respect to dominance and independence and with respect to dominance, independence and extension for arbitrary graphs.

An interesting implication of our results is that weakening strict independence to independence yields only a modest extension of the class of strongly orderable graphs. It points to strict independence being perhaps more essential for strong orderability than its weaker and more commonly studied version.

Although we have focused here on graphs to represent the sets of elements to be ordered in an implicit and compact way, we believe that our work has impact on further aspects of ongoing research on preferential reasoning in the field of AI. Indeed, implicit preference models are important for representing, eliciting and using preferences in practical applications. As an example, we mention here the work on preferences in Answer-Set Programming (see e.g. Brewka, Niemelä, & Truszczyński, 2003; Faber, Truszczyński, & Woltran, 2013) where logic programs compactly represent the sets to be ordered and languages as the one by Brewka et al. (2003) allow to express preferences over the individual elements, i.e. the atoms in the program. To this date, it is unclear whether the rankings obtained by such formalisms satisfy desirable properties as the ones discussed above. Our work thus can be seen as a starting point for more general investigations on (im)possibility results in formalisms from the areas in AI and KR.

More generally, our paper can also be seen as a contribution to an area of research in AI concerned with models of preferences for *combinatorial domains*. Alternatives or outcomes, both terms commonly used for objects of a combinatorial domain, are defined in terms of *attributes*, each alternative being a vector of attribute values. Collections of subsets of a given set X, which are central to our work, are examples of combinatorial domains (each element $x \in X$ can be seen an attribute that can take 0 or 1 as its value to indicate absence or presence of x in a particular subset of X). As we already observed for spaces of subsets of a set, the number of alternatives in combinatorial domains grows exponentially with the number of attributes. This precludes an explicit enumeration as a practical model of preferences over a combinatorial domain and brings about the need for concise *implicit* preference models. The topic has been extensively studied (Domshlak, Hüllermeier, Kaci, & Prade, 2011; Kaci, 2011). Most prominent approaches build on logical languages (Dubois & Prade, 1991; Brewka, Benferhat, & Berre, 2004; Brewka et al., 2003) or employ intuitive graphical representations such as lexicographic trees (Booth, Chevaleyre, Lang, Mengin, & Sombattheera, 2010; Bräuning & Hüllermeier, 2012; Liu & Truszczynski, 2015) and CP-nets (Boutilier, Brafman, Domshlak, Hoos, & Poole, 2004). Our research addresses the same issue but proposes a different approach. It can be seen as a study of the question whether a preference order on a collection of subsets of a set could be obtained by lifting a strict preference order on elements of that set in a way to satisfy some natural postulates *lifted* preference orders on collections of sets should satisfy. Whenever it is the case, the original order on elements of a set can serve as a concise representation of the one *lifted* to a much larger domain of subsets of that set. As we note above, in the present paper, we concentrate purely on the question whether lifting is possible at all. The question how to reason about lifted orders based on the "ground" information about preferences on elements is left for future research.

This paper extends the earlier conference version (Maly, Truszczyński, & Woltran, 2018). It contains all full proofs, additional examples and illustrations. We provide here also new results that sharpen the frontier between strongly orderable graphs under dominance and (the non-strict version of) independence and graphs which are not strongly orderable in this sense. First, we give a complete characterization of strong orderability under dominance and independence for two-connected graphs in Theorem 29 based on several propositions including two new ones, Propositions 26 and 27, and a new computer search argument given in the appendix. This characterization also holds if we add the extension axiom. Furthermore, we improve our understanding of strong orderability with respect to dominance, independence and extension for arbitrary graphs with a new result stated as Proposition 30 and give a new, more precise summary of our knowledge about strong orderability with respect to dominance and independence and with respect to dominance, independence and extension in Theorems 34 and 35.

2. Background

All sets we consider in the paper are finite. A binary relation is called an *order* if it is reflexive, transitive and total.[1] An order is *linear* if it is also antisymmetric. If $\preceq$ is an order on a set X, the corresponding *strict* order $\prec$ on X is defined by $x \prec y$ if $x \preceq y$ and $y \not\preceq x$, where x, y are arbitrary elements of X; the corresponding *equivalence* or *indifference* relation $\sim$ is defined by $x \sim y$ if $x \preceq y$ and $y \preceq x$. If $\preceq$ is linear then $x \sim y$ holds only if $x = y$. We call the linear order $1 < 2 < 3 < \dots$ the *natural linear order* on the natural numbers. If objects are identified with the natural numbers then we also call this order the natural order on these objects.

For a linear order $\preceq$ on a set A, we write $\max_{\preceq}(A)$ for the maximal element of A with respect to $\preceq$. Similarly, we write $\min_{\preceq}(A)$ for the minimal element of A with respect to $\preceq$. If no ambiguity arises, we drop the reference to the relation from the notation.

Given a set X and a linear order $\leq$ on X, the *order lifting* problem consists of deriving from $\leq$ an order $\preceq$ on a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ of non-empty *subsets* of X, guided by axioms formalizing some natural desiderata for such lifted orders. Observe that this does not mean that every axiom is desirable in every situation. We shortly discuss the conditions under which each of the axioms can be considered desirable below. For a nuanced discussion on the applicability of these axioms see the paper by Barberà et al. (2004). We recall several axioms in Figure 1. They are natural extensions of the versions of those axioms considered in the case when $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ (cf. Barberà et al., 2004). The extensions consist of adding conditions of the form $Y \in \mathcal{X}$ not needed in the original formulations (cf. Maly & Woltran, 2017).

The *extension rule* states that the singletons in $\mathcal{X}$ have to be ordered the same way as the elements in X are ordered. This is a natural requirement that is desirable in most interpretations of the order lifting problem, though there are some notable exceptions like the interpretation of $\preceq$ as a measure of freedom of choice (Pattanaik & Xu, 1990).

1. Orders are also called weak orders or total preorders.

Axiom 1 (Extension Rule)**.** For all $x, y \in X$, such that $\{x\}, \{y\} \in \mathcal{X}$:

$$x < y \text{ implies } \{x\} \prec \{y\}.$$

Axiom 2 (Strong Extension)**.** For all $A, B \in \mathcal{X}$:

$$\max(A) < \max(B) \text{ implies } A \prec B.$$

Axiom 3 (Dominance)**.** For all $A \in \mathcal{X}$ and all $x \in X$, such that $A \cup \{x\} \in \mathcal{X}$:

$$y < x \text{ for all } y \in A \text{ implies } A \prec A \cup \{x\};$$

$$x < y \text{ for all } y \in A \text{ implies } A \cup \{x\} \prec A.$$

Axiom 4 (Independence)**.** For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$, such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \prec B \text{ implies } A \cup \{x\} \preceq B \cup \{x\}.$$

Axiom 5 (Strict Independence)**.** For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$, such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \prec B \text{ implies } A \cup \{x\} \prec B \cup \{x\}.$$

Figure 1: Axioms for lifting orders

Dominance[2] states that adding a better element to a set increases the quality of the set and adding a worse element decreases the quality of the set. This principle is often desirable if the order $\preceq$ should reflect, to some extent, the average quality of the sets. If we assume, for example, that the sets represent incompatible alternatives from which one will be chosen randomly, then dominance is a natural desideratum (Can, Erdamar, & Sanver, 2009).

Independence and *strict independence* are natural monotonicity axioms that require that adding the same element to two sets does not reverse a strict preference in the case of independence, and does not change a strict preference in the case of strict independence. These are natural desiderata in many interpretations, for example if sets are bundles of objects that are compared according to their overall goodness according to some additive utility (Kraft, Pratt, & Seidenberg, 1959).

There is some tension between the motivations for dominance and (strict) independence, as dominance is more related to average utility while independence and strict independence are more related to total utility. Nevertheless, there are cases where both axioms are natural desiderata. These cases are often characterized by the fact that all elements may influence the quality of a set but the extent of this influence is unknown or unknowable. An example for such a situation might be choice under complete uncertainty:

2. The axion is often also called Gärdenfors principle after Peter Gärdenfors who introduced a version of the axiom (Gärdenfors, 1976).

Example 1. Consider a situation where an agent can perform different actions $a_1, \ldots, a_k$ for which he knows the (set of) possible outcomes but he is not able or not willing to determine the (approximate) probability of each outcome. Such a situation can be modeled as a family of outcomes $X = \{o_1, o_2, \ldots, o_l\}$ and a function $O : \{a_1, \ldots, a_k\} \to \mathcal{P}(X) \setminus \{\emptyset\}$ that maps every action to the set of possible outcomes of that action. If we assume that the agent has preferences over the set of possible outcomes X that can be modeled as a linear order, the problem of ranking the different actions can be modeled as an order lifting problem. Under this interpretation the extension rule, dominance and independence are usually considered natural desiderata (Bossert, Pattanaik, & Xu, 2000; Barberà et al., 2004). □

Finally, strong extension states that a set A is preferred to a set B if the maximal element of A is larger than the maximal element of B. This axiom can be considered reasonable, for example, in a choice under complete uncertainty situation where all outcomes have a positive but vastly different utility. Note that strong extension implies the extension rule. Furthermore, dominance and independence together imply strong extension, if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. One could also define a dual version of strong extension based on the minima of A and B. Because all problems in this paper are symmetric, we can use either version without loss of generality. The two dual strong extension axioms are strict versions of the well known Hoare and Smyth axioms (discussed in particular by Brewka, Truszczynski, and Woltran 2010) restricted to linear orders.

Example 2. Take $X = \{1, 2, 3, 4\}$ with the natural linear order $\leq$ and

$$\mathcal{X} = \{\{2\}, \{4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 4\}\}.$$

The axioms impose constraints on any lifted order $\preceq$ on $\mathcal{X}$. In particular, the extension rule implies $\{2\} \prec \{4\}$, while strong extension additionally implies $\{2\} \prec A$ for every $A \in \mathcal{X} \setminus \{\{2\}\}$. Dominance implies $\{2\} \prec \{2, 4\} \prec \{4\}$, $\{3, 4\} \prec \{4\}$, $\{1, 2, 4\} \prec \{2, 4\}$ and $\{1, 4\} \prec \{4\}$, and (strict) independence lifts the preference between $\{2, 4\}$ and $\{4\}$ to $\{1, 2, 4\}$ and $\{1, 4\}$. Thus, dominance and independence imply $\{1, 2, 4\} \preceq \{1, 4\}$, and dominance and strict independence imply $\{1, 2, 4\} \prec \{1, 4\}$. □

There are well known orders that satisfy most of the axioms that we consider. For example, it is easy to see that the lexicographic order satisfies the strong extension and hence the extension rule as well as independence and strict independence.

On the other hand, consider the minmax-order $\preceq_{mm}$ defined by $A \preceq_{mm} B$ if and only if one of the following holds:

- $\max(A) < \max(B)$,
- $\max(A) = \max(B)$ and $\min(B) \leq \min(B)$.

The minmax-order clearly satisfies strong extension and hence the extension rule. Furthermore, it satisfies dominance as $y < x$ for all $y \in A$ implies $\max(A) < \max(A \cup \{x\})$ and $x < y$ for all $y \in A$ implies $\max(A) = \max(A \cup \{x\})$ and $\min(A \cup \{x\}) < \max(A)$. However, the minmax-order does not necessarily satisfy independence. Take, for example,

$X = \{1, 2, 3, 4\}$ and let $\leq$ be the natural linear order on X. Then $\{2\} \prec_{mm} \{1, 3\}$ but $\{1, 3, 4\} \prec_{mm} \{2, 4\}$.

However, as it turns out, dominance and (strict) independence can rarely be satisfied together. In their seminal paper, Kannai and Peleg (1984) proved that if $|X| \geq 6$ then orders on $\mathcal{P}(X) \setminus \{\emptyset\}$ satisfying dominance and independence are not possible. Barberà and Pattanaik (1984) showed a similar impossibility result for $|X| \geq 3$, when dominance and *strict* independence are required.

We show in this paper that the picture for other families of non-empty subsets of X is much more interesting. In particular, we show it to be the case for collections of subsets of X that induce connected subgraphs in some graph on X. Namely, we describe non-trivial classes of graphs defining families of sets that allow for lifted orders satisfying dominance and (strict) independence. In many cases, these lifted orders also satisfy the extension rule or its stronger version. It is important as every reasonable lifted order should satisfy the extension rule.[3]

3. Problem Statement

We are interested in families of sets that are defined in terms of connectivity of subgraphs in a graph. We consider undirected graphs only. We write $G = (V, E)$ for a graph with the set of vertices V and the set of edges E. We denote an edge between vertices u and v as $\{u, v\}$. A graph $H = (W, F)$ is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. If a subgraph $H = (W, F)$ of G contains all edges in G connecting vertices in W, H is the subgraph *induced* by W. A *path* consists of a non-empty set S of vertices that can be enumerated so that every two consecutive vertices are connected with an edge; its length is given by $|S| - 1$, the number of edges the path contains. A *cycle* is a sequence of at least three different vertices that can be enumerated so that every two consecutive vertices, as well as the first and the last one, are connected with an edge; the length of a cycle is given by the number of vertices. A graph is *connected* if every two of its vertices are connected by a path. A *forest* is a graph with no cycles. A *tree* is a forest that is connected. A connected graph is called *unicyclic* if it contains at most one cycle. A *pseudoforest* is a graph whose every connected component is a unicyclic graph.

Definition 3. For a graph G we write $C(G)$ for the family of sets of vertices of all connected non-empty subgraphs of G. Moreover, $IT(G)$ denotes the family of sets of vertices V' such that the subgraphs induced by V' in G are trees.

Example 4. To illustrate these concepts, let us consider graphs G and G' shown in Figures 2 and 3. For the graph G in Figure 2, we get

$$C(G) = IT(G) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}, \\ \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

3. The standard statement of the lifting problem does not explicitly mention the extension rule since for $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ the extension rule is implied by two applications of dominance via $\{x_1\} \prec \{x_1, x_2\} \prec \{x_2\}$ for $x_1 < x_2$.

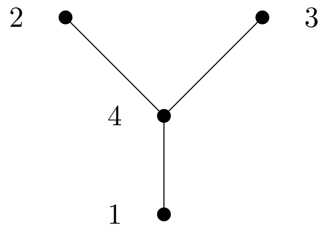

Figure 2: Graph G

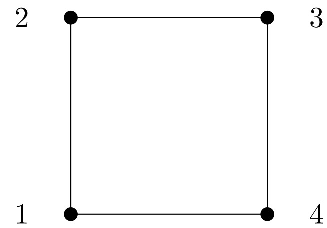

Figure 3: Graph G'

and for the graph G' in Figure 3, we have

$$C(G') = \mathcal{P}(\{1,2,3,4\}) \setminus \{\{1,3\},\{2,4\}\}$$

and

$$IT(G') = C(G') \setminus \{\{1,2,3,4\}\}.$$

$\square$

The next two definitions introduce the two types of orderability we study in this paper, namely strong and weak orderability. For each type, we consider six different versions depending on the usage of independence and extension axioms.

Definition 5. Let X be a set of elements and $\mathcal{X} \subseteq (\mathcal{P}(X) \setminus \{\emptyset\})$. We say $\mathcal{X}$ is

- *strongly (weakly) DI-orderable*, if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance and independence;
- *strongly (weakly) DI^S-orderable*, if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance and strict independence;
- *strongly (weakly) DIE-orderable* if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance, independence, and extension;
- *strongly (weakly) DI^SE-orderable* if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance, strict independence, and extension;
- *strongly (weakly) DIE^S-orderable* if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance, independence, and strong extension;
- *strongly (weakly) DI^SE^S-orderable* if for every (some) linear order on X there is an order on $\mathcal{X}$ satisfying dominance, strict independence, and strong extension.

Occasionally, we use the expression (S) in the superscript by the property symbol. We use this notation when we want to make statements that hold no matter whether we omit (S) or replace it with S. For instance, by stating that a family $\mathcal{X}$ of sets is weakly $DI^{(S)}$-orderable, we mean that $\mathcal{X}$ is both weakly DI-orderable and weakly DI^S-orderable.

We extend the concepts defined above to graphs. That is, if there is no ambiguity, we say that a graph G is strongly (or weakly) $DI^{(S)}$- or $DI^{(S)}E^{(S)}$-orderable if $C(G)$ has the corresponding property.

Remark 6. We observe that weak and strong orderability are equivalent concepts on complete graphs $K_i = (V, E)$, because $C(K_i) = \mathcal{P}(V) \setminus \{\emptyset\}$. Therefore, by symmetry, if for one order on the vertices there is an order on $C(K_i)$ satisfying a set of axioms, then such an order exists for every order on the vertices.

Example 7. Consider the graph G in Figure 2. One can check that G is strongly $DI^S E^S$-orderable. Indeed, without loss of generality we may assume that $1, 2$ and 3 are ordered so that $1 < 2 < 3$. Thus, there are four linear orders on $\{1, 2, 3, 4\}$ to consider:

$$\begin{gathered} 4 < 1 < 2 < 3 \\ 1 < 4 < 2 < 3 \\ 1 < 2 < 4 < 3 \\ 1 < 2 < 3 < 4. \end{gathered}$$

Now consider the order $\preceq$ with its strict variant $\prec$ given by

$$\begin{aligned} \{1\} \prec \{2\} \prec \{3\} \prec \{1,2,3,4\} \prec \{1,2,4\} \prec \{1,3,4\} \prec \{2,3,4\} & \\ \prec \{1,4\} \prec \{2,4\} \prec \{3,4\} \prec \{4\} & \end{aligned}$$

We claim that $\preceq$ satisfies dominance, strict independence and strong extension with respect to the last linear order. Strong extension implies $\{1\} \prec \{2\} \prec \{3\}$. Furthermore, for all $A \in C(G) \setminus \{\{1\}, \{2\}, \{3\}\}$ strong extension implies $\{1\}, \{2\}, \{3\} \prec A$ as $4 \in A$ holds. Dominance implies $\{i\} \prec \{i, 4\} \prec \{4\}$ for $i \in \{1, 2, 3\}$. Further, it implies $\{1, j, 4\} \prec \{j, 4\}$ for $j \in \{2, 3\}$ and $\{1, 2, 3, 4\} \prec \{2, 3, 4\} \prec \{3, 4\}$. Strict independence implies that $\{1, 4\}, \{2, 4\}$ and $\{3, 4\}$ are ordered like $\{1\}, \{2\}$ and $\{3\}$. Furthermore, strict independence implies that $\{l, k, 4\}$ and $\{l, 4\}$ are ordered like $\{k, 4\}$ and $\{4\}$ for $\{1, 2\} \ni l \neq k \in \{2, 3\}$. Additionally, $\{1, 4\} \prec \{2, 4\} \prec \{3, 4\}$ implies by strict independence $\{1, 2, 4\} \prec \{1, 3, 4\} \prec \{2, 3, 4\}$. Finally, there are sets $A, B \in C(G)$ such that $A \cup \{x\}, B \cup \{x\} \in C(G)$ where $|A| = 3$ and $|B| = 2$. We observe that in $\preceq$ this implies $A \prec B$ as well as $A \cup \{x\} \prec B \cup \{x\}$. It can be checked that these are all possible applications of the axioms and that they all are satisfied in $\preceq$. Similar lifted orders can be constructed for the three other orders, too. Thus, our claim follows.

On the other hand, the graph G' in Figure 3 is not strongly DI^S orderable. If we assume the natural order on the vertices of G', dominance implies $\{1\} \prec \{1, 2\}$ and $\{1, 2\} \prec \{1, 2, 3\}$, and transitivity implies $\{1\} \prec \{1, 2, 3\}$. Similarly, we can derive that $\{2, 3, 4\} \prec \{4\}$. Applying strict independence to these two relations yields $\{1, 4\} \prec \{1, 2, 3, 4\}$ and $\{1, 2, 3, 4\} \prec \{1, 4\}$, preventing $\prec$ from being a strict order. (This argument obviously does not work on $IT(G')$ because $\{1, 2, 3, 4\} \notin IT(G')$ and indeed $IT(G')$ is strongly DI^S-orderable.) However, the order

$$\begin{aligned} \{1\} \prec \{3\} \prec \{1,2,3\} \prec \{1,2\} \prec \{2,3\} \prec \{2\} \prec \{1,3,4\} \prec \{1,4\} \prec & \\ \{3,4\} \prec \{1,2,3,4\} \prec \{4\} \prec \{1,2,4\} \prec \{2,3,4\} & \end{aligned}$$

satisfies dominance and strict independence with respect to the linear order $1 < 3 < 2 < 4$. Hence G' is weakly DI^S-orderable. In fact, since the order we demonstrated also satisfies strong extension, G' is even weakly $DI^S E^S$-orderable. □

Graph G is ...	DI^S-orderable	DI^SE-orderable	DI^SE^S-orderable
... a forest	strongly	strongly	strongly
... two-colorable	weakly	weakly	weakly
... not two-colorable	not weakly	not weakly	not weakly

Table 1: Weak and Strong orderability with respect to strict independence.

It is evident that if a graph G is strongly or weakly orderable with respect to some collection of axioms selected from those discussed above, so are all its connected components. Indeed, we show below that if any set $\mathcal{X}$ is strongly or weakly orderable with respect to some collection of axioms selected from those discussed above, so are all its subsets.

Observation 8. Let $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ be a family of sets that is strongly DI-orderable and let $\mathcal{S} \subseteq \mathcal{X}$ be a subset of $\mathcal{X}$. Then $\mathcal{S}$ is strongly DI-orderable.

The same holds if we replace strongly DI-orderable by strongly DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable or by weakly DI-, DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable.

Proof. Since all axioms are universal statements, it follows that the restriction of an order to a subset satisfies all axioms that the original order satisfies. □

Importantly, the converse holds for disjoint subsets allowing us to restrict attention to connected graphs only.

Proposition 9. *Let $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{Y} \subseteq \mathcal{P}(Y) \setminus \{\emptyset\}$ be families of subsets of X and Y respectively such that $X \cap Y = \emptyset$. If $\mathcal{X}$ and $\mathcal{Y}$ are strongly DI-orderable, then $\mathcal{X} \cup \mathcal{Y}$ is strongly DI-orderable.*

The same holds if we replace strongly DI-orderable by strongly DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable or by weakly DI-, DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable.

We omit the proof here. A full proof can be found in Appendix A. The observation and Proposition 9 together immediately imply that a graph is strongly DI-orderable if and only if all its connected components are strongly DI-orderable. Again, the same holds if we replace strongly DI-orderable by strongly DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable or by weakly DI-, DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S- orderable.

In the forthcoming two sections we present our main results. Section 4 considers the combination of strict independence with dominance and optionally, extension or strong extension. Then in Section 5 we consider combinations of axioms containing independence.

4. Strict Independence

In this section, we present classification results for strong and weak DI^S-orderability, as well as strong and weak DI^SE- and DI^SE^S-orderability. These results are summarized Table 1. We start our investigations with strong orderability. Our first result concerns the family of all subsets of vertices of a graph that induce a tree.

Proposition 10. *For every graph G, $IT(G)$ is strongly $DI^S E^S$-orderable.*

Proof. Let V be the vertex set of a graph G and $n = |V|$. Further, let $\leq$ be any linear order on V. Wlog, we assume that $V = \{1, \dots, n\}$ and that $\leq$ is the natural linear order on $\{1, \dots, n\}$.

For every $A \in IT(G)$ and $i \in A$, we write $deg_A(i)$ for the degree of i in the subtree of G induced by A. We associate with every set $A \in IT(G)$ a vector

$$v_A = (a_1, \dots, a_n) \in (\mathbb{N} \cup \{\infty\})^n,$$

where $a_i = \infty$ if $i \notin A$, and $a_i = k$ if $i \in A$ and $deg_A(i) = k$.

Let $\leq^*$ be the linear order on $\mathbb{N} \cup \{\infty\}$ such that $\infty <^* \cdots <^* k <^* \cdots <^* 1$. We order $IT(G)$ by defining $A \preceq B$ precisely when $v_A \leq_{lex} v_B$, where $\leq_{lex}$ is the lexicographic order with respect to $\leq^*$, with the indices considered from n to 1. That is, $A \preceq B$ if $a_n <^* b_n$, or $a_n = b_n$ and $a_{n-1} <^* b_{n-1}$, and so on. Obviously, $\preceq$ is a (linear) order, and it satisfies strong extension. We will show that $\preceq$ satisfies dominance and strict independence.

Dominance. Assume that for every $y \in A$, $y < x$. It follows that $\max(A) < x$. Thus, $\max(A) < \max(A \cup \{x\})$ and $A \prec A \cup \{x\}$ holds by strong extension. So, assume that for every $y \in A$, $x < y$. Then, $x < \min(A)$. Let m be the neighbor of x in A (since $A, A \cup \{x\} \in IT(G)$, that is, each set induces a tree in G, it follows that x has exactly one neighbor in A). By the assumption, $x < m$ and $deg_{A\cup\{x\}}(m) = deg_A(m) + 1$. Therefore, $a_i = a_i^x$ for every i such that $m < i$, and $a_m^x <^* a_m$, where we write a_i and a_i^x for the elements of the vectors v_A and $v_{A\cup\{x\}}$. Hence, $A \cup \{x\} \prec A$.

Strict independence. Assume $A, B, A \cup \{x\}, B \cup \{x\} \in IT(G)$ and $A \prec B$. By the same argument as above, x has a unique neighbor in A and a unique neighbor in B. We will denote them by m_A and m_B, respectively. Using the same notation as above for the corresponding vectors for the sets $A, B, A \cup \{x\}, B \cup \{x\}$, we have that $a_x = b_x = \infty$ and $a_x^x = b_x^x = 1$. Further, we observe that $a_i = a_i^x$ and $b_i = b_i^x$ for $i \notin \{x, m_A, m_B\}$.

Assume first that $m_A = m_B = m$. Then $a_m^x = a_m + 1$ and $b_m^x = b_m + 1$. Hence, $a_m^x <^* b_m^x$ if and only if $a_m <^* b_m$. It follows that $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$ if and only if $v_A <_{lex} v_B$.

Next, assume $m_A \neq m_B$. Then $m_A \notin B$ and $m_B \notin A$. Hence, $b_{m_A} = b_{m_A}^x = \infty$ and $a_{m_B} = a_{m_B}^x = \infty$ by definition and, on the other hand, $a_{m_A}^x = a_{m_A} + 1 \neq \infty$ and $b_{m_B}^x = b_{m_B} + 1 \neq \infty$. Hence, by the definition of $\leq^*$ we have $b_{m_A} <^* a_{m_A}$ and $b_{m_A}^x <^* a_{m_A}^x$ Similarly, we have $a_{m_B} <^* b_{m_B}$ and $a_{m_B}^x <^* b_{m_B}^x$. This implies that $a_i^x <^* b_i^x$ holds if and only if $a_i <^* b_i$ holds and therefore $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$ if and only if $v_A <_{lex} v_B$. ☐

The following corollary follows immediately from the fact that $C(G) = IT(G)$ holds if G is a tree.

Corollary 11. *Every tree is strongly $DI^S E^S$-orderable.*

This result is optimal in the sense that cycles prevent a graph from being strongly DI^S-orderable.

Proposition 12. *If a graph G contains a cycle, then it is not strongly DI^S-orderable.*

Proof. Let $C = v_1, \ldots, v_n$ be a shortest cycle in G. Then $C(G)$ contains C and all connected subgraphs of C. In particular, $C(G)$ contains $\{v_1, v_n\}$ and all sets $\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$, where $1 \leq i \leq j \leq n$. Let $\leq$ be an order on V such that $v_1 < \cdots < v_n$. Let us assume that there is an order $\preceq$ on $C(G)$ that satisfies dominance and strict independence with respect to $\leq$. Then, by dominance

$$\{v_1\} \prec \{v_1, v_2\} \prec \cdots \prec \{v_1, \ldots, v_{n-1}\}$$

and

$$\{v_2, \ldots, v_n\} \prec \{v_3, \ldots, v_n\} \prec \cdots \prec \{v_n\}.$$

Therefore, by strict independence $\{v_1, v_n\} \prec \{v_1, \ldots, v_n\}$ and $\{v_1, \ldots, v_n\} \prec \{v_1, v_n\}$. Since $n \geq 3$, this is a contradiction! $\square$

The following theorem summarizes the previous results and follows from Corollary 11, Proposition 9, Proposition 12 and the fact that any graph that is not strongly DI^S-orderable is also not strongly $DI^S E^{(S)}$-orderable.

Theorem 13. *The set of strongly DI^S-, $DI^S E$- or $DI^S E^S$-orderable graphs is exactly given by the class of forests.*

This result states that every linear order on X can be lifted (with respect to dominance, strict independence and strong extension) to every family of sets of vertices inducing a connected subgraph in a forest on X. In other words, we know that a family of sets $\mathcal{X}$ is strongly $DI^S E^S$-orderable if there exists a forest F such that $\mathcal{X} \subseteq C(F)$ holds. For instance, no matter what linear order on $\{1, 2, \ldots, n\}$ we consider, it extends to a linear order on the family

$$\mathcal{I} = \{[i..j] \mid 1 \leq i < j \leq n\}$$

that satisfies dominance, strict independence and strong extension. It is so because every set in $\mathcal{I}$ induces a connected subgraph in the path in which elements $1, \ldots, n$ are listed in the natural order. The same is true for the family of sets

$$\mathcal{S} = \{X \subseteq \{1, \ldots, n\} \mid 1 \in X\}.$$

Indeed, each set in this family induces a connected subgraph in the "star" tree in which every vertex $i \geq 2$ is connected to 1 (and there are no other edges).

Furthermore, recall that a hypergraph H over a set X is called a hypertree or arboreal if there exists a tree T with nodes X such that all edges of H induce connected subtrees in T (Berge, 1984; Brandstädt, Dragan, Chepoi, & Voloshin, 1998). This means, in our notation, $H \subseteq C(T)$. Therefore, Theorem 13 implies that every hypertree is strongly $DI^S E^S$-orderable.

The converse is not necessarily true, however. Consider $\mathcal{X} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ for example. Then any graph G such that $\mathcal{X} \subseteq C(G)$ must contain the edges $(1, 2), (1, 3), (2, 3)$ and hence a cycle. However, because all sets in $\mathcal{X}$ have the same size, dominance and strict independence hold vacuously for any order on $\mathcal{X}$. Further, given any linear order $\leq$ on $\{1, 2, 3\}$, the relation $A \preceq B$ defined to hold when $\max(A) \leq \max(B)$ satisfies strong extension. Therefore $\mathcal{X}$ is strongly $DI^S E^S$-orderable.

We now turn to weak orderability and show in the forthcoming two results that the bipartite graphs form the crucial class for our characterization. We use the fact that a graph is bipartite if and only if it is two-colorable.

Proposition 14. *Every two-colorable graph is weakly DI^SE^S-orderable.*

Proof. Let us consider a two-colorable graph $G = (V, E)$. We color G with two colors *small* and *large* and call vertices of G *small* and *large* accordingly. Let $\leq$ be any linear order on V such that every *small* vertex is smaller than every *large* vertex.

For every $A \in C(G)$ we define $A_L = \{x \in A \mid x \text{ is } large\}$ and $A_S = \{x \in A \mid x \text{ is } small\}$. For $A, B \in C(G)$, we define $A \preceq B$ if and only if

- $A = B$; or
- $A_L \neq B_L$ and $\max(A_L \triangle B_L) \in B_L$; or
- $A_L = B_L$, $A_S \neq B_S$, and $\min(A_S \triangle B_S) \in A_S$.

(we write $\triangle$ for the symmetric difference of sets). We will prove that $\preceq$ is a linear order.

Indeed, it is easy to see that $\preceq$ is reflexive and total. Let us assume that $A \preceq B$. If $A \preceq B$ holds by the second condition of the definition, then $B \not\preceq A$, because we have $B_L \neq A_L$ and $\max(B_L \triangle A_L) = \max(A_L \triangle B_L) \notin A_L$. By a similar argument, $B \not\preceq A$ follows also if the third clause of the definition applies. Thus, if $A \preceq B$ and $B \preceq A$, it must be that the first condition holds, that is, $A = B$. It follows that $\preceq$ satisfies antisymmetry.

To prove transitivity, let us assume that $A \preceq B$ and $B \preceq C$. If $A = B$ or $B = C$, then we obtain $A \preceq C$ by substituting A for B in $B \preceq C$ or C for B in $A \preceq B$. Thus, from now on we assume that that $A \neq B$ and $B \neq C$. It follows that each of $A \preceq B$ and $B \preceq C$ holds because of the second or the third condition of the definition of $\preceq$.

Let us assume first that both $A \preceq B$ and $B \preceq C$ hold by the second condition and let

$$d = \max((A_L \setminus B_L) \cup (B_L \setminus C_L) \cup (C_L \setminus A_L)).$$

We note that

$$(A_L \setminus B_L) \cup (B_L \setminus C_L) \cup (C_L \setminus A_L) = (A_L \triangle B_L) \cup (B_L \triangle C_L) \cup (A_L \triangle C_L).$$

Clearly, $d \notin A_L \setminus B_L$. Indeed, let us assume that $d \in A_L \setminus B_L$. This would imply $d \in A_L$ as well as $d \in A_L \triangle B_L$. We would then have $\max(A_L \triangle B_L) = d \in A_L$, and, consequently, $B \preceq A$. Antisymmetry would then imply $A = B$, a contradiction. Similarly, $d \notin B_L \setminus C_L$. It follows that $d \in C_L \setminus A_L$. Thus, $A_L \neq C_L$, $d \in C_L$, $d \in A_L \triangle C_L$ and $d = \max(A_L \triangle C_L)$. Consequently, $\max(A_L \triangle C_L) \in C_L$ and $A \preceq C$.

The case when each of $A \preceq B$ and $B \preceq C$ holds because of the third clause in the definition of $\preceq$ can be dealt with in a similar way.

Thus, let us assume then that $A \preceq B$ holds by the second condition of the definition and $B \preceq C$ holds by the third condition. It follows that $A_L \neq B_L$, $\max(A_L \triangle B_L) \in B_L$ and $B_L = C_L$. Consequently, $A_L \neq C_L$ and $\max(A_L \triangle C_L) \in C_L$. Thus, $A \preceq C$. In the dual case, when $A \preceq B$ holds because of the third condition and $B \preceq C$ because of the second one, we obtain $A \preceq C$ in a similar way. This concludes the proof of transitivity.

We will now show that $\preceq$ satisfies dominance, strict independence and strong extension.

Dominance. Let $A, A \cup \{x\} \in C(G)$. By the connectivity of the subgraph induced in G by $A \cup \{x\}$, x has at least one neighbor in A. Let us fix any such neighbor of x and denote it by n. Clearly, the colors of x and n are different. Let us assume that $\max(A) < x$. It follows that x is large. Thus, $x \in (A \cup \{x\})_L$ and so, $A_L \neq A \cup \{x\}$. Since $A_L \triangle (A \cup \{x\})_L = \{x\}$, $\max(A_L \triangle (A \cup \{x\})_L) = x$. Thus, $\max(A_L \triangle (A \cup \{x\})_L) \in (A \cup \{x\})_L$ and $A \prec A \cup \{x\}$. The case $x < \min(A)$ can be dealt with in a similar way.

Strict independence. Assume $A, B, A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$ (thus, either the second or the third condition of the definition holds). As $A_L \triangle B_L = (A \cup \{x\})_L \triangle (B \cup \{x\})_L$ and $A_S \triangle B_S = (A \cup \{x\})_S \triangle (B \cup \{x\})_S$, we have $A \cup \{x\} \prec B \cup \{x\}$.

Strong Extension. Consider sets $A, B \in C(G)$ such that $\max(A) < \max(B)$. First assume that $B_L \neq \emptyset$. Clearly, $\max(B) \in B_L$ and $\max(B) \notin A_L$ (because $\max(B) \notin A$). It follows that $A_L \triangle B_L \neq \emptyset$ and $\max(A_L \triangle B_L) = \max(B)$. Thus, $\max(A_L \triangle B_L) \in B_L$ and so, $A \preceq B$. Since $A \neq B$ and $\preceq$ is a linear order, we have $A \prec B$.

Next, assume that $B_L = \emptyset$. It follows that $\max(B)$ is *small* and so, $\max(A)$ is *small*, too. Consequently, we have that A and B consist of *small* vertices only. As they induce connected subgraphs in G, $|A| = |B| = 1$. These observations imply that $A_L = B_L = \emptyset$, and $\min(A) = \max(A) < \max(B) = \min(B)$. Consequently, $\min(A) \in A_S$ and $\min(A) \notin B_S$. It follows that $A_S \triangle B_S \neq \emptyset$ and that $\min(A_S \triangle B_S) = \min(A_S)$. Hence, $\min(A_S \triangle B_S) \in A_S$ and so, $A \prec B$. $\square$

Note that a complete bipartite graph is also two-colorable. Hence, this result shows, in particular, that if X and Y are disjoint nonempty sets, then the family of sets

$$\{Z \subseteq X \cup Y \mid Z \cap X \neq \emptyset \neq Z \cap Y\}$$

is weakly $DI^S E^S$-orderable.

Proposition 14 is tight as graphs that are not two-colorable are not weakly DI^S-orderable.

Proposition 15. *If a graph is not two-colorable, then it is not weakly DI^S-orderable.*

Proof. Let V be the vertex set of G and let $\leq$ be a linear order on V. We say a vertex $x \in V$ is *large* (respectively, *small*) with respect to $\leq$ if for every neighbor n of x, $n < x$ (respectively, $n > x$) holds. We call $x \in V$ *intermediate* with respect to $\leq$ if x is neither *large* nor *small*. (When talking about large, small and intermediate vertices, we often drop references to $\leq$ if it is clear from the context.) Let us assume that every vertex in V is either *large* or *small*. Obviously no *large* vertex can be a neighbor of a *large* vertex and no *small* vertex can be the neighbor of a *small* vertex. Thus, the *large-small* labeling of nodes is a two-coloring of G, a contradiction.

Our argument shows that for every linear order $\leq$ on V, V contains at least one *intermediate* vertex. Let $\leq$ be an arbitrary linear order on V and let x be an *intermediate* vertex with respect to $\leq$. We call a neighbor n of x *small* if $n < x$ holds and *large* otherwise. Further, we call an intermediate x *critical* if at least one *small* neighbor of x is connected to at least one *large* neighbor of x by a path in G_x^-, the graph induced by $V \setminus \{x\}$.

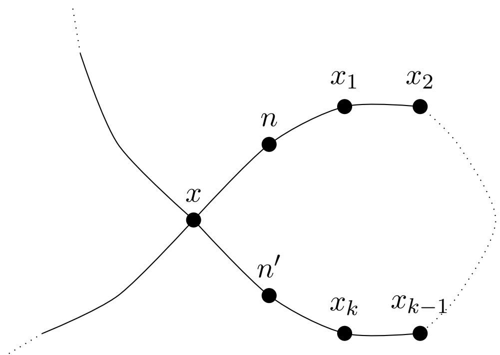

Figure 4: Vertex x with two neighbors n and n' connected by a path

We claim that every linear order contains at least one *critical* vertex. Indeed, let us assume otherwise and let $\leq$ be a counterexample order with the minimum number of intermediate vertices. That is, no vertex in V is critical and every linear order with fewer *intermediate* vertices than $\leq$ contains a *critical* vertex.

Let x be an *intermediate* vertex with respect to $\leq$, and let V' be the set of all vertices in V reachable in G from x by simple paths (no repetition of vertices) that start with an edge connecting x to a small neighbor of x. Let us define $V'' = V \setminus V'$. Clearly, x and all *small* neighbors of x belong to V' and all *large* neighbors of x belong to V''. To see the latter, let us assume that some large neighbor of x, say y, belongs to V'. It follows that there is a path from a small neighbor of x to y in G^-_x, contradicting that x is intermediate but not critical. In addition, by the definition of V', the only edges between V' and V'' are those that connect x and its *large* neighbors. We define linear order $\leq'$ on V by setting $y \leq' z$ if and only if

- $y, z \in V'$ and $y \leq z$,
- $y, z \in V''$ and $z \leq y$,
- $z \in V'$, $y \in V''$.

It is clear that $\leq'$ is a linear order on V. Moreover, x is not an *intermediate* vertex in G with respect to $\leq'$, because all neighbors of x that are *small* with respect to $\leq$ are also *small* with respect to $\leq'$ and all neighbors that are *large* with respect to $\leq$ are *small* with respect to $\leq'$. Furthermore, for all other vertices $y \neq x$, whether they are *intermediate* or not does not change. This is clear if $y \in V'$. If $y \in V''$ then the relation of y to all its neighbors is inverted, hence *small* vertices become *large* vertices, *large* vertices become *small* vertices and *intermediate* vertices stay *intermediate*. It follows that $\leq'$ has fewer intermediate vertices than $\leq$. Let y be any intermediate vertex with respect to $\leq'$. By construction either y and all its neighbors are all in V' or are all in V''. Since y is an intermediate but not critical vertex with respect to $\leq$, y is not *critical* with respect to $\leq'$. Thus, $\leq'$ is an order with fewer intermediate vertices than $\leq$ and with no critical vertices, a contradiction.

Let $\leq$ be any linear order on V and let x be a critical vertex under this order. Let n be a *small* neighbor of x connected in G_x^- to a *large* neighbor of x, say n', by a path $n, x_1, \ldots, x_k, n'$, as shown in Figure 4. Let us assume there is an order $\preceq$ on $C(G)$ satisfying dominance and strict independence with respect to $\leq$. Then, since $n < x$, we have $\{n\} \prec \{n, x\}$ by dominance. Further, by repeated application of strict independence and transitivity

$$\{n, x_1, \ldots, x_k, n'\} \prec \{n, x, x_1, \ldots, x_k, n'\}.$$

On the other hand, since $x < n'$, we have $\{x, n'\} \prec \{n'\}$ and hence, again by strict independence and transitivity,

$$\{n', x, x_1, \ldots, x_k, n\} \prec \{n', x_1, \ldots, x_k, n\}.$$

Thus,

$$\{n, x_1, \ldots, x_k, n'\} \prec \{n, x_1, \ldots, x_k, n'\},$$

a contradiction. □

The following theorem follows directly from Proposition 14 and Proposition 15.

Theorem 16. *The set of weakly DI^S-, DI^SE- or DI^SE^S-orderable graphs is exactly given by the class of two-colorable graphs (or equivalently, bipartite graphs).*

Remark 17. It is worth observing that it is possible to decide in polynomial time if a graph is a forest and if a graph is two-colorable. Therefore our results show that, for a given graph G, it is decidable in polynomial time if $C(G)$ is strongly/weakly DI^SE^S-orderable. Furthermore, for any two-colorable graph, we can compute a two-coloring in polynomial time. Therefore, for any weakly DI^SE^S-orderable graph $G = (V, E)$ we can compute, in polynomial time, an order $\leq$ on V such that there exists a linear order $\preceq$ on $C(G)$ satisfying dominance, strict independence and strong extension with respect to $\leq$ by using the construction used in the proof of Proposition 14.

Finally, we observe that the orders constructed in the proofs of Proposition 10 and Proposition 14 are linear orders. Furthermore, we observe that we did not use the totality of the order $\preceq$ in the proofs of Proposition 12 and Proposition 15. Therefore, we can conclude that if a graph G is not strongly (weakly) DI^SE^S-orderable, then for at least one (for every) linear order on its vertices there does not exist a *partial* order on $C(G)$ that satisfies dominance, strict independence and strong extension. In other words:

Corollary 18. *Let G be a graph. There exists a* linear *order on $C(G)$ satisfying dominance and strict independence (and extension or strong extension) for every (at least one) order on the vertices of G if and only if there exists a* partial *order on $C(G)$ satisfying dominance and strict independence (and extension or strong extension) for every (at least one) order on the vertices of G*

This result is especially interesting because it is known that for every family of sets $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ there is a partial order satisfying dominance, independence and extension for every linear order on X (Maly & Woltran, 2017). Hence, if we consider a clique with six or more vertices, there exists a partial order satisfying dominance, independence and

Graph G ...	strongly DI-ord.	strongly DIE-ord.	strongly DIE^S-ord.
...is a tree or K_3	Yes	Yes	Yes
...is a cycle of length ≥ 4	Yes	Yes	No
...is unicyclic	Yes	**?**	No
...contains 2 disjoint cycles	**?**	No[4]	No
...contains 2 cycles sharing a vertex	No[5]	No[5]	No

Table 2: Extent of strong orderability of a graph G, if G....

extension but no (total) order satisfying the same axioms. Furthermore, if we require the order $\preceq$ on $\mathcal{X}$ to be linear, independence and strict independence are equivalent. As we will see later there are graphs that are strongly DIE^S-orderable but not strongly DI^S-orderable. Hence, there are graphs that are DIE^S-orderable but there is no linear order on $C(G)$ that satisfies dominance and independence. In other words, if we consider strict independence, it makes no difference for orderability if we are looking for a partial order, an order or a linear order, whereas for independence all three notions result in different concepts of orderability.

5. Regular Independence

We now replace strict independence by independence. Table 2 summarizes some of our main results. For a complete list of the results, see Theorem 21, 29, 34 and 35. We focus first on strong DIE^S-orderability for which we give an exact characterization. The following is easy to see.

Proposition 19. *Let X be a set. If $|X| \leq 3$, then $\mathcal{P}(X) \setminus \{\emptyset\}$ is strongly DIE^S-orderable.*

Proof. Wlog we may assume that $X = \{1, 2, 3\}$ and that $\leq$ is the natural linear order on X. We define an order $\preceq$ by setting: $\{1\} \prec \{1, 2\} \prec \{2\} \prec \{1, 3\} \sim \{1, 2, 3\} \prec \{2, 3\} \prec \{3\}$. We claim that this order satisfies dominance, independence and strong extension. Strong extension can easily be checked. Dominance implies $\{1\} \prec \{1, 2\} \prec \{1, 2, 3\} \prec \{2, 3\} \prec \{3\}$ and $\{1\} \prec \{1, 3\} \prec \{3\}$ as well as $\{1, 2\} \prec \{2\} \prec \{2, 3\}$. All of these are satisfied. For independence, assume that $A, B \in \mathcal{P}(X) \setminus \{\emptyset\}$ and $A \cup \{x\}, B \cup \{x\} \in \mathcal{P}(X) \setminus \{\emptyset\}$ for some $x \notin A \cup B$. First we assume $|A| = |B| = 1$. In this case independence holds as $\{1, 3\}$ and $\{2, 3\}$ are ordered like $\{1\}$ and $\{2\}$, $\{1, 2\}$ and $\{2, 3\}$ are ordered like $\{1\}$ and $\{3\}$ and $\{1, 2\}$ and $\{1, 3\}$ are ordered like $\{2\}$ and $\{3\}$. Now assume $|A| = 1$ and $|B| = 2$ Then, $\{1\} \prec \{1, 2\}$ implies $\{1, 3\} \preceq \{1, 2, 3\}$, $\{1\} \prec \{1, 3\}$ implies $\{1, 2\} \preceq \{1, 2, 3\}$, $\{2\} \prec \{2, 3\}$ implies $\{1, 2\} \preceq \{1, 2, 3\}$, $\{1, 2\} \prec \{2\}$ implies $\{1, 2, 3\} \preceq \{2, 3\}$, $\{1, 3\} \prec \{3\}$ implies $\{1, 2, 3\} \preceq \{2, 3\}$ and $\{2, 3\} \prec \{3\}$ implies $\{1, 2, 3\} \preceq \{1, 3\}$. It is straightforward to check that all of these hold. Now $|A| = |B| = 2$ is not possible because then $A \cup \{x\} = B \cup \{x\}$ must hold and hence $A = B$. Furthermore, $|A| = 3$ or $|B| = 3$ is obviously not possible. □

4. If the size of at least two cycles is at least 4.
5. If the size of at least one of the cycles is at least 4.

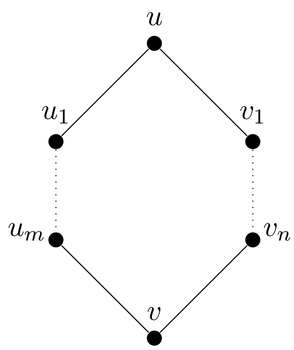

Figure 5: A circle with at least 4 vertices.

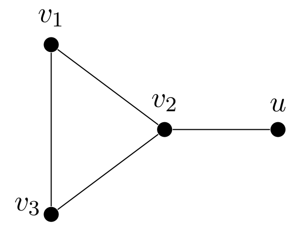

Figure 6: A circle with three vertices and connected to an additional vertex u.

While this result shows that cycles of length 3 are strongly DIE^S-orderable, the next result shows that we cannot go much beyond 3-cycles.

Proposition 20. *Let G be a connected graph with four or more vertices that contains at least one cycle. Then G is not strongly DIE^S-orderable.*

Proof. Either G contains a cycle of length at least four or a cycle of length three connected to an additional vertex. In the first case let $u, v \in V$ be two non-adjacent vertices contained in the cycle, and let $u, v_1, \dots, v_n, v$ and $u, u_1, \dots, u_m, v$ be the two paths from u to v (see Figure 5). Define $\leq$ by specifying its strict version $<$ as follows:

$$u < u_1 < \cdots < u_m < v_1 < \cdots < v_n < v.$$

Then there is no order on $C(G)$ satisfying dominance, independence and strong extension with respect to $\leq$. Indeed, let us assume otherwise and let $\preceq$ be such an order. Then, $\{u_m\} \prec V \setminus \{v\}$ by strong extension, and $\{u_m, v\} \preceq V$ by independence. However, by repeated application of dominance, $\{v_1, \dots, v_n, v\} \prec \{v\}$ and therefore, by independence, $\{u_m, v_1, \dots, v_n, v\} \preceq \{u_m, v\}$. It follows that $\{u_m, v_1, \dots, v_n, v\} \preceq V$. On the other hand, repeated application of dominance implies $V \prec \{u_m, v_1, \dots, v_n, v\}$, a contradiction.

In the second case let u be the additional vertex and let v_1, v_2, v_3 be the vertices in the cycle such that v_2 is connected to u (see Figure 6). Define $\leq$ by specifying its strict version $<$ as follows:

$$u < v_1 < v_2 < v_3.$$

Then there is no order on $C(G)$ satisfying dominance, independence and strong extension with respect to $\leq$. Indeed, let us assume otherwise and let $\preceq$ be such an order. Then $\{v_1\} \prec V \setminus \{v\}$ by strong extension, and $\{v_1, v_3\} \preceq V$ by independence. However, by dominance, $\{v_2, v_3\} \prec \{v\}$ and therefore, by independence, $\{v_1, v_2, v_3\} \preceq \{v_1, v_3\}$. It follows that $\{v_1, v_2, v_3\} \preceq V$. On the other hand, dominance implies $V \prec \{v_1, v_2, v_3\}$, a contradiction.

Recall that if a family of sets is strongly DIE^S-orderable, then all subsets of this family are also DIE^S-orderable. Hence, the negative result on the two types of graphs proven above carries over to all graphs which contain such graphs as subgraphs. $\square$

Therefore, K_3 is the only connected graph that is strongly DIE^S-orderable but not DI^SE^S-orderable (recall Proposition 12). Also recall that a graph is strongly DI^SE^S-orderable precisely when it is a forest. Thus, it follows from Propositions 9, 19 and 20 that the class of strongly DIE^S-orderable graphs is only marginally larger than the class of strongly DI^SE^S-orderable graphs.

Theorem 21. *The set of strongly DIE^S-orderable graphs consists precisely of graphs whose each connected component is a tree or a cycle K_3.*

We now turn to graphs that are strongly DIE-orderable. The next five results allow us to settle the matter of $DI(E)$-orderability for two-connected graphs. We recall that x is an *articulation* point in a graph G if the removal of x from G results in at least two connected components. Graphs without articulation points are called *two-connected*. The simplest two-connected graphs are cycles. The next result shows that all cycles are strongly DIE-orderable. Additionally, the result implies that replacing strong extension by extension leads to additional strongly orderable graphs.

Proposition 22. *Let $G = (V, E)$ be a graph and $\leq$ be a linear order on V. If there is an order on $C(G \setminus \{\min(V)\})$ satisfying dominance, independence and strong extension, then there exists an order on $C(G)$ satisfying dominance, independence and the extension rule.*

Proof. Wlog we may assume that $V = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $\leq$ is the natural linear order on V. Let $\preceq^*$ be an order on $C(G \setminus \{1\})$ satisfying dominance, independence and strong extension. We define an order $\preceq$ on $C(G)$ by setting $A \preceq B$ if and only if:

1. $1 \in A$ and $1 \notin B$
2. $1 \notin A \cup B$ and $A \preceq^* B$
3. $1 \in A \cap B$ and $\max(A) \leq \max(B)$.

It follows directly from the definition, that if $1 \notin A \cup B$ and $A \prec^* B$, then $A \prec B$. Similarly, if $1 \in A \cap B$ and $\max(A) < \max(B)$, then $A \prec B$. It can be checked that $\preceq$ is an order satisfying dominance, independence and the extension rule. The full proof can be found Appendix A. □

This shows that every cycle is strongly DIE-orderable. In fact, for a cycle G, $G \setminus \{\min_{\leq}(V(G))\}$ is a tree. Since trees are strongly DIE^S-orderable by Theorem 21, Proposition 22 implies that G is strongly DIE-orderable. We can generalize the result as follows using Proposition 9.

Corollary 23. *The set of strongly DIE-orderable graphs includes all graphs whose each connected component is a tree or a cycle.*

The following result shows that we can not go much beyond cycles if we want to preserve strong DIE-orderability. The result states that any graph that contains a cycle of length at least six and additionally any path (or edge) between two vertices contained in the cycle is not strongly DI-orderable. An immediate consequence of this result is that a two-connected graph that contains a cycle of length at least six is strongly DIE-orderable if and only if it is a cycle.

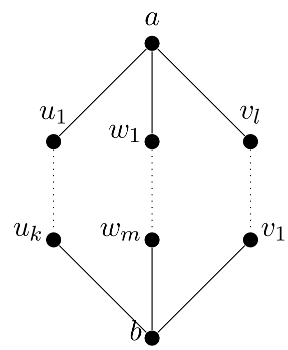

Figure 7: Vertices a, b connected by three mutually disjoint paths.

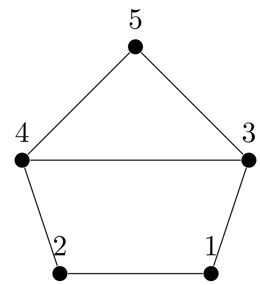

Figure 8: A graph where two vertices are connected by three disjoint path of length one, two and three

Proposition 24. *Let $G = (V, E)$ be a graph containing distinct vertices $a, b \in V$ connected by three paths that do not share any vertices except a and b such that two of them have length at least three, or one of the paths has length at least four and one of the remaining two paths is of length two. Then, G is not strongly DI-orderable.*

We omit the proof here. A full proof can be found in Appendix A. The graphs captured by this result are of the shape depicted in Figure 7. The assumption on the length of the paths amounts to $k, m \geq 2$ in the first case, and $k = 2$ and $m \geq 3$ in the second one. This result is optimal in the sense that there exists a strongly DIE-orderable (two-connected) graph that contains two vertices that are connected by three mutually disjoint paths such that one has length three and one has length two.

Example 25. Consider the graph shown in Figure 8. Obviously, this is a connected graph containing at least two cycles. However, we know that Kannai and Peleg's impossibility result is minimal in the sense that $\mathcal{P}(\{1, 2, 3, 4, 5\}) \setminus \{\emptyset\}$ can be ordered with an order satisfying dominance, independence and extension (Bandyopadhyay, 1988). This implies immediately that any graph with five or fewer vertices is strongly DIE-orderable, hence also the one shown in Figure 8. $\square$

However if we increase the number of paths we quickly run into impossibility again.

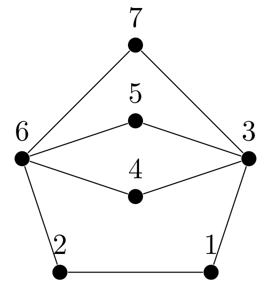

Figure 9: The ordering used in the proof of Proposition 26

Proposition 26. *If a graph G contains two distinct vertices a and b connected by four mutually disjoint paths (not counting a and b) of length at least two, with one of them of length at least three, then G is not strongly DI-orderable.*

Proof. Assume that two of the paths have three or more edges. Then, Proposition 24 applies and G is not strongly DI-orderable. Therefore, we may assume that three of the four paths have length 2. If the fourth path has length at least four, G is not strongly DI-orderable again by Proposition 24. Therefore, we will assume that this path has exactly three edges. It follows that G contains a subgraph like the one shown in Figure 9. Clearly, if that graph is not strongly DI-orderable, G is not strongly orderable either. Therefore, to prove the assertion, we will prove that the graph in Figure 9 is not strongly DI-orderable.

Let us consider the labeling of the vertices shown in Figure 9 and take for a linear order on this graph the order induced by the natural order of integers. We claim that there is no order $\preceq$ on $C(G)$ satisfying dominance and independence with respect to $\leq$.

First we assume $\{3\} \prec \{2,3,4,5,6\}$. Then, by independence, $\{3,7\} \preceq \{2,3,4,5,6,7\}$. However, we have $\{4,5,6,7\} \prec \{7\}$ by dominance and therefore $\{3,4,5,6,7\} \preceq \{3,7\}$ by independence. But then we have $\{3,4,5,6,7\} \preceq \{2,3,4,5,6,7\}$ by transitivity, which contradicts dominance.

Therefore, we must have $\{2,3,4,5,6\} \preceq \{3\}$. Now observe that $\{1\} \prec \{1,2\}$ by dominance and therefore, by independence and dominance, $\{1,3\} \preceq \{1,2,3\} \prec \{1,2,3,4\}$. Hence, by independence, $\{1,3,5\} \preceq \{1,2,3,4,5\}$. Further, by dominance $\{3\} \prec \{3,5\}$ and so, by transitivity, $\{2,3,4,5,6\} \prec \{3,5\}$. Independence implies $\{1,2,3,4,5,6\} \preceq \{1,3,5\}$ and transitivity implies $\{1,2,3,4,5,6\} \preceq \{1,2,3,4,5\}$. This contradicts $\{1,2,3,4,5\} \prec \{1,2,3,4,5,6\}$, which we have by dominance. □

Propositions 24 and 26 specify sufficient conditions for a graph not to be strongly DIE-orderable. The next result gives a sufficient condition for a graph to be strongly DIE-orderable.

Proposition 27. *Let G be a graph consisting of two vertices a, b, where $a \neq b$, connected by arbitrarily many paths of length at most two. Then G is strongly DIE-orderable*

Proof. Let us consider any linear order on the set V of vertices of G. Wlog we may assume that $a < b$. Under this assumption, we define the sets $L = \{v \in V \mid b < v\}$, $I = \{v \in V \mid a \leq v \leq b\}$ and $S = \{v \in V \mid v < a\}$.

Next, we define two orders $\prec^+$ and $\prec^-$ on the family of all subsets of V by setting

1. $A \prec^+ B$ if and only if $\max(A \triangle B) \in B$
2. $A \prec^- B$ if and only if $\min(A \triangle B) \in A$.

It is straightforward to verify that both relations are indeed orders.

We also define an order $\preceq^*$ on $C(G)\big|_I$, the set of all nonempty subsets of I inducing in G a connected subgraph. Clearly, every set in $C(G)\big|_I$ is either a singleton, or contains at least one of a and b. To define $\preceq^*$, for each $A \in C(G)\big|_I$ we define its type, written *type*(A):

$$type(A) := \begin{cases} a \text{ if } a \in A, b \notin A \\ b \text{ if } b \in A, a \notin A \\ ab \text{ if } a, b \in A \\ 0 \text{ if } a, b \notin A \end{cases}$$

We note that *type*$(A) = 0$ if and only if A is a singleton set other than $\{a\}$ and $\{b\}$. We order the types $a < 0 < ab < b$ (we point out that the order on types is unrelated to the order on V that we are considering; it will always be clear from the context whether we are comparing types or elements of V).

With these concepts in hand, for $A, B \in C(G)\big|_I$ we set $A \preceq^* B$ if and only if

1. $type(A) < type(B)$,
2. $type(A) = type(B) = a$ and $A \prec^+ B$,
3. $type(A) = type(B) = 0$, $A = \{v\}$, $B = \{w\}$ and $v < w$,
4. $type(A) = type(B) = ab$
5. $type(A) = type(B) = b$ and $A \prec^- B$,

This is an order, because $\preceq^*$ restricted to sets of any type is an order, and the types are linearly ordered.

Using the three orders defined above we now define an order $\preceq$ on $C(G)$. We set $A \preceq B$ if and only if

1. $A \cap L \prec^+ B \cap L$,
2. $A \cap L = B \cap L$ and $A \cap S \prec^- B \cap S$,
3. $A \cap L = B \cap L \neq \emptyset \neq A \cap S = B \cap S$,
4. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$ and $a \in A$,
5. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$, $a \notin A, B$ and $A \cap I \prec^- B \cap I$,
6. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$ and $b \in B$,
7. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$, $b \notin B$ and $A \cap I \prec^+ B \cap I$.

8. $A \cap L = B \cap L = \emptyset = A \cap S = B \cap S$ and $A \preceq^* B$.

It can be checked that $\preceq^*$ is an order on $C(G)\big|_I$ satisfying dominance, independence and extension. Again, a full proof is given in Appendix A. □

Proposition 27 implies that every two-connected graph with a longest cycle of length four is strongly DIE-orderable. We will make this observation formal later on as a part of a more general result on DIE-orderability of two-connected graphs.

The next result, while being of interest in its own right, is the last piece we need to classify all two-connected strongly DIE-orderable graphs.

Proposition 28. *Let $G = (V, E)$ be a graph containing two cycles C_1 and C_2 that have exactly one vertex in common, with one of the cycles having length at least 4. Then, G is not strongly DI-orderable.*

Proof. Let $v_1, \ldots, v_k$ and $v_k, v_{k+1}, \ldots, v_n$ be the vertices of the cycles C_1 and C_2 enumerated consistently with their order on the corresponding cycle, where v_k is the unique common vertex of the two cycles. By our assumptions, $k \geq 3$ and $n - k \geq 3$. Moreover, there are edges in G between v_1 and v_k, and between v_k and v_n.

Clearly, it suffices to show the assertion under the assumption that G has no other vertices. Thus, we adopt this assumption for the remainder of the proof. To simplify the presentation, let us identify v_i with i. In particular, $V = \{1, \ldots, n\}$.

Let us consider a linear order $\leq$ on V induced by the natural order on the integers. Let us assume that $\preceq$ is an order on $C(G)$ that satisfies dominance and independence with respect to $\leq$. We will derive a contradiction, which will prove the result.

All sets we use in the argument belong to $C(G)$. This is easy to see and we will not be making it explicit when we compare sets under $\preceq$. Let us assume that $\{k\} \prec V \setminus \{1, n\}$. By independence, $\{k, n\} \preceq V \setminus \{1\}$. In addition, by repeated application of dominance, we get $\{k+1, \ldots, n\} \prec \{n\}$ and, by independence, $\{k, k+1, \ldots, n\} \preceq \{k, n\}$.

However, since $k \geq 3$, repeated application of dominance implies $V \setminus \{1\} \prec \{k, k + 1, \ldots, n\}$. By transitivity, $\{k, n\} \prec \{k, n\}$, a contradiction.

Thus, we have $V \setminus \{1, n\} \preceq \{k\}$. Let us assume that $\{k\} \preceq \{2, \ldots, k+1\}$. By transitivity, $V \setminus \{1, n\} \preceq \{2, \ldots, k + 1\}$. On the other hand, since $n - k \geq 3$, $k + 1 < n - 1$. Thus, by repeated application of dominance $\{2, \ldots, k + 1\} \prec V \setminus \{1, n\}$, a contradiction.

It follows that $\{2, \ldots, k+1\} \prec \{k\}$. This implies $\{1, \ldots, k+1\} \preceq \{1, k\}$ by independence. However, we also have $\{1\} \prec \{1, \ldots, k - 1\}$ by dominance. Hence $\{1, k\} \preceq \{1, \ldots, k\} \prec \{1, \ldots, k + 1\}$ by independence and dominance, and we reach a contradiction! □

We are now ready to provide a complete characterization of the two-connected strongly $DI(E)$-orderable graphs. First, we know from Corollary 23 that cycles are strongly DIE-orderable. Further, Proposition 24 tells us that any two-connected graph *properly* containing a cycle of length at least six is not strongly DI-orderable.

Let now G be a two connected graph with a longest cycle having length 5. Let C be one such cycle. If G has only five vertices, it is strongly DIE-orderable as discussed in Example 25. Thus, let us assume that G has at least one vertex not on the cycle C. Let f be any vertex of $G \setminus C$ connected to (a vertex on) C by an edge. Such vertex exists as G is connected. Let a be a neighbor of f in C. Since G is two-connected, f is connected

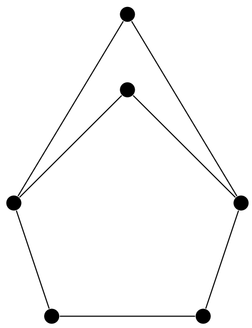

Figure 10: The graph T_5

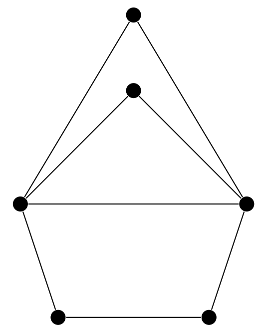

Figure 11: The graph T_5^+

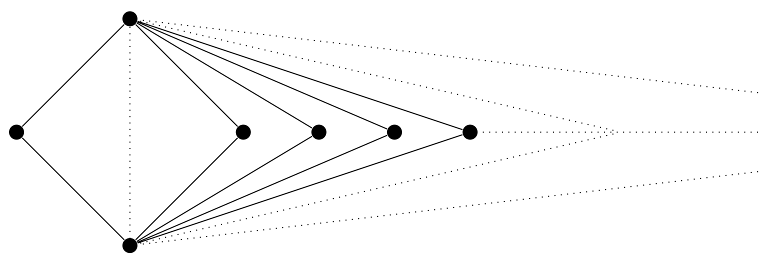

Figure 12: Two-connected graphs with longest cycle of length four

by a path in $G \setminus \{a\}$ to a vertex in C other than a. Let P be a shortest such path and let $b \in C$ be the end of P. If P has length at least two or if b is a neighbor of a, then G contains a cycle of length at least 6, a contradiction. Thus, b is not a neighbor of a and f is connected to a and b by edges. This situation is illustrated in Figure 10. Let us assume that G has yet another vertex. Then, by connectivity, it has a vertex, say g, connected by an edge to f or to a vertex in C. If g is connected to f, then G is connected to a vertex in C by a path in $G \setminus \{f\}$. In such case, G has a cycle of length at least 6, a contradiction. Thus, g is connected by an edge to a vertex in C. Reasoning as for f, we argue that g must be connected to two vertices in C that are not connected in C. Unless g is connected to a and b, G contains a cycle of length 6 or two cycles of length 4 that share exactly one vertex. The first possibility contradicts our assumption. In the second case, G is not strongly DI-orderable by Proposition 28. Thus, let us assume that g is connected by edges to a and b. In this case, G is not strongly DI-orderable by Proposition 26. This leaves us with the case when G is as shown in Figure 10, with possibly some more edges added. However, unless the added edge is just like the one shown in Figure 11, G contains a cycle of length 6. For the two graphs T_5 and T_5^+ shown in Figures 10 and 11, we found that they are strongly DIE-orderable by a computer search. We provide either a computer generated order satisfying dominance, independence and extension on $C(T_5^+)$ or a proof of existence of such an order for every possible order on the vertices of T_5^+ in Appendix B.

Next, let us assume then that a longest cycle in a two-connected graph G has length 4 and let C be one such cycle. If G has more than four vertices, there is a vertex in G, say e, connected by an edge to a vertex in C. Let us denote this vertex by a. Since G is two-connected, there is a path in $G \setminus \{a\}$ connecting e to a vertex in $C \setminus \{a\}$. Let P be a shortest path like that. If that path connects e to a neighbor of a in C, then G contains a cycle of length 5, a contradiction. If that path connects a to the only non-neighbor of a in C, say b, and has more than one edge, G contains a cycle of length 5, a contradiction again. Thus, e is connected by edges to a and b. If G has any other vertices, one can show reasoning as above that G has a cycle of length at least 5, or that each of these vertices is connected to a and b by edges. The first situation contradicts our assumption. Thus, G is of the form shown in Figure 12. Hence, it is strongly DIE-orderable by Proposition 27.

The only two-connected graphs with longest cycle less than four are the triangle and the graphs consisting of a single edge. These are all obviously strongly DIE-orderable.

Let us observe that graphs that are not strongly DI-orderable are not strongly DIE-orderable, and that graphs that are strongly DIE-orderable are also strongly DI-orderable. Together with the discussion above, this proves the following result on two-connected graphs.

Theorem 29. *A two-connected graph is strongly DI- and DIE-orderable if and only if it lies in one of the following classes:*

- *Cycles*
- *Graphs with fewer than six vertices*
- *Graphs that contain no cycle of length five or more*
- T_5 *and* T_5^+

Theorem 29 implies that for two-connected graphs the concepts of strong DI- and DIE-orderability coincide, and by Proposition 9 this result can be extended to graphs with two-connected components.

Next, we outline the extent of the strong $DI(E)$-orderability for graphs that are connected but not two-connected. To this end, we will need two additional auxiliary results. The first one shows that graphs containing two vertex-disjoint cycles connected with a path are not strongly DIE-orderable if both cycles have length at least four.

Proposition 30. *Let G be a graph containing two vertex-disjoint cycles of length at least four. If these cycles are connected by a path, then G is not strongly DIE-orderable.*

Proof. First, assume that the path connecting the two cycles has even length and let $u, p_1, \ldots, p_n, v$ be the path connecting the two cycles. Then, let $u, u^*, u_1, \ldots, u_k$ be the cycle containing u and $v, v_1, \ldots, v_l, v^*$ the circle containing v (see Figure 13). We define a linear order $\leq$ by

$$u_1 < \cdots < u_k < \cdots < p_{n-3} < p_{n-1} < v < v^* < u^* < u < \cdots < p_{n-2} < p_n < v_1 < \cdots < v_l.$$

If the path has odd length, then let $u_k, p_1, \ldots, p_n, v$ be the path, u be a neighbor of u_k and $u, u^*, u_1, \ldots, u_k$ be the cycle containing u. As above, let $v, v_1, \ldots v_l, v^*$ the circle containing

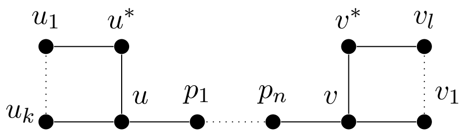

Figure 13: Two cycles connected by a path of even length.

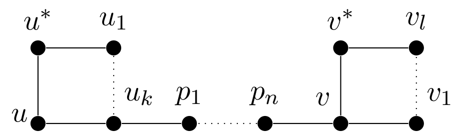

Figure 14: Two cycles connected by a path of odd length.

v (see Figure 14) and $\leq$ the same order as above. We claim that there is no order on $C(G)$ satisfying dominance, independence and the extension rule with respect to $\leq$.

Assume otherwise and let $\preceq$ be such an order. Assume that $\{v^*\} \prec \{p_n, v\}$ (for $n = 0$ replace p_n by u) By extension and independence we know that $\{p_n, v\} \preceq \{v, v_1\}$. Hence $\{v^*\} \prec \{v, v_1\}$. Further, by dominance, $\{v\} \prec \{v, v^*\}$ and, by independence, $\{v, v_1\} \preceq \{v, v^*, v_1\}$. By repeated application of dominance, $\{v, v^*, v_1\} \prec \{v, v^*, v_1, \ldots, v_{l-1}\}$. Thus, by transitivity, $\{v^*\} \prec \{v, v^*, v_1, \ldots, v_{l-1}\}$. By independence $\{v^*, v_l\} \preceq \{v, v^*, v_1, \ldots, v_l\}$. We also have $\{v_1, \ldots, v_l\} \prec \{v_l\}$ by dominance and $\{v^*, v_1, \ldots, v_l\} \preceq \{v^*, v_l\}$ by independence. Hence, we have $\{v^*, v_1, \ldots, v_l\} \preceq \{v, v^*, v_1, \ldots, v_l\}$ contradicting dominance.

It follows that $\{p_n, v\} \preceq \{v^*\}$ ($\{u, v\} \preceq \{v^*\}$, if $n = 0$). By extension, $\{v^*\} \prec \{u^*\}$. Thus, $\{p_n, v\} \prec \{u^*\}$ ($\{u, v\} \preceq \{u^*\}$, if $n = 0$). Observe that for $n = 0$ we have $\{u_k\} \prec \{v\}$ by extension and therefore by independence $\{u_k, u\} \preceq \{u, v\}$. By a sequence of similar arguments, for a path of even length we can derive $\{u_k, u\} \preceq \{u, p_1\} \preceq \cdots \preceq \{p_{n-1}, p_n\} \preceq \{p_n, v\}$ and, for a path of odd length, $\{u_k, u\} \preceq \{u_k, p_1\} \preceq \cdots \preceq \{p_{n-1}, p_n\} \preceq \{p_n, v\}$. Using this observation, we get $\{u_k, u\} \prec \{u^*\}$. From $u^* < u$ we get by dominance $\{u^*, u\} \prec \{u\}$ and hence by independence $\{u_k, u^*, u\} \preceq \{u_k, u\}$. By dominance, we can extend this to $\{u_2, \ldots, u_k, u^*, u\} \prec \{u_k, u^*, u\}$. By transitivity we get $\{u_2 \ldots, u_k, u^*, u\} \prec \{u^*\}$. Therefore, by independence $\{u_1, \ldots, u_k, u^*, u\} \preceq \{u_1, u^*\}$. Applying dominance we obtain $\{u_1, \ldots, u_k, u^*\} \prec \{u_1, \ldots, u_k, u^*, u\}$. Thus, by transitivity, $\{u_1, \ldots, u_k, u^*\} \prec \{u_1, u^*\}$.

On the other hand, by repeated application of dominance we have $\{u_1\} \prec \{u_1, \ldots, u_k\}$. Thus, by independence, $\{u_1, u^*\} \preceq \{u_1, \ldots, u_k, u^*\}$, a contradiction. □

Observe that this result does not tell us whether such graphs are strongly DI-orderable. Indeed we used a computer program to check that a graph consisting of two cycles of length four connected by an edge is strongly DI-orderable. This implies that strong DI- and strong DIE-orderability are not equivalent on arbitrary graphs.

The next result states that whenever removing an edge from a graph with a given order over its vertices leads to two disjoint graphs such that one can be ordered with respect to dominance and independence and the other can be ordered with respect to dominance and *strict* independence, then the original graph can also be ordered with respect to dominance and independence.

Proposition 31. *Let $G = (V, E)$ be a connected graph and $\leq$ a linear order on V. Let $vw \in E$ be an edge of G such that $(V, E \setminus \{vw\})$ is a graph with two connected components G' and G''. If $C(G')$ can be ordered satisfying dominance and independence and $C(G'')$ can be ordered satisfying dominance and strict independence then $C(G)$ can be ordered satisfying dominance and independence.*

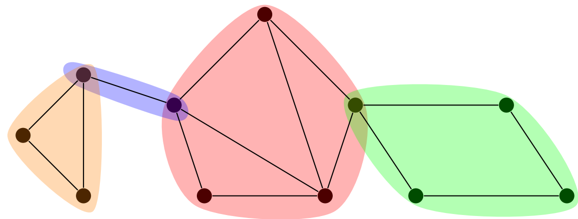

Figure 15: A graph and its biconnected components

Proof. Let us assume that $G' = (V', E')$ and $G'' = (V'', E'')$. Wlog we may assume that $v \in V'$, $w \in V''$. We will present the proof under the assumption that $v < w$. The other case, $w < v$, works analogously.

We partition $C(G)$ in three collections of sets: $P_1 = C(G')$, $P_2 = \{A \in C(G) \mid v, w \in A\}$ and $P_3 = C(G'')$. Let $\preceq_1$ be any order satisfying dominance and independence on P_1 (with respect to $\leq$ restricted to V'), and $\preceq_3$ any *linear* order satisfying dominance and strict independence on P_3 (with respect to $\leq$ restricted to V''). We define an order $\preceq$ on $C(G)$ by setting $A \preceq B$ (where $A, B \in C(G)$) if and only if

1. $A, B \in P_1$, and $A \preceq_1 B$
2. $A, B \in P_3$, and $A \preceq_3 B$
3. $A, B \in P_2$ and $A \cap V'' \prec_3 B \cap V''$
4. $A, B \in P_2$ and $A \cap V'' = B \cap V''$ and $A \cap V' \preceq_1 B \cap V'$
5. $A \in P_i$, $B \in P_j$ and $i < j$ (in fact, in this case, $A \prec B$ holds).

The relation $\preceq$ is obviously an order. It can be shown that this order satisfies dominance and independence. A full proof can be found in Appendix A. □

The next two results describe our knowledge of the extent of strong DIE- and DI-orderability. To formulate them we need more notation. We recall that a *biconnected* component of a graph G is any maximal two-connected subgraph of G. We note that a single edge is two-connected and may appear in a graph as its biconnected component (it is the case, when removing this edge disconnects the graph). Every graph G can be viewed as a tree-like structure composed of its biconnected components, in which whenever two biconnected components share a node, this node must be an articulation point. This representation of a graph is illustrated in Figure 15. We denote by $\mathcal{B}$ the set of all two-connected graphs that are strongly DIE-orderable (cf. Theorem 29). Next, we call a biconnected component of G *large* if it contains a cycle of length at least four. Thus, if a biconnected component is not large, it consists of a single edge or is a cycle of length three.

We define now four classes of graphs, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. A graph G is in $\mathcal{C}_3$ if all its biconnected components belong to $\mathcal{B}$ and no large biconnected component of G shares an articulation point with another non-edge biconnected component of G. A graph G is in $\mathcal{C}_2$ if $G \in \mathcal{C}_3$ and G has at most one large biconnected component. A graph G is in $\mathcal{C}_1$ if $G \in \mathcal{C}_2$

and has at most one non-edge biconnected component. Finally, a graph G is in $\mathcal{C}_0$ if $G \in \mathcal{C}_1$ and either every biconnected component of G is an edge or no biconnected component of G is an edge (i.e. G is either a tree or in $\mathcal{B}$). Clearly, $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$. Figures 16–18 show examples of graphs in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Proposition 32. *All graphs in $\mathcal{C}_1$ are strongly DI-orderable. If a graph is not in $\mathcal{C}_3$ then it is not strongly DI-orderable.*

Proof. First we prove that all graphs in $\mathcal{C}_1$ are strongly orderable. We proceed by contradiction and consider the smallest graph in $\mathcal{C}_1$ that is a counterexample to the assertion. Since strict independence implies independence, trees are strongly DI-orderable (cf. Corollary 11). It follows that G contains a unique DIE-orderable two-connected subgraph, say C.

Let $\leq$ be a linear order on V that cannot be lifted to an order $\preceq$ on $C(G)$ so that to satisfy dominance and independence. First assume $C = G$. Then G is strongly DIE-orderable by assumption. So assume there is a vertex $v \in G$ such that $v \notin C$. Let u be any neighbor of v in G. It follows that $G \setminus \{uv\}$ consists of a graph in $\mathcal{C}_1$, say G', and a tree, say T. By the way G was chosen, G' is strongly DI-orderable. Moreover, T is strongly DI^S-orderable (Corollary 11). Thus, Proposition 31 implies that G is strongly DI-orderable, a contradiction.

Now assume G is not in $\mathcal{C}_3$. G can either not be in $\mathcal{C}_3$ because it contains a biconnected component that is not in $\mathcal{B}$ or because it contains a large biconnected component that shares a node with a non-edge biconnected component. In the first case, G is obviously not strongly DIE-orderable. In the second case, the large biconnected component contains by definition a longest cycle of length at least four. Observe that in a two-connected graph with longest cycle of length at least four, every vertex is contained in a cycle of length at least four, because every vertex must be connected to two different vertices in the cycle of length at least four. Hence in the second case, G contains a cycle of length at least three and a cycle of length at least four that share one vertex. Therefore G is not DIE-orderable by Proposition 28. $\square$

It follows that for a connected graph G, its DI-orderability is open only if $G \in \mathcal{C}_3 \setminus \mathcal{C}_1$.

Proposition 33. *All graphs in $\mathcal{C}_0$ are strongly DIE-orderable. If a graph is not in $\mathcal{C}_2$ then it is not strongly DIE-orderable.*

Proof. All graphs in $\mathcal{C}_0$ are either trees or in $\mathcal{B}$. Hence, they are strongly DIE-orderable either by Corollary 11 or by Theorem 29. By Proposition 32 a graph that is not in $\mathcal{C}_3$ can not be strongly DI-orderable, therefore it can also not be DIE-orderable. Now assume G is in $\mathcal{C}_3 \setminus \mathcal{C}_2$. Then it contains two two-connected subgraphs with longest cycle four or longer. Hence it is not strongly DIE-orderable by Proposition 30. It follows that any graph that is not in $\mathcal{C}_2$ can not be strongly DIE-orderable. $\square$

For a connected graph G, its DIE-orderability is open only if $G \in \mathcal{C}_2 \setminus \mathcal{C}_0$. By Proposition 9 we can extend these results to arbitrary graphs as follows:

Theorem 34. *If every connected component of a graph G is in $\mathcal{C}_1$ then G is strongly DI-orderable. If a graph G contains at least one component not in $\mathcal{C}_3$ then G is not strongly DI-orderable.*

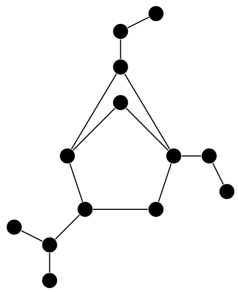

Figure 16: An example of a graph in the class $\mathcal{C}_1$

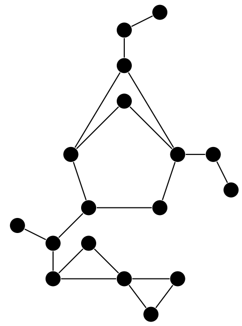

Figure 17: An example of a graph in the class $\mathcal{C}_2$

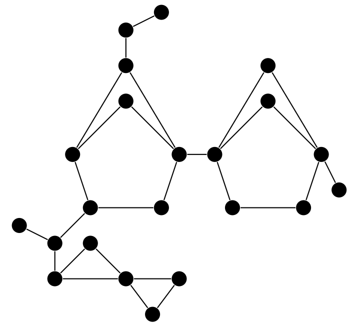

Figure 18: An example of a graph in the class $\mathcal{C}_3$

Theorem 35. *If every connected component of a graph G is in $\mathcal{C}_0$ then G is strongly DIE-orderable. If a graph G contains at least one component not in $\mathcal{C}_2$ then G is not strongly DIE-orderable.*

In particular, this implies, for example, that every pseudoforest is strongly DI-orderable.

Finally, we provide one preliminary result on graphs that are weakly orderable with respect to dominance and independence. The result of Kannai and Peleg (1984) implies that the complete graph K_n is not weakly DIE-orderable for $n \geq 6$ (recall that for fully symmetric graphs weak and strong orderability coincide). Interestingly, in our setting of orderable graphs, it turns out this result is optimal in the sense that every proper subgraph of K_6 is weakly DIE-orderable.

Proposition 36. *Every proper subgraph G of the complete graph K_6 is weakly DIE-orderable and thus weakly DI-orderable.*

The proof is provided in Appendix A. Interestingly, this result can not be extended to strong extension. This is because K_4 is a proper subgraph of K_6 and not weakly DIE^S-orderable, which follows from Remark 6 and Proposition 20.

6. Discussion

Lifting a preference order on elements of some universe to a preference order on *subsets* of this universe respecting certain axioms is a fundamental problem, but well-known impossibility results pose severe limits on when such liftings exist. Bossert (1995) observed that these impossibility results may be avoided by considering families of subsets of the same fixed cardinality. Maly and Woltran (2017) showed that, deciding whether a given linear order on a set of objects X can be lifted to an order on a given collection of subsets of X is NP-complete. Bouveret et al. (2017) were the first to consider graph topologies and subsets inducing connected subsets. They proposed this model for the problem of fair allocation of indivisible goods. Our work adopts their idea for an implicit representation of classes of families of non-empty subsets (in contrast to the explicit representation considered by

Bossert 1995 and Maly and Woltran 2017). It turns out that for several interesting families of sets definable in terms of graphs the impossibility results observed for the family of all non-empty subsets of a set can be avoided!

Our main results characterize strongly and weakly DI^SE^S-orderable graphs, and moreover, show that the same classes are obtained for strongly and weakly DI^SE-orderable and strongly and weakly DI^S-orderable graphs. In other words, the two versions of the extension rule we have considered do not affect our results when strict independence is considered. The picture is different if independence is used. We obtain a complete characterization of strongly DIE^S-orderable graphs. For strong DI-orderability and DIE-orderability we have an almost complete picture. Our results show rich classes of well-motivated families of sets that allow for lifting of linear orders in ways that combine dominance and (strict) independence. They also suggest that independence, despite being much less restrictive than strict independence, does not significantly extend that class of strongly orderable graphs. This suggests that it is strict independence that might be the axiom to focus on. Finally, we only touched on weak DI-orderability and showed that all proper subgraphs of a complete graph K_6 are weakly DI-orderable but have as of yet no general results on weakly DI-orderable graphs. We have to leave more detailed analysis on those graphs for future work.

In fact, our research opens several directions for future studies. Using graphs as implicit representations of families of sets is just one of many possibilities. Knowledge representation often uses logic formalisms towards this end. For instance, formulas can be viewed as concise representations of the families of their models. Together with an order of the atoms in the formulas, it is natural to ask how to rank these models and for which classes of formulas such a lifting respects certain criteria. A particular formalism where lifting orders is inherently needed can be found in the area of formal argumentation where ranking semantics (Amgoud & Ben-Naim, 2013; Bonzon, Delobelle, Konieczny, & Maudet, 2016) have received increasing interest within the last years. Hereby, a total (not necessarily linear) order on arguments is obtained from the structure of an argumentation framework which then can be used to rank the standard extensions (i.e. certain sets of arguments) of that framework (Yun, Vesic, Croitoru, & Bisquert, 2018). It is evident that this is exactly the setting of lifting we have studied here and our results can provide additional insight in which scenarios such liftings satisfy certain criteria. Further, for all mentioned representations of families of sets it is a key challenge to establish the complexity of deciding the existence of lifted orders, and study algorithms for computing lifted orders in some concise representation. Finally, another direction for future work, is to investigate how our findings apply to the "reverse" problem of social ranking (Moretti & Öztürk, 2017), where an order over individuals needs to be obtained from a given order over sets of individuals.

Acknowledgments

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Appendix A. Full Proofs

Proof of Proposition 9

Proposition. *Let $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{Y} \subseteq \mathcal{P}(Y) \setminus \{\emptyset\}$ be families of subsets of X and Y respectively such that $X \cap Y = \emptyset$. If $\mathcal{X}$ and $\mathcal{Y}$ are strongly DI-orderable, then $\mathcal{X} \cup \mathcal{Y}$ is strongly DI-orderable.*

The same holds if we replace strongly DI-orderable by strongly DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable or by weakly DI-, DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable.

Proof. Let us define $Z = X \cup Y$. Let us assume that $\leq_X$ and $\leq_Y$ are linear orders on X and Y such that some orders $\preceq_X$ on $\mathcal{X}$ and $\preceq_Y$ on $\mathcal{Y}$ satisfy all necessary axioms with respect to $\leq_X$ and $\leq_Y$. To prove the claim in all its versions, it suffices to show that for every linear order $\leq$ on Z such that $\leq_X$ and $\leq_Y$ are restrictions of $\leq$ to X and Y, respectively, there is an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$ satisfying all necessary axioms with respect to $\leq$.

DIE- and DI^SE-orderability. We first handle the case that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DIE- or DI^SE-orderable, i.e. the case that $\preceq_X$ and $\preceq_Y$ satisfy the extension axiom. Let $\{x_1, \ldots, x_k\}$ be an enumeration of all elements in X such that $x_1 <_X x_2 <_X \ldots <_X x_k$. By the extension axiom, $\{x_1\} \prec_X \{x_2\} \prec_X \ldots \prec_X \{x_k\}$ Similarly, let $\{y_1, \ldots, y_l\}$ be an enumeration of all elements of Y such that $y_1 <_Y y_2 <_Y \ldots <_Y y_l$ which, also by the extension axiom, implies $\{y_1\} \prec_Y \{y_2\} \prec_Y \ldots \prec_Y \{y_l\}$. Let $z \in Z$. If $z = x_i$, where $1 \leq i \leq k-1$, we define

$$C_z = \{A \in \mathcal{X} \mid \{x_i\} \preceq_X A \prec_X \{x_{i+1}\}\}.$$

If $z = x_k$, we define

$$C_z = \{A \in \mathcal{X} \mid \{x_k\} \preceq_X A\}.$$

We define sets C_z for $z = y_i$, where $1 \leq i \leq l$, analogously, with $\mathcal{Y}$, l, y_i, and $\preceq_Y$ in place of $\mathcal{X}$, k, x_i, and $\preceq_X$, respectively. It is clear that for every $z \in Z$, $\{z\} \in C_z$.

Further, to simplify the notation later on, we assume the existence of a "dummy" element 0 (not in Z) such that $0 < z$, for every $z \in Z$, and we define

$$C_0 = \{A \in \mathcal{X} \mid A \prec_X \{x_1\}\} \cup \{A \in \mathcal{Y} \mid A \prec_Y \{y_1\}\}.$$

Clearly, the sets C_z, $z \in \{0\} \cup Z$ are pairwise disjoint. Moreover, since $Z = X \cup Y$, it follows that, $\mathcal{X} \cup \mathcal{Y} = \bigcup_{z \in \{0\} \cup Z} C_z$.

Let $\leq$ be any linear order on Z such that $\leq_X$ and $\leq_Y$ are the restrictions of $\leq$ to X and Y, respectively. To define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$, for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in C_z$, for some $z \in X$, and $A \preceq_X B$
- $A, B \in C_z$, for some $z \in Y$, and $A \preceq_Y B$
- $A, B \in C_0 \cap \mathcal{X}$, and $A \preceq_X B$
- $A, B \in C_0 \cap \mathcal{Y}$, and $A \preceq_Y B$

- $A, B \in C_0$, $A \in \mathcal{X}$, $B \in \mathcal{Y}$
- $A \in C_z$, $B \in C_{z'}$, for $z, z' \in \{0\} \cup Z$, and $z < z'$.

It is straightforward to verify that the relation $\preceq$ is total, reflexive and transitive. Hence, it is an order. It is also clear that if $z, z' \in \{0\} \cup Z$, $z < z'$, $A \in C_z$ and $B \in C_{z'}$, then $A \prec B$ holds. Indeed, in such case, by the definition we have $A \preceq B$. Moreover, $B \preceq A$ is impossible (none of the six cases applies).

We claim that $\preceq$ is an order satisfying the same axioms as $\preceq_X$ and $\preceq_Y$. First, we will prove that $\preceq$ satisfies the extension axiom. Thus, let us consider elements $z, z' \in Z$ such that $z < z'$. By our earlier observation, $\{z\} \in C_z$ and $\{z'\} \in C_{z'}$. Thus, $\{z\} \preceq \{z'\}$ (by the last clause of the definition). Since we do not have $z' < z$ (because $\leq$ is a linear order), $\{z'\} \preceq \{z\}$ does not hold. It follows that $\preceq$ satisfies the extension axiom.

Next, we note that $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will prove it for $\preceq_X$; the other case is similar. Therefore, let us assume that $A, B \in \mathcal{X}$. If $A, B \in C_z$, where $z \in \{0\} \cup X$, then by definition, $A \preceq B$ if and only if $A \preceq_X B$. Thus, let $A \in C_z$ and $B \in C_{z'}$, where $z, z' \in \{0\} \cup X$ and $z \neq z'$. If $A \preceq B$ then it must be because of the last clause in the definition of $\preceq$. Consequently, $z < z'$. Since $\leq_X$ is the restriction of $\leq$ to X, $z <_X z'$. Since $A \in C_z$ and $B \in C_{z'}$, $A \prec_X \{z'\}$ and $\{z'\} \preceq_X B$. Thus, $A \preceq_X B$ by transitivity. Conversely, assume that $A \preceq_X B$. If $z' < z$, then $z' <_X z$. Since $A \in C_z$ and $B \in C_{z'}$, $B \prec_X \{z\} \preceq_X A$. By transitivity, $B \prec_X A$, a contradiction. Since $z \neq z'$, we have $z < z'$ and so, $A \preceq B$.

Using this claim, it is easy to show that $\preceq$ satisfies dominance and (strict) independence if $\preceq_X$ and $\preceq_Y$ satisfy the corresponding axiom(s). Indeed, $A, A \cup \{x\} \in \mathcal{X} \cup \mathcal{Y}$ implies $A, A \cup \{x\} \in \mathcal{X}$ or $A, A \cup \{x\} \in \mathcal{Y}$, and $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X} \cup \mathcal{Y}$ implies $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ or $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{Y}$.

DIE^S- and DI^SE^S-orderability. Now assume that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DIE^S- or DI^SE^S-orderable, i.e. that $\prec_X$ and $\prec_Y$ satisfy strong extension. Let $z \in Z$. We define

$$C_z = \begin{cases} \{A \in \mathcal{X} \mid \max_{\leq_X}(A) = z\} & \text{if } z \in X \\ \{A \in \mathcal{Y} \mid \max_{\leq_Y}(A) = z\} & \text{if } z \in Y \end{cases}$$

Let us now assume that the orders $\preceq_X$ and $\preceq_Y$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively, satisfy strong extension with respect to $\leq_X$ and $\leq_Y$, respectively, and let $\leq$ be any linear order on Z such that $\leq_X$ and $\leq_Y$ are the restrictions of $\leq$ to X and Y. To define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$, for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in C_z$, $z \in X$ and $A \preceq_X B$
- $A, B \in C_z$, $z \in Y$ and $A \preceq_Y B$
- $A \in C_z$, $B \in C_{z'}$ and $z < z'$.

It is straightforward to show that $\preceq$ is total, reflexive and transitive. Further, directly from the definition, it follows that $\preceq$ satisfies the strong extension property.

Next, we note that $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will prove it for $\preceq_X$; the other case is similar. Thus, let us consider sets $A, B \in \mathcal{X}$. By definition, $A \in C_x$ and $B \in C_y$, where $x = \max(A)$ and $y = \max(B)$. If $A \preceq B$ then $x \leq y$ ($y < x$ is impossible by the strong expansion axiom). If $x = y$, then $A \preceq_X B$ (the first condition is the only one that can imply $A \preceq B$ in this case). If $x < y$, then strong extension of $\preceq_X$ implies $A \preceq_X B$. Conversely, if $A \preceq_X B$ then $x \leq y$ ($y < x$ is impossible by strong extension of $\preceq_X$). If $x = y$, then $A \preceq B$ by the first condition. If $x < y$ then $A \preceq B$, by the third condition. Thus, dominance and (strict) independence can be argued as above.

DI- and DI^S-orderability. Finally, we will consider the case that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DI- or DI^S-orderable, i.e. the case that $\preceq_X$ and $\preceq_Y$ satisfy dominance and (strict) independence but no assumptions are made about extension or strong extension. In this case, to define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$, for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in \mathcal{X}$ and $A \preceq_X B$
- $A, B \in \mathcal{Y}$ and $A \preceq_Y B$
- $A \in \mathcal{X}$, $B \in \mathcal{Y}$.

It is straightforward to show that $\preceq$ is total, reflexive and transitive. Further, it is clear that the relations $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. Thus, we can derive dominance (independence and strict independence, respectively) of $\preceq$ from dominance (independence or strict independence, respectively) of $\preceq_X$ and $\preceq_Y$ in the same way as before. □

Proof of Proposition 22

Proposition. *Let $G = (V, E)$ be a graph and $\leq$ be a linear order on V. If there is an order on $C(G \setminus \{\min(V)\})$ satisfying dominance, independence and strong extension, then there exists an order on $C(G)$ satisfying dominance, independence and the extension rule.*

Proof. Wlog we may assume that $V = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $\leq$ is the natural linear order on V. Let $\preceq^*$ be an order on $C(G \setminus \{1\})$ satisfying dominance, independence and strong extension. We define an order $\preceq$ on $C(G)$ by setting $A \preceq B$ if and only if:

1. $1 \in A$ and $1 \notin B$
2. $1 \notin A \cup B$ and $A \preceq^* B$
3. $1 \in A \cap B$ and $\max(A) \leq \max(B)$.

It follows directly from the definition, that if $1 \notin A \cup B$ and $A \prec^* B$, then $A \prec B$. Similarly, if $1 \in A \cap B$ and $\max(A) < \max(B)$, then $A \prec B$.

We claim that $\preceq$ is an order satisfying dominance, independence and the extension rule.

Order. Obviously, $\preceq$ is reflexive. To prove transitivity, let $A, B, C \in C(G)$ satisfy $A \preceq B$ and $B \preceq C$. If $A \preceq B$ holds by (1), we have $1 \in A$ and $1 \notin B$. Since $B \preceq C$, it follows that $1 \notin C$ and $B \preceq^* C$. Thus, $A \preceq C$ by (1).

If $A \preceq B$ holds by (2), we know $A \preceq^* B$, $1 \notin A$, and $1 \notin B$. As before, the latter implies that $1 \notin C$ and $B \preceq^* C$. By the transitivity of $\preceq^*$, $A \preceq^* C$. Consequently, $A \preceq C$ by (2).

Finally, if $A \preceq B$ holds by (3), then $1 \in A$, $1 \in B$, and $\max(A) \leq \max(B)$. Since $B \preceq C$, $1 \notin C$, or $1 \in C$ and $\max(B) \leq \max(C)$. In the first case, $A \preceq C$ by (1). In the second case, $\max(A) \leq \max(C)$. Thus, $A \preceq C$ holds by (3).

To show that the relation $\preceq$ is total, let us consider sets $A, B \in C(G)$. If $1 \in A \setminus B$ then $A \preceq B$ by (1). The case $1 \in B \setminus A$ is symmetric. If $1 \notin A \cup B$, $A \preceq B$ or $B \preceq A$ follows as $\preceq^*$ is total. Lastly, if $1 \in A \cap B$, $A \preceq B$ or $B \preceq A$ follows as $\leq$ is total. Thus, $\preceq$ is an order.

Extension rule. Since $\preceq^*$ satisfies the strong extension rule, (2) implies that $\{i\} \prec \{j\}$, for all i, j such that $2 \leq i < j \leq n$ (as a matter of fact, here we need only the extension rule). Further, (1) implies that $\{1\} \prec \{j\}$, for $j = 2, 3, \ldots, n$.

Dominance. Let us consider sets $A, A \cup \{x\} \in C(G)$ such that $x < \min(A)$. If $x = 1$ we have $A \cup \{x\} \preceq A$ by (1). Furthermore, we have $A \not\preceq A \cup \{x\}$ (clearly, neither of the conditions (1)-(3) applies). Thus, $A \cup \{x\} \prec A$. If $x \neq 1$, $1 \notin A$ therefore $1 \notin A \cup \{x\}$. Since $\preceq^*$ satisfies dominance, we have $A \cup \{x\} \prec^* A$. By the observation above, it follows that $A \cup \{x\} \prec A$.

Next, let us consider sets $A, A \cup \{x\} \in C(G)$ such that $\max(A) < x$. Then we know $x \neq 1$. Assume that $1 \notin A$. Then, since $\preceq^*$ satisfies dominance, $A \prec^* A \cup \{x\}$. Recall that we have $1 \notin A$ and $1 \notin A \cup \{x\}$. Thus, by the observation above, $A \prec A \cup \{x\}$. Let us assume then that $1 \in A$. Then $1 \in A \cup \{x\}$. Moreover, we have $\max(A) < x = \max(A \cup \{x\})$. Thus, using the observation above, $A \prec A \cup \{x\}$.

Independence. Let us consider sets $A, B \in C(G)$ and an element $x \in V$ such that $x \notin A \cup B$, $A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$.

We first assume that $1 \notin A \cup B$. If $x \neq 1$, we have $A \cup \{x\} \preceq^* B \cup \{x\}$ because $\preceq^*$ satisfies independence. Hence, $A \cup \{x\} \preceq B \cup \{x\}$ by (2). If, on the other hand, $x = 1$, we observe that $\max(A) \leq \max(B)$. Indeed, since $\preceq^*$ satisfies strong extension, $\max(B) < \max(A)$ would imply $B \prec^* A$ which, in turn would imply $B \prec A$ (as $1 \notin A \cup B$), a contradiction. Therefore, $A \cup \{x\} \preceq B \cup \{x\}$ by (3).

Next, assume that $1 \in A$ and $1 \notin B$. Then $x \neq 1$ and hence, $A \cup \{x\} \preceq B \cup \{x\}$ by (1).

Finally, assume that $1 \in A, B$. Then $A \prec B$ implies $\max(A) < \max(B)$. Assume first that $x < \max(B)$. Then $\max(A \cup \{x\}) < \max(B) = \max(B \cup \{x\})$ and so, $A \cup \{x\} \preceq B \cup \{x\}$ by (3). If $\max(B) < x$, then $\max(A \cup \{x\}) = \max(B \cup \{x\}) = x$ and hence $A \cup \{x\} \preceq B \cup \{x\}$. Finally, the case $x = \max(B)$ is impossible as $x \notin B$. □

Proof of Proposition 24

Proposition. *Let $G = (V, E)$ be a graph containing distinct vertices $a, b \in V$ connected by three paths that do not share any vertices except a and b such that two of them have length at least three, or one of the paths has length at least four and one of the remaining two paths is of length two. Then, G is not strongly DI-orderable.*

Proof. Let the three paths be $a, u_1, \ldots, u_k, b$; $b, v_1, \ldots, v_l, a$; and $a, w_1, \ldots, w_m, b$ (see Figure 19). By the assumption on the lengths of the paths, wlog we will assume $k, m \geq 2$ in the first case, and $k = 2$ and $m \geq 3$ in the second one. Let us also define

$$W = \{a, u_1, \ldots, u_k, v_1, \ldots, v_l, w_1, \ldots, w_m, b\}.$$

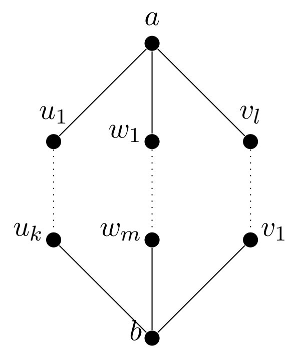

Figure 19: Vertices a, b connected by three mutually disjoint paths.

We first consider the case when $k, m \geq 2$. Let $\leq$ be any linear order on G such that

$$u_1 < \cdots < u_k < b < v_1 < \ldots < v_l < a < w_1 < \cdots < w_m.$$

It is clear that such orders exist.

Let us assume that there is an order on $C(G)$, say $\preceq$, that satisfies dominance and independence. All sets we use in the following belong to $C(G)$. This is easy to see and we will not be making it explicit when we compare sets under $\preceq$.

Suppose $\{b\} \prec W \setminus \{u_1, w_m\}$. By independence,

$$\{b, w_m\} \preceq W \setminus \{u_1\}.$$

Further, since $v_1 < \ldots < v_l < a < w_1 < \cdots < w_m$, repeated application of dominance implies $\{v_1, \ldots, v_l, a, w_1, \ldots, w_m\} \prec \{w_m\}$. Thus, independence implies

$$\{b, v_1, \ldots, v_l, a, w_1, \ldots, w_m\} \preceq \{b, w_m\}.$$

However, since $k > 1$, dominance also implies that

$$W \setminus \{u_1\} \prec \{b, v_1, \ldots, v_l, a, w_1, \ldots, w_m\}.$$

By transitivity, $\{b, w_m\} \prec \{b, w_m\}$, a contradiction.

Therefore, we must have $W \setminus \{u_1, w_m\} \preceq \{b\}$. By repeated application of dominance,

$$\{b\} \prec \{b, v_1, \ldots, v_l, a\} \prec \{a\}.$$

Thus, $W \setminus \{u_1, w_m\} \prec \{a\}$ and so, by independence,

$$W \setminus \{w_m\} \preceq \{u_1, a\}.$$

Further, dominance also implies $\{u_1\} \prec \{u_1, \ldots, u_k, b, v_1, \ldots, v_l\}$. Hence, by independence,

$$\{u_1, a\} \preceq \{u_1, \ldots, u_k, b, v_1, \ldots, v_l, a\}.$$

Since $m > 1$, dominance implies

$$\{u_1, \ldots, u_k, b, v_1, \ldots, v_l, a\} \prec W \setminus \{w_m\}.$$

Thus, by transitivity, $\{u_1, a\} \prec \{u_1, a\}$, a contradiction.

Next, we consider the remaining case $k = 1$ and $m \geq 3$. This time, we assume the order

$$u_1 < b < v_1 < \ldots < v_l < a < w_m < \cdots < w_1.$$

Let us assume that $\{a\} \prec W \setminus \{u_1, w_1\}$. Reasoning as before yields a contradiction. Namely, by independence,

$$\{a, w_1\} \preceq W \setminus \{u_1\}.$$

Further, since $w_1 > \cdots > w_m$, repeated application of dominance implies $\{w_1, \ldots, w_m\} \prec \{w_1\}$. Thus, independence implies

$$\{a, w_1, \ldots, w_m\} \preceq \{a, w_1\}.$$

However, dominance also implies that

$$W \setminus \{u_1\} \prec \{a, w_1, \ldots, w_m\}.$$

By transitivity, $\{a, w_1\} \prec \{a, w_1\}$, a contradiction. Hence, we must have $W \setminus \{u_1, w_1\} \preceq \{a\}$. Since $k = 1$ and $m \geq 3$, dominance implies

$$\{b, v_1, \ldots, v_l, a, w_m\} \prec \{b, v_1, \ldots, v_l, a, w_2, \ldots, w_m\} = W \setminus \{u_1, w_1\}.$$

Thus, by transitivity, $\{b, v_1, \ldots, v_l, a, w_m\} \prec \{a\}$ and, by independence,

$$\{u_1, b, v_1, \ldots, v_l, a, w_m\} \preceq \{u_1, a\}.$$

On the other hand, dominance implies $\{u_1\} \prec \{u_1, b, v_1, \ldots, v_l\}$. Thus, by independence and dominance

$$\{u_1, a\} \preceq \{u_1, b, v_1, \ldots, v_l, a\} \prec \{u_1, b, v_1, \ldots, v_l, a, w_m\},$$

a contradiction. □

Proof of Proposition 27

Proposition. *Let G be a graph consisting of two vertices a, b, where $a \neq b$, connected by arbitrarily many paths of length at most two. Then G is strongly DIE-orderable*

Proof. Let us consider any linear order on the set V of vertices of G. Wlog we may assume that $a < b$. Under this assumption, we define the sets $L = \{v \in V \mid b < v\}$, $I = \{v \in V \mid a \leq v \leq b\}$ and $S = \{v \in V \mid v < a\}$.

Next, we define two orders $\prec^+$ and $\prec^-$ on the family of all subsets of V by setting

1. $A \prec^+ B$ if and only if $\max(A \triangle B) \in B$
2. $A \prec^- B$ if and only if $\min(A \triangle B) \in A$.

It is straightforward to verify that both relations are indeed orders.

We also define an order $\preceq^*$ on $C(G)\big|_I$, the set of all nonempty subsets of I inducing in G a connected subgraph. Clearly, every set in $C(G)\big|_I$ is either a singleton, or contains at least one of a and b. To define $\preceq^*$, for each $A \in C(G)\big|_I$ we define its type, written *type*(A):

$$\mathit{type}(A) := \begin{cases} a \text{ if } a \in A, b \notin A \\ b \text{ if } b \in A, a \notin A \\ ab \text{ if } a, b \in A \\ 0 \text{ if } a, b \notin A \end{cases}$$

We note that $\mathit{type}(A) = 0$ if and only if A is a singleton set other than $\{a\}$ and $\{b\}$. We order the types $a < 0 < ab < b$ (we point out that the order on types is unrelated to the order on V that we are considering; it will always be clear from the context whether we are comparing types or elements of V).

With these concepts in hand, for $A, B \in C(G)\big|_I$ we set $A \preceq^* B$ if and only if

1. $\mathit{type}(A) < \mathit{type}(B)$,
2. $\mathit{type}(A) = \mathit{type}(B) = a$ and $A \prec^+ B$,
3. $\mathit{type}(A) = \mathit{type}(B) = 0$, $A = \{v\}$, $B = \{w\}$ and $v < w$,
4. $\mathit{type}(A) = \mathit{type}(B) = ab$
5. $\mathit{type}(A) = \mathit{type}(B) = b$ and $A \prec^- B$,

We claim that $\preceq^*$ is an order on $C(G)\big|_I$ satisfying dominance, independence and extension. It is indeed an order, because $\preceq^*$ restricted to sets of any type is an order, and the types are linearly ordered.

Extension Assume $v, w \in V$ and $v < w$. If $w = b$ or $v = a$ then $\mathit{type}(\{v\}) < \mathit{type}(\{w\})$. Thus, $\{v\} \prec^* \{w\}$ by (1). If $v, w \notin \{a, b\}$ then $\mathit{type}(\{v\}) = \mathit{type}(\{w\}) = 0$ and $\{v\} \prec^* \{w\}$ holds by (3).

Dominance Let us consider a set $A \in C(G)\big|_I$ and an element $x \in I \setminus A$ such that $A \cup \{x\} \in C(G)\big|_I$. We need to show that if $x < \min(A)$ then $A \cup \{x\} \prec^* A$, and if $x > \max(A)$, then $A \prec^* A \cup \{x\}$.

We consider the case $x < \min(A)$. The other one is dual. Clearly, $x < \min(A)$ implies that $\mathit{type}(A) \neq a, ab$. Let us assume first that $\mathit{type}(A) = 0$. It follows that $x = a$. Thus, $\mathit{type}(A \cup \{x\}) = a$ and $A \cup \{x\} \prec^* A$ holds by (1). The only other possibility is that $\mathit{type}(A) = b$. If $x = a$, we have $\mathit{type}(A \cup \{x\}) = ab$. Hence, $A \cup \{x\} \prec^* A$ holds by (1). If $x \neq a$, $\mathit{type}(A \cup \{x\}) = b$. Moreover, $A \cup \{x\} \prec^- A$ (because $\min(A \triangle (A \cup \{x\})) = x \in A \cup \{x\}$). Hence, $A \prec^* A \cup \{x\}$ holds by (5).

Independence Let us consider sets $A, B \in C(G)\big|_I$ and an element $x \in I \setminus (A \cup B)$ such that $A \prec^* B$ and $A \cup \{x\}, B \cup \{x\} \in C(G)\big|_I$. First, let us assume that $x \notin \{a, b\}$. Then $A \prec^* B$ holds by one of the conditions (1), (2), (4) or (5). It also follows that $\mathit{type}(A) = \mathit{type}(A \cup \{x\})$, $\mathit{type}(B) = \mathit{type}(B \cup \{x\})$, and $A \triangle B = (A \cup \{x\}) \triangle (B \cup \{x\})$. It is now easy to see that if $A \prec^* B$ holds by the condition (i), where i=1, 2, 4, or 5, then the same condition (i) implies that $A \cup \{x\} \preceq^* B \cup \{x\}$.

If $x = a$, then $a \notin A \cup B$. It follows that $type(A) = 0$ or b, $type(B) = 0$ or b, and $A \prec^* B$ holds by the condition (1), (3) or (5). In the first case, $type(A) = 0$ and $type(B) = b$. Thus, $type(A \cup \{x\}) = a$, $type(B \cup \{x\}) = ab$ and, consequently, $A \cup \{x\} \preceq^* B \cup \{x\}$ holds by (1). In the second case, there are $v, w \in I \setminus \{a, b\}$ such that $A = \{v\}$, $B = \{w\}$ and $v < w$. It follows that $type(A \cup \{x\}) = type(B \cup \{x\}) = a$, and $\max((A \cup \{x\}) \triangle (B \cup \{x\}) = w \in B$. Thus, $A \cup \{x\} \preceq^* B \cup \{x\}$ by (2). In the third case, $type(A \cup \{x\}) = type(B \cup \{x\}) = ab$, and $A \cup \{x\} \preceq^* B \cup \{x\}$ holds by (4). The case $x = b$ is similar.

Using the three orders defined above we now define an order $\preceq$ on $C(G)$. We set $A \preceq B$ if and only if

1. $A \cap L \prec^+ B \cap L$,
2. $A \cap L = B \cap L$ and $A \cap S \prec^- B \cap S$,
3. $A \cap L = B \cap L \neq \emptyset \neq A \cap S = B \cap S$,
4. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$ and $a \in A$,
5. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$, $a \notin A, B$ and $A \cap I \prec^- B \cap I$,
6. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$ and $b \in B$,
7. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$, $b \notin B$ and $A \cap I \prec^+ B \cap I$.
8. $A \cap L = B \cap L = \emptyset = A \cap S = B \cap S$ and $A \preceq^* B$.

The relation $\preceq$ is indeed an order. To see it, we observe that for every two sets $S' \subseteq S$ and $L' \subseteq L$, the family of sets $X \in C(G)$ such that $X \cap S = S'$ and $X \cap L = L'$ is ordered (for each family, there is a condition among the conditions (3) - (8) that is used to compare any two of its sets). Further, each set $X \in C(G)$ belongs to one of these families. Finally, the conditions (1) and (2) impose on these families a linear order that implies an ordering for pairs of sets coming from different families. We will now prove that the order $\preceq$ satisfies extension, dominance, and independence.

Extension: Assume $v, w \in V$ and $v < w$. If $w \in L$ then $\{v\} \prec \{w\}$ by (1). Otherwise, if $v \in S$ then $\{v\} \prec \{w\}$ by (2). Hence assume $v, w \in I$. Then $\{v\} \prec^* \{w\}$ because $\preceq^*$ satisfies extension. Therefore $\{v\} \prec \{w\}$ by (8).

Dominance: Let us consider a set $A \in C(G)$ and an element $x \notin A$ such that $A \cup \{x\} \in C(G)$. First we assume $x \in L$. Then $x < \min(A)$ is impossible. Indeed, it would imply that $A \subseteq L$ and $A \cup \{x\} \subseteq L$. The latter set has at least two elements. However, the only sets in $C(G)$ contained in L are singletons, a contradiction. So, assume $\max(A) < x$. Then $\max(A \triangle (A \cup \{x\})) = x \in A \cup \{x\}$ and we obtain $A \prec A \cup \{x\}$ by (1). The case $x \in S$ is dual.

Hence, assume that $x \in I$. Furthermore, assume $\max(A) < x$. Then $A \cap L = \emptyset = (A \cup \{x\}) \cap L$ and $A \cap S = (A \cap \{x\}) \cap S$. Let us assume that $A \cap S \neq \emptyset$. Then if $x = b$ we have $A \prec A \cup \{x\}$ by (6). On the other hand, if $x \neq b$ then $b > x > \max(A)$. It follows that

$b \notin A \cup \{x\}$ and $\max((A \cap I) \triangle ((A \cup \{x\}) \cap I)) = x \in (A \cup \{x\}) \cap I$. Hence, $A \prec A \cup \{x\}$ by (7). Finally, if $A \cap S = \emptyset$, we have $A \prec^* A \cup \{x\}$ as $\preceq^*$ satisfies dominance. Hence $A \prec A \cup \{x\}$ by (8). The case $x < \min(A)$ is similar.

Independence: Let us consider sets $A, B \in C(G)$ and an element $x \notin A \cup B$ such that $A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$. We distinguish eight cases based on the reason $A \prec B$ holds.

First assume $A \prec B$ by (1). Then, $A \cup \{x\} \prec B\{x\}$ by (1) as $(A \cup \{x\} \triangle B \cup \{x\}) = (A \triangle B)$. The same identity also shows that if $A \prec B$ holds by (2), then $A \cup \{x\} \prec B\{x\}$ holds also by (2). Next, we note that $A \prec B$ cannot hold by (3) (otherwise, we would also have $B \prec A$, a contradiction). Let us assume then that $A \prec B$ holds by (4). If $x \notin S$, then it follows immediately that (4) applies to imply $A \cup \{x\} \prec B\{x\}$. So assume that $x \in S$. Then, $A \cup \{x\} \preceq B \cup \{x\}$ follows from (3). Next, let $A \prec B$ hold by (5). If $x \in S$, we reason as above and derive $A \cup \{x\} \preceq B \cup \{x\}$ from (3). So, let us assume that $x \in L$. Since $((A \cup \{x\}) \cap I) \triangle ((B \cup \{x\}) \cap I) = (A \cap I) \triangle (B \cap I)$, $(A \cup \{x\}) \cap I \prec^- (B \cup \{x\}) \cap I$. It follows that (5) applies to imply $A \cup \{x\} \preceq B \cup \{x\}$. In the case $x \in I$, we have either $x = a$ or $a \notin A \cup \{x\}, B \cup \{x\}$. In the first case, $A \cup \{x\} \prec B \cup \{x\}$ holds by (4). In the second case, we have $(A \cup \{x\}) \cap I \prec^- (B \cup \{x\}) \cap I$, which follows from the identity $((A \cup \{x\}) \cap I) \triangle ((B \cup \{x\}) \cap I) = (A \cap I) \triangle (B \cap I)$. Thus, $A \cup \{x\} \prec B \cup \{x\}$ by (5). The cases $A \prec B$ by (6) or (7) are similar.

Finally assume that $A \prec B$ by (8). It follows that $A, B \in C(G)\big|_I$. If $x \in I$, then $A \cup \{x\} \preceq B \cup \{x\}$ by (8) because $\preceq^*$ satisfies independence. So assume $x \in S \cup L$. Observe that $type(A) \neq 0 \neq type(B)$ is impossible as elements in L and S are only connected to a and b. Assume $b \in A$. Then we know $b \in B$ and $A \prec^- B$. Therefore $x \in L$ implies $A \cup \{x\} \preceq B \cup \{x\}$ either by (4) or (5). If $x \in S$, we have $A \cup \{x\} \preceq B \cup \{x\}$ by (6) as $b \in B$. The case $a \in A$ is similar. $\square$

Proof of Proposition 31

Proposition. *Let $G = (V, E)$ be a connected graph and $\leq$ a linear order on V. Let $vw \in E$, such that $(V, E \setminus \{vw\})$ results in two connected components $G' = (V', E')$ and $G'' = (V'', E'')$. If $C(G')$ and can be ordered satisfying dominance and independence and $C(G'')$ can be ordered satisfying dominance and strict independence then G can be ordered with an order satisfying dominance and independence.*

Proof. Let us assume that $G' = (V', E')$ and $G'' = (V'', E'')$. Wlog we may assume that $v \in V'$, $w \in V''$. We will present the proof under the assumption that $v < w$. The other case, $w < v$, works analogously.

We partition $C(G)$ in three collections of sets: $P_1 = C(G')$, $P_2 = \{A \in C(G) \mid v, w \in A\}$ and $P_3 = C(G'')$. Let $\preceq_1$ be any order satisfying dominance and independence on P_1 (with respect to $\leq$ restricted to V'), and $\preceq_3$ any *linear* order satisfying dominance and strict independence on P_3 (with respect to $\leq$ restricted to V''). We define an order $\preceq$ on $C(G)$ by setting $A \preceq B$ (where $A, B \in C(G)$) if and only if

1. $A, B \in P_1$, and $A \preceq_1 B$

2. $A, B \in P_3$, and $A \preceq_3 B$

3. $A, B \in P_2$ and $A \cap V'' \prec_3 B \cap V''$
4. $A, B \in P_2$ and $A \cap V'' = B \cap V''$ and $A \cap V' \preceq_1 B \cap V'$
5. $A \in P_i$, $B \in P_j$ and $i < j$ (in fact, in this case, $A \prec B$ holds).

The relation $\preceq$ is obviously an order. We claim that it satisfies dominance and independence.

Dominance. Assume that $A, A \cup \{x\} \in C(G)$ and $\max(A) < x$. If $A, A \cup \{x\} \in P_i$ for some i, the result is clear. If $A \in P_3$ and $A \cup \{x\} \notin P_3$, then we have $w \in A$ and $x = v$. Since $\max(A) < x$, we have $w < v$, a contradiction. If $A \in P_1$ and $A \cup \{x\} \notin P_1$ then $A \prec A \cup \{x\}$ by the condition (5). Finally, if $A \in P_2$ then $A \cup \{x\} \notin P_2$ is impossible. The case $\min(A) > x$ is symmetric.

Independence. Assume that $A, B, A \cup \{x\}, B \cup \{x\} \in C(G)$, $x \notin A \cup B$, and $A \prec B$.
Case 1. $A, B \in P_1$ or $A, B \in P_3$. If $A, B \in P_1$, then $x = w$ or $x \in V'$. If $A, B \in P_3$, then $x = v$ of $x \in V''$. Let us assume that $A, B \in P_1$ and $x = w$, or $A, B \in P_3$ and $x = v$. In each case, $A \cup \{x\}, B \cup \{x\} \in P_2$. In the first case, $(A \cup \{x\}) \cap V'' = \{x\} = (B \cup \{x\})$. Clearly, $A \prec B$ implies $B \not\preceq A$. Since $A, B \in P_1$, we have $A \preceq_1 B$ and $B \not\preceq_1 A$. Thus, $A \prec_1 B$. Clearly, $(A \cup \{x\}) \cap V' = A$ and $(B \cup \{x\}) \cap V' = B$. It follows that $A \cup \{x\} \prec_1 B \cup \{x\}$. Therefore, $A \cup \{x\} \preceq B \cup \{x\}$ holds by (4). In the second case, since $A \prec B$, we have $A \prec_3 B$. Reasoning as before, we obtain $(A \cup \{x\}) \cap V'' \prec_3 (B \cup \{x\}) \cap V''$ and so, $A \cup \{x\} \preceq B \cup \{x\}$ holds by (3).

Thus, let us assume that $A, B \in P_1$ and $x \in V'$, or $A, B \in P_3$ and $x \in V''$. In the first case, $A \prec B$ implies $A \preceq_1 B$. Therefore, since $A \cup \{x\}, B \cup \{x\} \in P_1$, we have $A \cup \{x\} \preceq B \cup \{x\}$ by (1). The case $A, B \in P_3$ and $x \in V''$ can be dealt with in a similar way.
Case 2. $A, B \in P_2$. This implies that $A \cup \{x\}, B \cup \{x\} \in P_2$. Let us assume that $x \in V''$. Clearly, $A \prec B$ is either by (3) or (4). Assume $A \prec B$ is by (3). It follows that $A \cap V'' \prec_3 B \cap V''$. Hence, by strict independence of $\preceq_3$, we have $(A \cup \{x\}) \cap V'' \prec_3 (B \cup \{x\}) \cap V''$ (indeed, we note that $(A \cap V'') \cup \{x\} = (A \cup \{x\}) \cap V''$, and similarly for B). This implies that $(A \cup \{x\}) \cap V'' \preceq (B \cup \{x\}) \cap V''$ by (3). Let us assume then that $A \prec B$ by (4). Then $A \cap V'' = B \cap V''$ and $A \cap V' \prec_1 B \cap V'$. But then also $(A \cup \{x\}) \cap V'' = (B \cup \{x\}) \cap V''$ and $(A \cup \{x\}) \cap V' = A \cap V' \prec_1 B \cap V' = (B \cup \{x\}) \cap V'$. This implies $A \cup \{x\} \prec B \cup \{x\}$ by (4). The case $x \in V'$ is similar.
Case 3. $A \in P_2$ and $B \notin P_2$ or $B \in P_2$ and $A \notin P_2$. Let us assume that $A \in P_2$ and $B \notin P_2$. Since $A \prec B$, $B \in P_3$. Let us assume that $x \in V'$. Since $B \cup \{x\} \in C(G)$, $x = v$. Thus, $x \in A$, a contradiction. It follows that $x \in V''$ and, consequently, $B \cup \{x\} \in P_3$. Since $A \cup \{x\} \in P_2$, $A \cup \{x\} \prec B \cup \{x\}$ follows by (5). The case when $B \in P_2$ and $A \notin P_2$ is similar.
Case 4. $A \in P_1$ and $B \in P_3$. Then either $x = v$ and $w \in B$ or $x = w$ and $v \in A$. In the first case $A \cup \{x\} \in P_1$ and $B \cup \{x\} \in P_2$ and hence $A \cup \{x\} \prec B \cup \{x\}$ by (5). The other case is similar. □

Proof of Proposition 36

Proposition. *Every proper subgraph G of the complete graph K_6 is weakly DIE-orderable and thus weakly DI-orderable.*

Proof. Wlog we assume that the vertices of G are $1 \dots 6$, and that G contains edges between every pair of vertices except for 1 and 2. Consequently, $C(G) = \mathcal{P}(X) \setminus \{\emptyset, \{1,2\}\}$.

Let us now consider the standard linear order $\leq$ on $1, \dots, 6$. We will show that there is an order $\preceq$ on $C(G)$ satisfying dominance and independence with respect to $\leq$. We construct an order on $C(G)$ in two steps. First, we partition $C(G)$ in $P_1 = \{S \in C(G) \mid 1,2 \in S\}$, $P_2 = \{S \in C(G) \mid 1 \in S,\ 2 \notin S\}$, $P_3 = \{S \in C(G) \mid 2 \in S,\ 1 \notin S\}$ and $P_4 = \{S \in C(G) \mid 1,2 \notin S\}$.

For every $i \in \{1,2,3\}$, we define an order $\preceq_i$ on P_i by setting $S \preceq_i T$ if $\max(S) \leq \max(T)$. For example, for $i = 2$, we get the following order:

$$\begin{aligned}
\{1\} \prec_2 \{1,3\} \prec_2 \{1,3,4\} \sim_2 \{1,4\} \prec_2 & \\
\{1,3,5\} \sim_2 \{1,3,4,5\} \sim_2 \{1,4,5\} \sim_2 \{1,5\} \prec_2 & \\
\{1,3,6\} \sim_2 \{1,3,4,6\} \sim_2 \{1,3,5,6\} \sim_2 \{1,3,4,5,6\} & \\
\sim_2 \{1,4,5,6\} \sim_2 \{1,4,6\} \sim_2 \{1,5,6\} \sim_2 \{1,6\} &
\end{aligned}$$

Observe that for every $S, T \in P_i$, where $i \in \{1,2,3\}$, $S \prec_i T$ if and only if $\max(S) < \max(T)$.

It is easy to see that every $\preceq_i$, $i \in \{1,2,3\}$, satisfies dominance and independence. To show dominance, let us consider $S \in P_i$ and $x \in \{1, \dots, 6\}$ such that $S \cup \{x\} \in P_i$ and either $x < \min(S)$ or $\max(S) < x$. Since for all $S, T \in P_i$, $\min(S) = \min(T)$, $x < \min(S)$ is impossible. Thus, to verify dominance we only need to consider the case $\max(S) < x$. But then we have $\max(S) < \max(S \cup \{x\})$. Thus, $S \prec_i S \cup \{x\}$.

To show independence, let us consider $x \in \{1, \dots, 6\}$ and sets $S, T \in P_i$ such that $x \notin S \cup T$, $S \cup \{x\}, T \cup \{x\} \in P_i$ and $S \prec_i T$. The latter implies that $\max(S) < \max(T)$. Hence $\max(S \cup \{x\}) \leq \max(T \cup \{x\})$ and, by definition, $S \cup \{x\} \preceq_i T \cup \{x\}$.

Next, we define an order $\preceq_4$ on P_4 as follows (we present it in terms of the strict preference relation $\prec_4$ and the equivalence relation $\sim_4$):

$$\begin{aligned}
\{3\} \prec_4 \{3,4\} \prec_4 \{4\} \sim_4 \{3,5\} \sim_4 \{3,4,5\} \prec_4 \{4,5\} \prec_4 & \\
\{5\} \sim_4 \{3,6\} \sim_4 \{3,4,6\} \sim_4 \{3,5,6\} \sim_4 \{3,4,5,6\} & \\
\prec_4 \{4,5,6\} \sim_4 \{4,6\} \prec_4 \{5,6\} \prec_4 \{6\} &
\end{aligned}$$

It can be checked that the order $\preceq_4$ satisfies dominance and independence. In addition, it is evident that $S \prec_4 T$ implies $\max(S) \leq \max(T)$.

We now define a relation $\preceq$ on $C(G)$ by $S \preceq T$ if for some $i \in \{1,2,3,4\}$, $S, T \in P_i$ and $S \preceq_i T$, or if $S \in P_i$ and $T \in P_j$ for $i < j$. Since the relations $\preceq_i$, $1 \leq i \leq 4$, are orders, it is clear that $\preceq$ is an order. We claim that $\preceq$ satisfies dominance, independence and the extension rule.

Extension rule. We note that $\{1\} \in P_2$, $\{2\} \in P_3$ and $\{3\}, \{4\}, \{5\}, \{6\} \in P_4$. Directly by the definition of $\preceq_4$ we have $\{3\} \prec_4 \{4\} \prec_4 \{5\} \prec_4 \{6\}$. Thus, by the definition of $\preceq$, $\{1\} \prec \{2\} \prec \{3\} \prec \{4\} \prec \{5\} \prec \{6\}$.

Dominance. Let us consider $A \in C(G)$ and $x \in \{1, \dots, 6\}$ such that $A \cup \{x\} \in C(G)$ and (i) $\max(A) < x$ or (ii) $x < \min(A)$. In the first case, $x \neq 1$ ($x = 1$ would imply $A = \emptyset$, a contradiction) and $x \neq 2$ ($x = 2$ would imply $A = \{1\}$; in such case, $A \cup \{x\} = \{1,2\} \notin C(G)$; a contradiction). Therefore, $A \cap \{1,2\} = (A \cup \{x\}) \cap \{1,2\}$ and, consequently, $A \cup \{x\} \in P_i$

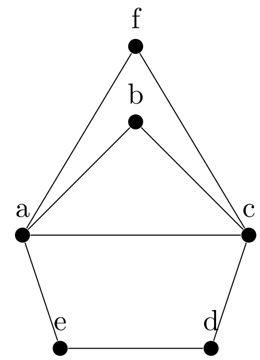

Figure 20: A labeled T_5^+

for some i. Since $\preceq_i$ satisfies dominance, $A \prec_i A \cup \{x\}$. Thus, by the definition of $\preceq$, $A \prec A \cup \{x\}$.

Let us then assume assume (ii). If $x \neq 1, 2$, $A, A \cup \{x\} \in P_4$ and so $A \cup \{x\} \prec_4 A$. By the definition of $\preceq$, $A \cup \{x\} \prec A$. If $x = 2$, we have $A \in P_4$ and $A \cup \{x\} \in P_3$. Hence, $A \cup \{x\} \prec A$. If $x = 1$ there are two cases: $2 \in A$ and $2 \notin A$. In the first case, $A \in P_3$ and $A \cup \{x\} \in P_1$. Hence $A \cup \{x\} \prec A$. In the second case, $A \in P_4$ and $A \cup \{x\} \in P_2$. Hence, $A \cup \{x\} \prec A$.

Independence. Let us consider sets $A, B \in C(G)$ and $x \in \{1, \ldots, 6\}$ such that $x \notin A \cup B$, $A \prec B$ and $A \cup \{x\}, B \cup \{x\} \in C(G)$. If $x \neq 1, 2$, reasoning as above, we conclude that $A, A \cup \{x\} \in P_i$ and $B, B \cup \{x\} \in P_j$ for some $i, j \in \{1, 2, 3, 4\}$. If $i = j$ then, by the definition of $\preceq$ and by the independence of $\preceq_i$, we have $A \cup \{x\} \preceq_i B \cup \{x\}$ and, consequently, $A \cup \{x\} \preceq B \cup \{x\}$. If $i \neq j$ then, $i < j$. Consequently, $A \cup \{x\} \prec B \cup \{x\}$.

So assume next that $x = 1$. There are four cases to consider. First, let us assume $2 \notin A, B$. Then $A, B \in P_4$ and, by the construction of $\preceq$, $A \prec_4 B$. Therefore, by the observation above, $\max(A) \leq \max(B)$. Moreover, $A \cup \{x\}, B \cup \{x\} \in P_2$. Thus, $A \cup \{x\} \preceq_2 B \cup \{x\}$ by the definition of $\preceq_2$. By the definition of $\preceq$, $A \cup \{x\} \preceq B \cup \{x\}$.

Next, let us assume $2 \in A, B$. Then, $A, B \in P_3$ and so, by the definition of $\preceq$, $A \prec B$ implies $A \prec_3 B$. Consequently, $\max(A) < \max(B)$, which implies $\max(A \cup \{x\}) \leq \max(B \cup \{x\})$. Since, $A \cup \{x\}, B \cup \{x\} \in P_1$, $A \cup \{x\} \preceq_1 B \cup \{x\}$ and, by construction, $A \cup \{x\} \preceq B \cup \{x\}$.

The third case to consider is when $2 \in A$ and $2 \notin B$. It follows that $A \cup \{x\} \in P_1$ and $B \cup \{x\} \in P_2$. Hence $A \cup \{x\} \prec B \cup \{x\}$.

Finally, $2 \notin A$ and $2 \in B$ is impossible as it contradicts $A \prec B$. Indeed, in this case, we would have $A \in P_4$, $B \in P_1 \cup P_3$ and $B \prec A$.

The case $x = 2$ can be dealt with similarly; we omit the details. $\square$

Appendix B. Computer generated orders on T_5^+

In the following we refer to the vertices of T_5^+ according to the labels given in Figure 20. In general, there are 720 ways to order the vertices of T_5^+. However, we can use the symmetry of the graph as well as some lemmas to significantly reduce the number of orders that need to be checked. First, we define for any relation R on a set X the reverse order R^{-1} by $xR^{-1}y$ iff yRx for all $x, y \in X$. Then the following result holds:

Lemma 37. *Let X be a set of objects and $\mathcal{X} \subseteq \mathcal{P}(X)$ a family of sets. Assume that there exists an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to a linear order $\leq$. Then, there exists an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to $\leq^{-1}$.*

Proof. Let $\preceq$ be an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to $\leq$. Then we claim that $\preceq^{-1}$ satisfies dominance, independence and extension with respect to $\leq^{-1}$. Assume $x <^{-1} y$ for $x, y \in X$. Then $y < x$, which implies by assumption $\{y\} \prec \{x\}$ and hence $\{x\} \prec^{-1} \{y\}$. Assume $A, A \cup \{x\} \in \mathcal{X}$, then $\forall y \in A(y <^{-1} x)$ implies $\forall y \in A(y > x)$, which implies $A \cup \{x\} \prec A$ by assumption, hence $A \prec^{-1} A \cup \{x\}$. Similarly, $\forall y \in A(x <^{-1} y)$ implies $A \cup \{x\} \prec^{-1} A$.

Now assume $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ and $A \prec^{-1} B$. Then $B \prec A$ and hence by assumption $B \cup \{x\} \preceq A \cup \{x\}$ which implies $A \cup \{x\} \preceq^{-1} B \cup \{x\}$. $\square$

Therefore, we can only consider linear orders where the vertex f is on position 4, 5 or 6 because for every linear order $\leq$ where f is on position 1, 2 or 3 we already consider the inverse order.

Now, obviously, switching the places of b and f produces a completely symmetric instance, therefore we can always assume $b < f$. Similarly, switching a and c and e and d at the same time creates a symmetric instance. Hence, we can always assume $d < e$.

Finally, observe that $T_5^+ \setminus \{a\}$ and $T_5^+ \setminus \{c\}$ are trees. Therefore, we know for every linear order $\leq$ for which either a or c are the minimal element that there exists a order on $C(T_5^+)$ that satisfies dominance, independence and extension with respect to $\leq$ by Proposition 22. By symmetry, we know that this also holds if a or c are the maximum of an order.

This leaves us with 58 linear orders. They are listed in the following as vectors where the first entry denotes the position of a in the order, the second entry gives the position of b in the order and so on. Additionally, a computer generated order on $C(T_5^+)$ is given that satisfies dominance, independence and extension with respect to that linear order. The source code used to generate these orders is provided under the following url: `https://www.dbai.tuwien.ac.at/software/pref/T5.zip`

(2, 1, 3, 4, 5, 6)
$1 \leq 12 \leq 13 = 123 \leq 1234 \leq 134 \leq 1245 \leq 125 \leq 2 \leq 23 = 12345 = 1235 \leq 1345 \leq 3 \leq 234 \leq 34 \leq 4 \leq 126 \leq 1236 \leq 25 = 245 = 136 = 235 = 12346 = 2345 \leq 1346 \leq 345 \leq 45 = 1256 = 12456 \leq 5 \leq 26 = 236 = 12356 = 123456 \leq 13456 \leq 36 \leq 6 \leq 2346 \leq 346 \leq 2456 \leq 256 = 2356 = 23456 \leq 3456$

(2, 1, 3, 4, 6, 5)
$1 \leq 12 \leq 13 = 123 \leq 1234 \leq 134 \leq 126 \leq 125 \leq 1236 \leq 2 \leq 23 = 1235 \leq 1246 \leq 3 = 135 = 234 = 12346 = 12345 \leq 34 \leq 4 \leq 1346 = 1345 \leq 25 = 26 = 246 = 235 = 236 = 2346 = 1256 = 2345 = 12456 = 12356 \leq 123456 \leq 35 \leq 5 \leq 346 \leq 345 \leq 46 \leq 6 \leq 13456 \leq 256 \leq 2456 \leq 2356 \leq 23456 \leq 3456$

(2, 1, 3, 5, 6, 4)
$1 \leq 12 \leq 13 = 123 \leq 124 \leq 1234 \leq 1235 \leq 134 \leq 126 \leq 1236 \leq 135 = 12345 \leq 1246 \leq 2 \leq 23 = 12346 \leq 1256 = 12356 = 1345 \leq 3 \leq 24 = 234 = 235 = 12456 = 123456 \leq 1356 \leq 26 = 34 = 35 = 256 = 236 = 246 = 2356 = 2346 = 2345 = 13456 \leq 4 \leq 5 \leq 345 \leq 2456 \leq 23456 \leq 356 \leq 56 \leq 6 \leq 3456$

(2, 1, 4, 3, 5, 6)
$1 \leq 12 = 1234 \leq 14 = 124 = 134 = 1235 \leq 234 \leq 126 \leq 2 \leq 125 \leq 12346 \leq 24 = 1246 = 12345 \leq 3 \leq 1245 \leq 34 \leq 235 \leq 1346 \leq 4 \leq 12356 \leq 1345 \leq 25 = 26 = 246 = 146 = 2346 = 1256 = 2345 = 123456 \leq 245 \leq 12456 \leq 35 = 346 \leq 2356 \leq 345 \leq 46 = 23456 = 13456 \leq 5 \leq 6 \leq 256 \leq 2456 \leq 3456$

(2, 1, 4, 3, 6, 5)
$1 \leq 12 = 1234 \leq 14 = 124 = 134 \leq 234 \leq 1236 \leq 2 \leq 3 \leq 24 = 12346 \leq 34 = 12345 \leq 126 \leq 1246 \leq 4 \leq 236 = 145 = 125 = 2346 = 1346 = 1345 = 2345 = 12356 = 1245 \leq 25 = 26 = 36 = 245 = 345 = 246 = 123456 \leq 346 \leq 1256 \leq 12456 \leq 45 = 13456 \leq 5 \leq 6 \leq 2356 \leq 23456 \leq 256 \leq 2456 \leq 3456$
(2, 1, 5, 3, 4, 6)
$1 \leq 12 \leq 1234 \leq 1235 \leq 124 \leq 135 \leq 15 = 125 = 126 = 12345 = 12346 \leq 1245 = 1345 \leq 2 \leq 3 \leq 12356 \leq 1246 \leq 1256 = 1356 \leq 234 \leq 235 \leq 156 \leq 24 = 34 = 123456 \leq 25 = 35 \leq 12456 \leq 4 \leq 13456 \leq 5 \leq 2345 \leq 26 = 256 = 345 = 245 = 246 = 2356 = 2346 \leq 356 \leq 56 = 2456 = 23456 \leq 6 \leq 3456$
(2, 1, 5, 3, 6, 4)
$1 \leq 12 \leq 124 \leq 2 \leq 1235 \leq 24 = 3 = 12345 \leq 135 \leq 4 \leq 1345 \leq 15 = 125 = 145 = 235 = 126 = 1236 = 12346 = 1245 \leq 2345 \leq 1246 \leq 25 = 35 = 245 = 345 \leq 45 \leq 5 \leq 236 \leq 26 = 36 = 2346 = 12356 = 123456 \leq 6 \leq 246 \leq 1356 = 13456 \leq 1256 \leq 12456 \leq 2356 = 23456 \leq 3456 \leq 356 \leq 256 \leq 2456$
(2, 3, 4, 1, 5, 6)
$1234 \leq 1235 \leq 125 = 124 = 12345 \leq 1245 \leq 134 = 12346 = 12356 = 123456 \leq 1246 \leq 1256 \leq 1345 \leq 12456 \leq 1 \leq 14 \leq 15 \leq 145 \leq 2 \leq 23 \leq 1346 \leq 24 = 25 = 146 = 234 = 235 = 13456 \leq 1456 \leq 2345 \leq 3 \leq 245 \leq 34 \leq 26 = 4 = 246 = 236 = 256 = 2346 = 2456 = 23456 = 2356 \leq 5 \leq 346 \leq 46 \leq 6$
(2, 3, 4, 1, 6, 5)
$1234 \leq 124 \leq 1245 = 12345 \leq 126 \leq 1236 \leq 1246 = 12346 \leq 1256 = 12356 \leq 123456 \leq 1 \leq 14 = 134 = 12456 \leq 1345 \leq 145 \leq 16 \leq 146 \leq 2 \leq 23 = 1346 \leq 24 = 234 \leq 25 = 245 = 235 = 2345 = 1456 = 13456 \leq 26 = 236 \leq 2346 \leq 3 \leq 246 \leq 2356 \leq 256 \leq 34 = 23456 \leq 2456 \leq 4 \leq 345 \leq 45 \leq 5 \leq 6$
(2, 3, 5, 1, 4, 6)
$125 \leq 1235 \leq 124 \leq 1234 \leq 1245 \leq 12345 \leq 1 \leq 14 = 15 = 135 = 1256 = 12356 = 1246 \leq 12346 \leq 145 \leq 12456 \leq 2 \leq 23 = 1345 = 123456 \leq 1356 \leq 156 \leq 24 = 25 = 235 \leq 234 \leq 26 = 256 = 236 = 245 = 2356 = 1456 = 2345 = 13456 \leq 3 \leq 35 \leq 4 \leq 5 \leq 246 \leq 2346 \leq 2456 \leq 23456 \leq 356 \leq 56 \leq 6$
(2, 3, 5, 1, 6, 4)
$126 \leq 1246 \leq 1236 \leq 12346 \leq 1 \leq 2 \leq 23 = 24 = 125 = 1235 = 1245 = 12345 \leq 234 \leq 15 = 16 = 25 = 26 = 135 = 145 = 235 = 245 = 236 = 246 = 1256 = 2346 = 12356 = 12456 = 2345 = 1345 = 123456 \leq 3 \leq 4 \leq 13456 \leq 1456 \leq 1356 \leq 156 \leq 23456 \leq 2356 \leq 345 \leq 2456 \leq 35 = 256 \leq 45 \leq 5 \leq 6$
(2, 4, 3, 1, 5, 6)
$123 \leq 1234 \leq 125 \leq 1235 \leq 1236 \leq 1245 \leq 12345 \leq 12346 \leq 1256 \leq 12356 \leq 1 \leq 2 \leq 13 = 23 = 12456 = 123456 \leq 24 = 234 \leq 134 \leq 15 = 25 \leq 235 \leq 135 \leq 26 = 236 \leq 136 \leq 245 \leq 2345 \leq 1345 \leq 246 \leq 2346 \leq 1346 \leq 256 \leq 2356 \leq 1356 \leq 2456 \leq 23456 = 13456 \leq 3 \leq 34 \leq 4 \leq 5 \leq 36 \leq 6 \leq 346$
(2, 4, 3, 1, 6, 5)
$123 \leq 1234 \leq 126 = 1236 = 1246 = 12346 = 1235 = 12345 \leq 1256 = 12456 \leq 123456 \leq 1 \leq 13 = 12356 \leq 134 \leq 135 \leq 2 \leq 16 \leq 23 = 136 \leq 1345 \leq 24 = 234 \leq 25 = 235 \leq 1346 \leq 26 = 236 = 245 = 2345 = 1356 = 13456 \leq 246 \leq 2346 \leq 256 \leq 2356 = 2456 = 23456 \leq 3 \leq 34 \leq 4 \leq 35 \leq 5 \leq 6 \leq 345$
(2, 4, 5, 1, 3, 6)
$123 \leq 1234 \leq 125 \leq 1245 \leq 1235 \leq 12345 \leq 1236 \leq 1256 = 12456 = 12346 \leq 12356 \leq 1 \leq 13 = 123456 \leq 2 \leq 15 = 23 = 24 = 145 = 135 = 1345 \leq 25 = 245 = 234 \leq 235 \leq 2345 \leq 26 = 256 = 156 = 236 = 246 = 2456 = 1456 = 1356 = 13456 \leq 2346 \leq 2356 \leq 23456 \leq 3 \leq 4 \leq 45 \leq 5 \leq 456 \leq 56 \leq 6$
(2, 5, 3, 1, 4, 6)
$123 \leq 124 \leq 1234 \leq 1235 \leq 1245 \leq 1236 = 12345 \leq 1246 \leq 12346 \leq 12356 \leq 12456 = 123456 \leq 1 \leq 13 \leq 14 \leq 134 \leq 135 \leq 136 \leq 2 \leq 23 \leq 1345 \leq 24 = 234 = 1346 \leq 25 = 235 = 245 = 1356 = 2345 \leq 26 = 236 = 246 = 2346 = 13456 \leq 2356 = 23456 \leq 2456 \leq 3 \leq 256 \leq 4 \leq 35 \leq 5 \leq 36 \leq 6 \leq 356$
(2, 5, 4, 1, 3, 6)
$123 \leq 124 \leq 1234 \leq 1235 \leq 1245 = 12345 \leq 1236 \leq 1246 \leq 12346 \leq 12356 \leq 1 \leq 13 = 12456 = 123456 \leq 14 = 134 \leq 2 \leq 23 = 1345 \leq 145 \leq 146 \leq 24 \leq 234 = 1346 \leq 25 = 26 = 246 = 245 = 235 = 236 = 2346 = 1456 = 2345 = 13456 \leq 23456 = 2356 \leq 2456 \leq 3 \leq 4 \leq 256 \leq 45 \leq 5 \leq 46 \leq 6 \leq 456$
(3, 1, 2, 4, 5, 6)
$1 \leq 12 \leq 13 = 123 \leq 124 = 1234 \leq 2 \leq 23 \leq 1235 \leq 24 = 3 = 135 = 234 = 12345 \leq 1345 \leq 1245 \leq 235 \leq 2345 \leq 4 \leq 35 = 345 \leq 245 \leq 45 \leq 5 \leq 126 = 1236 \leq 136 = 12346 \leq 1246 \leq 26 = 236 = 12356 \leq 36 \leq 6 \leq 123456 \leq 1356 \leq 2346 \leq 246 = 13456 = 12456 \leq 2356 \leq 356 = 23456 \leq 3456 \leq 2456$
(3, 1, 2, 4, 6, 5)

$1 \leq 12 \leq 124 \leq 13 = 123 \leq 1234 \leq 2 \leq 23 = 24 \leq 125 \leq 3 = 234 = 1245 = 1235 \leq 4 = 135 = 12345 \leq 1236 = 1246 \leq 136 = 12346 \leq 1346 \leq 25 = 235 \leq 245 \leq 2345 \leq 35 = 246 = 236 = 2346 = 12356 \leq 5 \leq 12456 \leq 36 = 346 = 123456 \leq 46 \leq 6 \leq 1356 \leq 13456 \leq 2356 \leq 2456 \leq 23456 \leq 356 \leq 3456$

(3, 1, 2, 5, 6, 4)

$1 \leq 12 \leq 13 = 123 \leq 1236 \leq 124 = 1234 \leq 125 = 1235 \leq 134 = 136 = 12356 = 12346 \leq 1256 \leq 2 \leq 23 = 12345 \leq 1245 \leq 1356 \leq 1346 \leq 123456 \leq 24 = 25 = 3 = 234 = 236 = 235 = 13456 = 12456 \leq 34 \leq 2356 \leq 4 \leq 36 = 5 = 356 = 256 = 245 = 2346 = 2345 \leq 346 = 23456 \leq 56 = 3456 \leq 6 \leq 2456$

(3, 1, 4, 2, 5, 6)

$1234 \leq 124 = 1235 \leq 1 \leq 13 \leq 234 \leq 2 \leq 12345 \leq 14 = 24 = 134 = 12346 = 1245 = 12356 \leq 135 \leq 235 \leq 1246 \leq 3 \leq 136 \leq 25 = 34 = 2346 = 1345 = 2345 = 123456 \leq 1346 \leq 4 \leq 35 = 36 = 346 = 146 = 246 = 345 = 245 = 13456 = 23456 = 1356 = 12456 = 2356 \leq 5 \leq 46 = 3456 \leq 6 \leq 2456 \leq 356$

(3, 1, 4, 2, 6, 5)

$1 \leq 13 = 1234 \leq 2 \leq 14 = 3 = 124 = 134 = 234 = 135 = 1236 = 12345 \leq 24 = 34 \leq 1345 \leq 2345 \leq 1245 \leq 12346 \leq 236 \leq 136 \leq 4 \leq 12356 \leq 26 = 35 = 36 = 345 = 145 = 245 = 2346 = 2356 = 1346 = 1246 = 123456 \leq 1356 \leq 346 \leq 246 \leq 45 = 23456 = 13456 \leq 5 \leq 6 \leq 12456 \leq 356 \leq 3456 = 2456$

(3, 1, 5, 2, 4, 6)

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