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Preferences Single-Peaked on a Tree: Multiwinner Elections and Structural Results

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Abstract

A preference profile is single-peaked on a tree if the candidate set can be equipped with a tree structure so that the preferences of each voter are decreasing from their top candidate along all paths in the tree. This notion was introduced by Demange (1982), and subsequently Trick (1989b) described an efficient algorithm for deciding if a given profile is single-peaked on a tree. We study the complexity of multiwinner elections under several variants of the Chamberlin–Courant rule for preferences single-peaked on trees. We show that in this setting the egalitarian version of this rule admits a polynomial-time winner determination algorithm. For the utilitarian version, we prove that winner determination remains NP-hard for the Borda scoring function; indeed, this hardness results extends to a large family of scoring functions. However, a winning committee can be found in polynomial time if either the number of leaves or the number of internal vertices of the underlying tree is bounded by a constant. To benefit from these positive results, we need a procedure that can determine whether a given profile is single-peaked on a tree that has additional desirable properties (such as, e.g., a small number of leaves). To address this challenge, we develop a structural approach that enables us to compactly represent all trees with respect to which a given profile is single-peaked. We show how to use this representation to efficiently find the best tree for a given profile for use with our winner determination algorithms: Given a profile, we can efficiently find a tree with the minimum number of leaves, or a tree with the minimum number of internal vertices among trees on which the profile is single-peaked. We then explore the power and limitations of this framework: we develop polynomial-time algorithms to find trees with the smallest maximum degree, diameter, or pathwidth, but show that it is NP-hard to check whether a given profile is single-peaked on a tree that is isomorphic to a given tree, or on a regular tree.

1. Introduction

Computational social choice deals with algorithmic aspects of collective decision-making. One of the fundamental questions studied in this area is the complexity of determining the election winner(s) for voting rules: indeed, for a rule to be practically applicable, it has to be the case that we can find the winner of an election in a reasonable amount of time.

Most common rules that are designed to output a single winner admit polynomial-time winner determination algorithms; examples include such diverse rules as Plurality, Borda, Maximin, Copeland, and Bucklin (for definitions, see, e.g., the handbook by Arrow et al., 2002). However, there are also some intuitively appealing single-winner rules for which winner determination is known to be computationally hard: this is the case, for instance, for Dodgson's rule (Bartholdi et al., 1989; Hemaspaandra et al., 1997), Young's rule (Rothe et al., 2003), and Kemeny's rule (Bartholdi et al., 1989; Hemaspaandra et al., 2005). More recently, there has been much interest in the computational complexity of voting rules whose purpose is to elect a representative *committee* of candidates rather than select a single winner (Faliszewski et al., 2017). One can adapt common single-winner rules to this setting, for example by appointing the candidates with the top k scores, where k is the target committee size. Such rules will pick candidates of high "quality", and are useful for shortlisting purposes. However, if we aim for a representative committee, it is preferable to use a voting rule that is specifically designed for this purpose. We note that Faliszewski et al. (2017) provide a detailed discussion of different goals in multi-winner elections, and which types of rules are suitable for each goal.

An important representation-focused rule was proposed by Chamberlin and Courant (1983). Given a committee A of k candidates, the rule assumes that each voter i is *represented* by her most-preferred candidate in A, that is, the member of A ranked highest in her preferences. Voter i is assumed to obtain utility from this representation. This utility is non-decreasing in the rank of her representative in her preference ranking. For example, her utility could be obtained as the Borda score she assigns to her representative (i.e., the number of candidates she ranks below that representative), but other scoring functions can be used as well. There are no constraints on the number of voters that can be represented by a single candidate; the assumption is that the committee will make its decisions by weighted voting, where the weight of each candidate is proportional to the fraction of the electorate that she represents (or, alternatively, that the purpose of the committee is deliberation rather than decision-making, so the goal is to select a diverse committee that represents many voters). Given a target committee size k, Chamberlin and Courant's scheme outputs a size-k committee that maximizes the sum of voters' utilities according to the chosen scoring function (see Section 2 for a formal definition).[1] Subsequently, Betzler et al. (2013) suggested an egalitarian, or maximin, variant, where the quality of a committee is measured by the utility of the worst-off voter rather than the sum of individual utilities.

Unfortunately, the problem of identifying an optimal committee under the Chamberlin–Courant rule is known to be computationally hard, even for fairly simple scoring functions. In particular, Procaccia et al. (2008) show that this is the case under r-approval scoring functions, where a voter obtains utility 1 if her representative is one of her r highest-ranked candidates, and utility 0 otherwise. Lu and Boutilier (2011) give an NP-hardness proof for the Chamberlin–Courant rule under the Borda scoring function. Betzler et al. (2013) extend these hardness results to the egalitarian variant.

1. Monroe (1995) has subsequently proposed a variant of this scheme where the committee is assumed to use non-weighted voting, and, consequently, each member of the committee is required to represent approximately the same number of voters (up to a rounding error).

Clearly, this is bad news if we want to use the Chamberlin–Courant rule in practice: elections may involve millions of voters and hundreds of candidates, and the election outcome needs to be announced soon after the votes have been cast. On the other hand, simply abandoning these voting rules in favor of easy-to-compute adaptations of single-winner rules is not acceptable if the goal is to select a truly representative committee. Thus, it is natural to try to circumvent the hardness results, either by designing efficient algorithms that compute an *approximately optimal* committee or by identifying reasonable assumptions on the structure of the election that ensure computational tractability. The former approach was initiated by Lu and Boutilier (2011), and subsequently Skowron et al. (2015a) and Munagala et al. (2021) have developed algorithms with strong approximation guarantees; see the survey by Faliszewski et al. (2017). The latter approach was proposed by Betzler et al. (2013), who provide an extensive analysis of the fixed-parameter tractability of the winner determination problem under both utilitarian and egalitarian variants of the Chamberlin–Courant rule. They also investigate the complexity of this problem for *single-peaked electorates*.

A profile is said to be *single-peaked* (Black, 1958) if the set of candidates can be placed on a one-dimensional axis, so that a voter prefers candidates that are close to her top choice on the axis. We can expect a profile to be single-peaked when every voter evaluates the candidates according to their position on a numerical issue, such as the income tax rate or minimum wage level, or by their position on the left-right ideological axis. Many voting-related problems that are known to be computationally hard for general preferences become easy when voters' preferences are assumed to be single-peaked (Elkind et al., 2017). For instance, this is the case for the winner determination problem under Dodgson's, Young's and Kemeny's rules (Brandt et al., 2015). Betzler et al. (2013) show that this is also the case for winner determination of both the utilitarian and the egalitarian version of the Chamberlin–Courant rule.

Our Contribution The goal of this paper is to investigate whether the easiness results of Betzler et al. (2013) for single-peaked electorates can be extended to a more general class of profiles. To this end, we explore a generalization of single-peaked preferences introduced by Demange (1982), which captures a much broader class of voters' preferences, while still implying the existence of a Condorcet winner. This is the class of preference profiles that are single-peaked *on a tree*. Informally, an election belongs to this class if we can construct a tree whose vertices are candidates in the election, and each voter ranks all candidates according to their perceived distance along this tree from her most-preferred candidate, with closer candidates preferred to those who are further away. A profile is single-peaked if and only if it is single-peaked on a path. Preferences that are single-peaked on a tree capture, e.g., the setting where voters' preferences are single-peaked over non-extreme candidates, but a small number of extreme candidates prove to be difficult to order; the resulting tree may then have each of the extreme candidates as a leaf. They also arise in the context of choosing a location for a facility (such as a hospital or a convenience store) given an acyclic road network. Further examples are provided by Demange (1982). Moreover, this preference domain is a natural and well-studied extension of the single-peaked domain, and checking if the algorithms of Betzler et al. (2013) extend to preferences that are single-peaked on trees helps us understand whether the linear structure is essential for tractability. As we will see, it turns out that the answer to this question is 'no'.

We focus on the Chamberlin–Courant rule. We first show that, for the egalitarian variant of this rule, winner determination is easy for an arbitrary scoring function when voters' preferences are single-peaked on a tree. Our proof proceeds by reducing our problem to an easy variant of the Hitting Set problem. For the utilitarian setting, we show that winner determination for the Chamberlin–Courant rule remains NP-complete if preferences are single-peaked on a tree, for many scoring functions, including the Borda scoring function. Hardness holds even if preferences are single-peaked on a tree of bounded diameter and bounded pathwidth. However, we present an efficient winner determination algorithm for preferences that are single-peaked on a tree with a *small number of leaves*: the running time of our algorithm is polynomial in the size of the profile, but exponential in the number of leaves. Formally, the problem is in XP with respect to the number of leaves. Further, we give an algorithm that works for trees with a small number of *internal vertices* (i.e., with a *large* number of leaves) when using the Borda scoring function. This algorithm places the problem in XP with respect to the number of internal vertices and in FPT with respect to the combined parameter 'committee size+the number of internal vertices'.

Now, these parameterized algorithms assume that the tree with respect to which the preferences are single-peaked is given as an input. However, in practice we cannot expect this to be the case: typically, we are only given the voters' preferences and have to construct such a tree (if it exists) ourselves. While the algorithm of Betzler *et al.* faces the same issue (i.e., it needs to know the societal axis), there exist efficient algorithms for determining the societal axis given the voters' preferences (Bartholdi & Trick, 1986; Doignon & Falmagne, 1994; Escoffier et al., 2008). In contrast, for trees the situation is more complicated. Trick (1989b) describes a polynomial-time algorithm that decides whether there exists a tree such that a given election is single-peaked with respect to it, and constructs *some* such tree if this is indeed the case. However, Trick's algorithm leaves us a lot of freedom when constructing the tree. As a result, if the election is single-peaked with respect to several different trees, the output of Trick's algorithm will be dependent on the implementation details. In particular, there is no guarantee that an arbitrary implementation will find a tree that caters to the demands of the winner determination algorithms that we present: for example, the algorithm may return a tree with many leaves, while we wish to find one that has as few leaves as possible. Indeed, Trick's algorithm may output a complex tree even when the input profile is single-peaked on a line.

To address this issue, we propose a general framework for finding trees with desired properties, and use it to obtain polynomial-time algorithms for identifying 'nice' trees when they exist, for several appealing notions of 'niceness'. Specifically, we define a digraph that encodes, in a compact fashion, all trees with respect to which a given profile is single-peaked. This digraph enables us to count and/or enumerate all such trees. Moreover, we show that it has many useful structural properties. These properties can be exploited to efficiently find trees with the minimum number of leaves, or the number of internal vertices, or the degree, or diameter, or pathwidth, among all trees with respect to which a given profile is single-peaked. These recognition algorithms complement our parameterized algorithms for winner determination. However, not all interesting questions about finding special trees are easy to solve. In particular, we show that it is NP-hard to decide whether a given profile is single-peaked on a regular tree, i.e., a tree all of whose internal vertices have the same degree. It is also NP-complete to decide whether a profile is single-peaked on a tree which is isomorphic to a given tree.

Related Work The recent literature on the use of structured preferences in computational social choice is surveyed by Elkind et al. (2016, 2017).

Demange (1982) introduced the domain of preferences single-peaked on a tree and showed that every profile in this domain admits a Condorcet winner. Thus, there exists a strategyproof voting rule on this domain. Danilov (1994) characterized the set of all strategyproof voting rules on this domain, generalizing the classic characterization for preferences single-peaked on a line by Moulin (1980). Schummer and Vohra (2002) consider strategyproofness for the case where the tree is embedded in $\mathbb{R}^2$ and preferences are Euclidean. Peters et al. (2019) characterize strategyproof probabilistic voting rules for preferences single-peaked on trees and other graphs. The domain of single-*crossing* preferences (Mirrlees, 1971; Roberts, 1977) has also been extended to trees and other graphs (Kung, 2015; Clearwater et al., 2015; Puppe & Slinko, 2019).

A polynomial-time algorithm for recognizing whether a profile is single-peaked on a line was given by Bartholdi and Trick (1986). Subsequently, faster algorithms were developed by Doignon and Falmagne (1994) and Escoffier et al. (2008). Fitzsimmons and Lackner (2020) put forward an efficient algorithm for preferences that may contain ties. For single-peakedness on trees, Trick (1989b) gives an algorithm that we describe in detail later. Trick's algorithm only works when voters' preferences are strict. For preferences that may contain ties, more complicated algorithms have been proposed (Trick, 1989a; Conitzer et al., 2004; Tarjan & Yannakakis, 1984; Sheng Bao & Zhang, 2012). Peters and Lackner (2020) give a polynomial-time algorithm for recognizing preferences single-peaked on a circle; very recently, these results have been extended to pseudotrees (Escoffier et al., 2020). On the other hand, a result of Gottlob and Greco (2013) implies that recognizing whether preferences are single-peaked on a graph of treewidth 3 is NP-hard.

The complexity of winner determination under the Chamberlin–Courant rule has been investigated by a number of authors, starting with the work of Procaccia et al. (2008) and Lu and Boutilier (2011); see the survey of Faliszewski et al. (2017). The first paper to consider this problem for a structured preference domain was the work of Betzler et al. (2013), who gave a dynamic programming algorithm for single-peaked preferences. This result was extended by Cornaz et al. (2012) to profiles with bounded *single-peaked width*. Skowron et al. (2015b) show that a winning committee under the Chamberlin–Courant rule can be computed in polynomial time for preferences that are single-crossing, or, more generally, have bounded single-crossing width. Constantinescu and Elkind (2021) build on the work of Clearwater et al. (2015) to extend this result to preferences that are single-crossing on a tree. Peters and Lackner (2020) develop a polynomial-time winner determination algorithm for profiles that are single-peaked on a circle, via an integer linear program that is totally unimodular if preferences are single-peaked on a circle, and hence optimally solved by its linear programming relaxation. In contrast, for 2D-Euclidean preferences, Godziszewski et al. (2021) obtain an NP-hardness result for a variant of the Chamberlin–Courant rule that uses approval ballots. The computational complexity of the winner determination problem under structured preferences has also been studied for other voting rules: for example, Brandt et al.

(2015) consider the complexity of the Dodgson rule and the Kemeny rule under single-peaked preferences.

Conitzer (2009) showed that single-peakedness offers advantages if we wish to elicit voters' preferences using comparison queries: Without any information about the structure of the preferences, one requires $\Omega(nm \log m)$ comparison queries to discover the preferences of n voters over m alternatives. If preferences are known to be single-peaked on a line, then $O(nm)$ queries suffice. Dey and Misra (2016) show that single-peakedness on trees can also be used to speed up elicitation, as long as we know the underlying tree and this tree is well-structured (the relevant notions of structure are similar to the ones considered in Section 8). Sliwinski and Elkind (2019) consider the problem of sampling preferences that are single-peaked on a given tree uniformly at random, and explain how to identify a tree that is most likely to generate a given profile, assuming that preferences are sampled uniformly at random.

2. Preliminaries

Let C be a finite set of *candidates*, and let $N = \{1, \dots, n\}$ be a set of *voters*. A *preference profile* P assigns to each voter a strict total order over C. For each $i \in N$ we write $\succ_i$ for the preference order of i. If $a \succ_i b$, then we say that voter i prefers a to b.

Given a profile P, we denote by $\text{pos}(i, a)$ the position of candidate $a \in C$ in the preference order of voter $i \in N$:

$$\text{pos}(i, a) = |\{b \in C : b \succ_i a\}| + 1.$$

We write $\text{top}(i)$ for voter i's most-preferred candidate, i.e., the candidate in position 1, we write $\text{second}(i)$ for the candidate in position 2, and $\text{bottom}(i)$ for i's least-preferred candidate, i.e., the candidate in position m. Given a subset of candidates $W \subseteq C$, we extend this notation and let $\text{top}(i, W)$, $\text{second}(i, W)$, and $\text{bottom}(i, W)$ denote voter i's most-, second-most- and least-preferred candidate in W, respectively, provided that $|W| \geqslant 3$.

Given a subset $W \subseteq C$, we write $P|_W$ for the profile obtained from P by restricting the candidate set to W.

Multi-Winner Elections A *scoring function* for a given N and C is a mapping $\mu : N \times C \to \mathbb{Z}$ such that $\text{pos}(i, a) < \text{pos}(i, b)$ implies $\mu(i, a) \geqslant \mu(i, b)$. Intuitively, $\mu(i, a)$ indicates how well candidate a represents voter i. A scoring function is said to be *positional* if there exists a vector $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m$ with $s_1 \geqslant s_2 \geqslant \dots \geqslant s_m$ such that $\mu(i, a) = s_{\text{pos}(i,a)}$; when this is the case, we will say that the scoring function is *induced* by the vector $\mathbf{s}$. It will be convenient to work with vectors $\mathbf{s}$ such that $s_1 = 0$ and $s_2, \dots, s_m \leqslant 0$, where negative values correspond to 'misrepresentation'. This choice is without loss of generality, as applying a positive affine transformation to $\mathbf{s}$ does not change the output of the voting rules we introduce below. We will refer to the positional scoring function that corresponds to the vector $(0, -1, \dots, -m + 1)$ as the *Borda scoring function*.

Given a preference profile P, a committee of candidates $W \subseteq C$, and a scoring function $\mu : N \times C \to \mathbb{Z}$, we take voter i's utility from the committee W to be $\mu(i, \text{top}(i, W))$, that is, the score she gives to her favorite candidate in W. We also write

$$\text{score}^+_\mu(P, W) = \sum_{i \in N} \mu(i, \text{top}(i, W))$$

for the sum of the utilities of all voters (the *utilitarian Chamberlin–Courant score*), and

$$\text{score}_\mu^{\min}(P, W) = \min_{i \in N} \mu(i, \text{top}(i, W))$$

for the utility obtained by the worst-off voter (the *egalitarian Chamberlin–Courant score*). Given a committee size k with $1 \leqslant k \leqslant |C|$, the *utilitarian Chamberlin–Courant rule* elects all committees $W \subseteq C$ with $|W| = k$ such that $\text{score}_\mu^+(P, W)$ is maximum. The *egalitarian* Chamberlin–Courant rule elects all committees $W \subseteq C$ with $|W| = k$ such that $\text{score}_\mu^{\min}(P, W)$ is maximum. When referring to the Chamberlin–Courant rule, we will mean the utilitarian version by default. Sometimes it is useful to think of this rule as minimizing costs rather than maximizing scores, so we will write $\text{cost}_\mu^+(P, W) = -\text{score}_\mu^+(P, W)$ and $\text{cost}_\mu^{\min}(P, W) = -\text{score}_\mu^{\min}(P, W)$. The Chamberlin–Courant rule minimizes these costs.

To study the computation of winning committees under these rules, we now formally define the decision problems associated with their optimization versions.

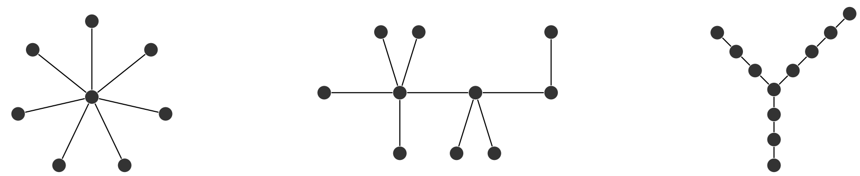
Definition 2.1. *An instance of the* Utilitarian CC *(respectively,* Egalitarian CC*) problem is given by a preference profile P, a committee size k, $1 \leqslant k \leqslant |C|$, a scoring function $\mu : N \times C \to \mathbb{Z}$, and a bound $B \in \mathbb{Z}$. It is a 'yes'-instance if there is a subset of candidates $W \subseteq C$ with $|W| = k$ such that* $\text{score}_\mu^+(P, W) \geqslant B$ *(respectively,* $\text{score}_\mu^{\min}(P, W) \geqslant B$*) and a 'no'-instance otherwise.*[2]

We will sometimes consider the complexity of these problems for specific families of scoring functions. Note that a scoring function is defined for fixed C and N, so the question of asymptotic complexity makes sense for families of scoring functions (parameterized by C and N), but not for individual scoring functions. For instance, the Borda scoring function can be viewed as a family of scoring functions, as it is well-defined for any C and N.

Graphs and Digraphs A *digraph* $D = (C, A)$ is given by a set C of vertices and a set $A \subseteq C \times C$ of pairs, which we call *arcs*. If $(a, b) \in A$, we say that (a, b) is an *outgoing arc* of a. An *acyclic* digraph (a *dag*) is a digraph with no directed cycles. For a vertex $a \in C$, its *out-degree* $d^+(a) = |\{b \in C : (a, b) \in A\}|$ is the number of outgoing arcs of a. A *sink* is a vertex a with $d^+(a) = 0$, i.e., a vertex without outgoing arcs. It is easy to see that every dag has at least one sink. Given a digraph $D = (C, A)$, we write $\mathcal{G}(D)$ for the undirected graph (C, E) where for all $a, b \in C$ we have $\{a, b\} \in E$ if and only if $(a, b) \in A$ or $(b, a) \in A$. Thus, $\mathcal{G}(D)$ is the graph obtained from D when we forget about the orientations of the arcs of D.

Given a digraph $D = (C, A)$ and a set $W \subseteq C$, we write $D|_W$ for the induced subdigraph. Similarly, for a graph $G = (C, E)$, we write $G|_W$ for the induced subgraph. We say that a set $W \subseteq C$ is *connected* in a graph G if $G|_W$ is connected.

2. Under our definition it may happen that some candidate in the committee does not represent any voter, i.e., there exists an $a \in W$ such that $a \neq \text{top}(i, W)$ for all $i \in N$; equivalently, we allow for committees of size $k' < k$. It is assumed that the voting weight of such candidate in the resulting committee will be 0. This definition is also used by, e.g., Cornaz et al. (2012) and Skowron et al. (2015a). In contrast, Betzler et al. (2013) define the Chamberlin–Courant rule by explicitly specifying an assignment of voters to candidates, so that each candidate in W has at least one voter who is assigned to her. The resulting voting rule is somewhat harder to analyze algorithmically. Note that when $|\{\text{top}(i, C) : i \in N\}| \geqslant k$, the two variants of the Chamberlin–Courant rule coincide.



(a) Star (b) Caterpillar (c) Subdivision of a star

Figure 1: Examples of different classes of trees

Classes of Trees Recall that a *tree* is a connected graph that has no cycles. A *leaf* of a tree is a vertex of degree 1. Vertices that are not leaves are *internal* vertices. A *path* is a tree that has exactly two leaves. A *star* $K_{1,n}$ is a tree that has exactly one internal vertex and n leaves. The internal vertex is called the *center* of the star. The *diameter* of a tree T is the maximum distance between two vertices of T; e.g., the diameter of a star is 2. A k*-regular tree* is a tree in which every internal vertex has degree k. Note that paths are 2-regular, and the star $K_{1,n}$ is n-regular. A *caterpillar* is a tree in which every vertex is within distance 1 of a central path. A *subdivision of a star* is a tree obtained from a star by means of replacing each edge by a path. Figure 1 illustrates some of these concepts.

Pathwidth The *pathwidth* of a tree T is a measure of how close T is to being a path. A *path decomposition* of $T = (C, E)$ is given by a sequence $S_1, \dots, S_r$ of subsets of C (called *bags*) such that

- for each edge $\{a, b\} \in E$, there is a bag S_i with $a, b \in S_i$, and
- for each $a \in C$, the bags containing a form an interval of the sequence, so that if $a \in S_i$ and $a \in S_j$ for $i < j$, then a also belongs to each of $S_{i+1}, S_{i+2}, \dots, S_{j-1}$.

The *width* of the path decomposition is $\max_{i\in[r]} |S_i| - 1$. The *pathwidth* of T is the minimum width of a path decomposition of T. For more on pathwidth and the related concept of treewidth, see, e.g., the survey by Bodlaender (1994).

Preferences that are Single-Peaked on a Tree Consider a tree T with vertex set C. A preference profile P is said to be *single-peaked on* T (Demange, 1982) if for every voter $i \in N$ and every pair of distinct candidates $a, b \in C$ such that b lies on the unique path from $\mathrm{top}(i)$ to a in T it holds that $\mathrm{top}(i) \succ_i b \succ_i a$. The profile P is said to be *single-peaked on a tree* if there exists a tree T with vertex set C such that P is single-peaked on T. The profile P is said to be *single-peaked* if P is single-peaked on some tree T that is a path.

The following proposition considers alternative ways of defining preferences single-peaked on a tree T. The (known) proof is straightforward from the definitions.

Proposition 2.2. *Let P be a preference profile and let T be a tree on vertex set C. The following properties are equivalent:*

- *P is single-peaked on T.*
- *For every $W \subseteq C$ that is connected in T, $P|_W$ is single-peaked on $T|_W$.*

- *For every $i \in N$ and every $a \in C$, the* top-initial segment $\{b \in C : b \succ_i a\}$ *is connected in T.*

Given a profile P, we denote the set of all trees T such that P is single-peaked on T by $\mathcal{T}(P)$.

3. Egalitarian Chamberlin–Courant on Arbitrary Trees

We start by presenting a simple algorithm for finding a winning committee under the egalitarian Chamberlin–Courant rule that works for preferences single-peaked on arbitrary trees. Our algorithm proceeds by finding a committee of minimum size that satisfies a given worst-case utility bound.

First, we show that the winner determination problem in the egalitarian case can be reduced to the following variant of the Hitting Set problem, where the ground set is the vertex set of a tree T, and we need to hit a collection of connected subsets of T.

Definition 3.1. *An instance of the* Tree Hitting Set *problem is given by a tree T on a vertex set C, a family $\mathcal{C} = \{C_1, \dots, C_n\}$ of subsets of C such that each C_i is connected in T, and a target cover size $k \in \mathbb{Z}_+$. It is a 'yes'-instance if there is a subset of vertices $W \subseteq C$ with $|W| \leqslant k$ such that $W \cap C_i \neq \varnothing$ for $i = 1, \dots, n$, and a 'no'-instance otherwise.*

Guo and Niedermeier (2006) show that the Tree Hitting Set problem can be solved in polynomial time. Since they consider a dual formulation (in terms of set cover), we present an adaptation of the short argument here.

Theorem 3.2 (Guo and Niedermeier (2006))**.** Tree Hitting Set *can be solved in polynomial time.*

Proof. Consider a vertex $a \in C$ that is a leaf of T, and let $b \in C$ be the (unique) vertex that a is adjacent to. Suppose that $a \in C_i$ for some i. Then, because C_i is a connected subset of T, we either have $C_i = \{a\}$ or $b \in C_i$.

With this observation, we can now give a simple algorithm that constructs a minimum hitting set: Consider a leaf vertex $a \in C$ adjacent to $b \in C$. If there exists some $C_i \in \mathcal{C}$ with $C_i = \{a\}$, then any hitting set must include a, so add a to the hitting set under construction, remove a from T and remove all copies of $\{a\}$ from $\mathcal{C}$. Otherwise, every set C_i that would be hit by a is also hit by b, so any hitting set including a remains a hitting set when a is replaced by b. Hence it is safe to delete a from T and from each $C_i \in \mathcal{C}$. Now repeat the process on the smaller instance constructed. Once all vertices have been deleted, return the constructed hitting set, which is minimum by our argument. □

Now we show how to reduce our winner determination problem to the hitting set problem. Suppose we are given an instance of the Egalitarian CC problem consisting of a profile P, a tree T on which P is single-peaked, a target committee size k, and the bound B. We construct a Tree Hitting Set instance as follows.

The ground set is the candidate set C, the tree T is the tree with respect to which voters' preferences are single-peaked, and the target cover size equals the committee size k. For each $i \in N = \{1, \dots, n\}$, construct the set

$$C_i = \{a \in C : \mu(i, a) \geqslant B\}.$$

Since μ is monotone, the set C_i is a top-initial segment of i's preference order, i.e., is of the form $\{a \in C : a \succ_i b\}$ for some $b \in C$. By Proposition 2.2, since P is single-peaked on T, each set C_i is connected in T, so we have constructed an instance of Tree Hitting Set. Now note that for every set $W \in C$ we have

$$\text{score}_\mu^{\max}(P, W) \geqslant B \text{ if and only if } W \cap C_i = W \cap \{a \in C : \mu(i, a) \geqslant B\} \neq \varnothing \text{ for all } i.$$

It follows that our reduction is correct.

Using this reduction and the algorithm for Tree Hitting Set, we can solve Egalitarian CC in polynomial time.

Theorem 3.3. *For profiles that are single-peaked on a tree, we can find a winning committee under the egalitarian Chamberlin–Courant rule in polynomial time.*

4. Hardness of Utilitarian Chamberlin–Courant on Arbitrary Trees

For preferences single-peaked on a *path*, the utilitarian version of the Chamberlin–Courant rule is computationally easy: a winning committee can be computed using a dynamic programming algorithm (Betzler et al., 2013). While we are able to generalize this algorithm to work for some other trees (see Section 5), it is not clear how to extend it to arbitrary trees. Indeed, we will now show that for the utilitarian Chamberlin–Courant rule the winner determination problem remains NP-complete for preferences single-peaked on a tree. This hardness result holds for the Borda scoring function, and applies even to trees that have diameter 4 and pathwidth 2.

We have defined the Borda scoring function as the vector $(0, -1, \dots, -(m-1))$. Recall that we defined $\text{cost}_\mu^+(P, W) = -\text{score}_\mu^+(P, W)$, and we will use costs in the following proof to avoid negative numbers.

Theorem 4.1. *Given a profile P that is single-peaked on a tree, a target committee size k, and a target score B, it is* NP*-complete to decide whether there exists a committee of size k with score at least B under the utilitarian Chamberlin–Courant rule with the Borda scoring function. Hardness holds even restricted to profiles single-peaked on a tree with diameter 4 and pathwidth 2.*

Proof. We will reduce the classic Vertex Cover problem to Utilitarian CC. An instance of Vertex Cover is given by an undirected graph $G = (V, E)$ and a positive integer t. It is a 'yes'-instance if it admits a vertex cover of size t, i.e., a subset of vertices $S \subseteq V$ such that for each $\{u, v\} \in E$ we have that $u \in S$ or $v \in S$. This problem is known to be NP-hard (Karp, 1972).

Given an instance (G, t) of Vertex Cover such that $G = (V, E)$, $V = \{u_1, \dots, u_p\}$ and $E = \{e_1, \dots, e_q\}$, we construct an instance of Utilitarian CC as follows.

Let $M = 5pq$; intuitively, M is a large number. We introduce a candidate a, two candidates y_i and z_i for each vertex $u_i \in V$, and M dummy candidates. Formally, we set

$$Y = \{y_1, \dots, y_p\}, \quad Z = \{z_1, \dots, z_p\}, \quad D = \{d_1, \dots, d_M\},$$

and define the candidate set to be

$$C = \{a\} \cup Y \cup Z \cup D.$$

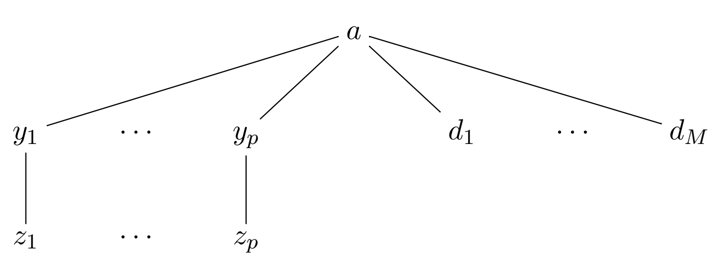
We set the target committee size to be $k = p + t$.

We now introduce the voters, who will come in three types $N = N_1 \cup N_2 \cup N_3$.

N_1			N_2			N_3		
$5q$	$\cdots$	$5q$	1	$\cdots$	1	M	$\cdots$	M
y_1		y_p	a		a	z_1		z_p
z_1		z_p	$y_{j_{1,1}}$		$y_{j_{q,1}}$	y_1		y_p
a		a	$y_{j_{1,2}}$		$y_{j_{q,2}}$	a		a
$\vdots$		$\vdots$	d_1		d_1	$\vdots$		$\vdots$
			$\vdots$		$\vdots$			
			d_M		d_M			
			$\vdots$		$\vdots$			

- The set N_1 consists of $5q$ identical voters for each $u_i \in V$: they rank y_i first, z_i second, and a third, followed by all other candidates as specified below. Intuitively, the purpose of these voters is to ensure that good committees contain representatives y_i of vertices in V.
- The set N_2 consists of a single voter for each edge $e_j \in E$: this voter ranks a first, followed by the two candidates from Y that correspond to the endpoints of e_j (in an arbitrary order), followed by the dummy candidates $d_1, \dots, d_M$, followed by all other candidates as specified below. The purpose of these voters is to ensure that every edge is covered by one of the vertices that correspond to a committee member, and to incur a heavy penalty of M if the edge is uncovered.
- The set N_3 is a set of M identical voters for each $u_i \in V$ who all rank z_i first, y_i second, and a third, followed by all other candidates as specified below. The purpose of these voters is to force every good committee to include *all* the z_i candidates.

We complete the voters' preferences so that the resulting profile is single-peaked on the following tree:



This tree is obtained by taking a star with center a and leaves $Y \cup D$, and then attaching z_i as a leaf onto y_i for every $i = 1, \dots, p$. It is easy to see that it has diameter 4 and pathwidth 2 (with bags $\{a, y_1, z_1\}, \dots, \{a, y_p, z_p\}, \{a, d_1\}, \dots, \{a, d_M\}$). We will now specify how to complete each vote in our profile to ensure that the resulting profile is single-peaked on this tree. Inspecting the tree, we see that it suffices to ensure that for each $i = 1, \dots, p$ it holds that in all votes where the positions of y_i and z_i are not given explicitly, candidate y_i is ranked above z_i.

This completes the construction of the profile P with voter set N and candidate set C. We use the Borda scoring function $\mathbf{s} = (0, -1, -2, \dots)$, and set the cost bound to be $B = (5q)(p - t) + 2q$ and ask whether there exists a committee with $\text{cost}^+_\mu(P, W) \leqslant B$. Note that, by construction, $M > B$. This completes the description of our instance of the Utilitarian CC problem. Intuitively, the 'correct committee' we have in mind consists of all z_i candidates (of which there are p) and a selection of y_i candidates that corresponds to a vertex cover (of which there should be t), should a vertex cover of size t exist. Now let us prove that the reduction is correct.

Suppose we have started with a 'yes'-instance of Vertex Cover, and let S be a collection of t vertices that form a vertex cover of G. Consider the committee $W = Z \cup \{y_i : u_i \in S\}$; note that $|W| = p + t = k$. The voters in N_3 and $5qt$ voters in N_1 have their most-preferred candidate in W, so they contribute 0 to the cost of W. The remaining $(5q)(p - t)$ voters in N_1 contribute 1 to the cost of W, since $z_i \in W$ for all i. Further, each voter in N_2 contributes at most 2 to the cost of W. Indeed, the candidates that correspond to the endpoints of the respective edge are ranked in positions 2 and 3 in this voter's ranking, and since S is a vertex cover for G, one of these candidates is in W. We conclude that $\text{cost}^+_\mu(P, W) \leqslant (5q)(p - t) + 2q = B$.

Conversely, suppose there exists a committee W of size $k = p + t$ with $\text{cost}^+_\mu(P, W) \leqslant B$. Note first that W has to contain all candidates in Z: otherwise, there are M voters in N_3 who contribute at least 1 to the cost of W, and then the utilitarian Chamberlin–Courant cost of W is at least $M > B$. Thus $Z \subseteq W$. We will now argue that $W \setminus Z$ is a subset of Y, and that $S' = \{u_i : y_i \in W \setminus Z\}$ is a vertex cover of G. Suppose that $W \setminus Z$ contains too few candidates from Y, i.e., at most $t - 1$ candidates from Y. Then N_1 contains at least $(5q)(p - (t - 1))$ voters who contribute at least 1 to the cost of W, so $\text{cost}^+_\mu(P, W) \geqslant (5q)(p - t + 1) > (5q)(p - t) + 2q = B$, a contradiction. Thus, we have $W \setminus Z \subseteq Y$. Now, suppose that S' is not a vertex cover for G. Let $e_j \in E$ be an edge that is not covered by S', and consider the voter in N_2 corresponding to e_j. Clearly, none of the candidates ranked in positions $1, \dots, M + 3$ by this voter appear in W. Thus, this voter contributes more than M to the cost of W, so the total cost of W is more than $M > B$, a contradiction. Thus, a 'yes'-instance of Utilitarian CC corresponds to a 'yes'-instance of Vertex Cover. □

In the appendix, we modify this reduction to establish that Utilitarian CC remains hard even on trees with maximum degree 3 (Theorem A.1); intuitively, it suffices to 'clone' candidate a.

5. Utilitarian Chamberlin–Courant on Trees with Few Leaves

The hardness result in Section 4 shows that single-peakedness on trees is not a strong enough assumption to make Utilitarian CC tractable. However, we will now show that it is possible to achieve tractability by placing further constraints on the shape of the underlying tree.

Specifically, in this section, we present an algorithm for the utilitarian Chamberlin–Courant rule whose running time is polynomial on any profile that is single-peaked on a tree with a constant number of leaves. The algorithm proceeds by dynamic programming. It can be viewed as a generalization of the algorithm due to Betzler et al. (2013) for preferences single-peaked on a path (i.e., a tree with two leaves).

Theorem 5.1. *Given a profile P with $|C| = m$ and $|N| = n$, a tree T with λ leaves such that P is single-peaked on T, and a target committee size k, we can find a winning committee under the utilitarian Chamberlin–Courant rule in time* $\text{poly}(n, m^\lambda, k^\lambda)$.

Proof. We use dynamic programming to find a committee of size k that maximizes the utilitarian Chamberlin–Courant score.

We pick an arbitrary vertex r^* to be the root of T. This choice induces a partial order $\succ$ on C: we set $a \succ b$ if a lies on the (unique) path from r^* to b in T. Thus, $r^* \succ a$ for every $a \in C \setminus \{r^*\}$. A set $A \subseteq C$ is said to be an *anti-chain* if no two elements of A are comparable with respect to $\succ$. Figure 2 on the right provides an example; if we added the left child of r^* to the set, it would no longer be an anti-chain. Observe that for every subset of C its set of maximal elements with respect to $\succ$ forms an anti-chain. Note also that every two ancestors of a leaf are comparable with respect to $\succ$. Hence, if a and b belong to an anti-chain $A \subseteq C$ and c is a leaf of T, then it cannot be the case that both a and b are ancestors of c. Therefore, $|A| \leqslant \lambda$.

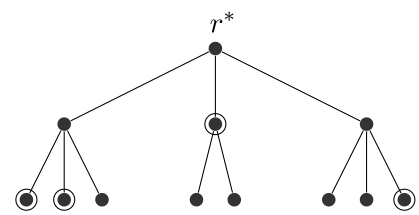


Figure 2: An anti-chain

Given a vertex r, let T_r be the subtree of T rooted at r. The vertex set of T_r is $C_r = \{r\} \cup \{a : r \succ a\}$. Let $N_r = \{i \in N : \text{top}(i) \in C_r\}$ be the set of all voters whose most-preferred candidate belongs to C_r. Let P_r be the profile obtained from P by restricting the candidate set to C_r and the voter set to N_r. For each $r \in C$ and each $\ell = 1, \ldots, k$ let

$$M(r, \ell) = \max\left\{\text{score}^+_\mu(P_r, W) : W \subseteq C_r \text{ with } |W| = \ell \text{ and } r \in W\right\}$$

be the highest Chamberlin–Courant score obtainable in P_r by a committee from C_r of size at most ℓ, subject to r being selected.

Suppose that we have computed these quantities for all descendants of r. We will now explain how to compute them for r. Let $W \subseteq C_r$ be an optimal committee for P_r that has size ℓ and includes r, so that $\text{score}^+_\mu(P_r, W) = M(r, \ell)$. Let $A = \{r_1, \ldots, r_s\}$ be the set of maximal elements of $W \setminus \{r\}$ with respect to $\succ$ and let $\ell_j = |W \cap C_{r_j}|$ for $j = 1, \ldots, s$; we have $\ell_1 + \cdots + \ell_s = \ell - 1$. Since P_r is single-peaked on T, for each $j = 1, \ldots, s$ it holds that each voter in N_{r_j} is better represented by r_j than by any candidate not in C_{r_j}. Thus, the contribution of voters in N_{r_j} to the total score $M(r, \ell)$ of W is given by $\text{score}^+_\mu(P_{r_j}, W \cap C_{r_j})$. In fact, this quantity must equal $M(r_j, \ell_j)$, since otherwise we could replace the candidates in $W \cap C_{r_j}$ by an optimizer of $M(r_j, \ell_j)$, thereby increasing the score of W, which would be a contradiction. On the other hand, consider a voter i in $N_r \setminus (N_{r_1} \cup \cdots \cup N_{r_s})$. Since P_r is single-peaked on T, for each $j = 1, \ldots, s$ it holds that candidate r_j is a better representative for i than any other candidate in C_{r_j}. Thus, voter i's most-preferred candidate in W must be one of $r, r_1, \ldots, r_s$.

This suggests the following procedure for computing $M(r, \ell)$. The case $\ell = 1$ is straightforward, as the unique optimal committee in this case is $\{r\}$. For $\ell > 1$, let $\mathcal{T}_r$ be the set of all anti-chains in T_r. A *t-division scheme* for an anti-chain $A = \{r_1, \dots, r_s\} \in \mathcal{T}_r$ is a list $L = (\ell_1, \dots, \ell_s)$ such that $\ell_j \geqslant 1$ for all $j = 1, \dots, s$ and $\ell_1 + \dots + \ell_s = t$. We denote by $\mathcal{L}_t^A$ the set of all t-division schemes for A. Now, for every anti-chain $A = \{r_1, \dots, r_s\} \in \mathcal{T}_r \setminus \{\{r\}\}$ and every $(\ell-1)$-division scheme $L = (\ell_1, \dots, \ell_s) \in \mathcal{L}_{\ell-1}^A$, we set $N_r' = N_r \setminus (N_{r_1} \cup \ldots \cup N_{r_s})$ and

$$M(A, L) = \sum_{j=1}^{s} M(r_j, \ell_j) + \sum_{i \in N_r'} \mu(i, \mathrm{top}(i, A \cup \{r\})).$$

We then have $M(r, \ell) = \max_{A \in \mathcal{T}_r \setminus \{\{r\}\}, L \in \mathcal{L}_{\ell-1}^A} M(A, L)$, where we maximize over all anti-chains in T_r except $\{r\}$ and over all ways of dividing the $\ell - 1$ slots among the elements of the anti-chain. The base case for this recurrence corresponds to the case where r is a leaf, and is easy to deal with.

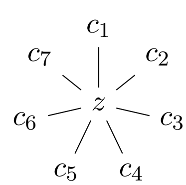
Now, the score of an optimum Chamberlin–Courant committee containing r^* is $M(r^*, k)$. We can repeat the algorithm for all possible choices of r^*. Then, the optimum Chamberlin–Courant score is the highest value of $M(r^*, k)$ that we have encountered.

We have argued that the size of each anti-chain is at most λ. Therefore, to calculate each $M(r, \ell)$, we enumerate at most m^λ anti-chains and at most k^λ divisions. This calculation needs to be performed for every vertex r (proceeding from the leaves towards the root) and for each $\ell \in [k]$. This establishes our bound on the running time. □

Notice that the time bound in Theorem 5.1 implies that our problem is in XP with respect to the number λ of leaves in the underlying tree. Whether there is an FPT algorithm for this parameter, or even for the combined parameter (k, λ), remains an open problem.

6. Utilitarian Chamberlin–Courant on Trees with Few Internal Vertices

Consider the star with center candidate z and leaf candidates $c_1, \dots, c_7$. Which preference orders are single-peaked on this tree?



A ranking could begin with z. After z, we can rank the other candidates in an arbitrary order without violating single-peakedness. But suppose we begin the ranking with a leaf candidate such as c_1. Then z must be the second candidate, because the set consisting of the top two candidates must be connected in the tree. After ranking c_1 and z, we can then order the remaining candidates arbitrarily without violating single-peakedness. Thus, the rankings that are single-peaked on the star are precisely the rankings in which the center vertex is ranked first or second.

Proposition 6.1. *A preference profile is single-peaked on a star if and only if there exists a candidate that every voter ranks in first or second position.*

This observation implies that, in some sense, the restriction of being single-peaked on a tree does not give us much information. For example, consider the problem of computing an optimal Kemeny ranking, i.e., a ranking that minimizes the sum of Kendall tau distances to the rankings in the input profile. This problem is NP-hard in general (Bartholdi et al., 1989), and we can easily see that it remains hard for preferences single-peaked on a star. Indeed, we can transform a general instance of this problem into one that is single-peaked on a star by adding a new candidate that is ranked in the first position by every voter; the resulting problem is clearly as hard as the original one.

For some other problems, though, the restriction to stars makes the problem easy. In particular, this is the case for the utilitarian Chamberlin–Courant rule with the Borda scoring function. To see this, note that it will often be a good idea to include the candidate who is the center vertex of the star in the committee. Once we have done so, every voter is already quite well represented: the Borda score of each voter's representative is either 0 or -1. Thus, it remains to identify $k-1$ candidates whose inclusion in the committee would bring the score of as many voters as possible up to 0, which amounts to simply selecting $k-1$ candidates with highest Plurality scores.[3] Finally, we need to consider the case where an optimal committee does not include the center vertex. One can check that in this case the committee necessarily consists of k candidates with highest Plurality scores (see the proof of Theorem 6.2 below). By selecting the better of these two committees, we find a winning committee. This procedure works for many scoring functions other than Borda (see the end of this section). However, as we show in the appendix, this argument does not extend to *all* scoring functions: For some positional scoring functions, winner determination for utilitarian Chamberlin–Courant remains hard even for preferences single-peaked on a star (Theorem B.1).

The algorithm we have sketched for the Borda scoring function on stars can be generalized to trees that have a small number of internal vertices (and thus a large number of leaves). While for stars it suffices to guess whether the center vertex would be part of the winning committee, we now have to make a similar guess for *each* internal vertex.

Theorem 6.2. *Given a profile P with $|C| = m$ and $|N| = n$, a tree $T \in \mathcal{T}(P)$ with η internal vertices such that P is single-peaked on T, and a target committee size $k \geqslant 1$, we can find a winning committee of size k for P under the Chamberlin–Courant rule with the Borda scoring function in time* $\mathrm{poly}(n, m, (k+1)^\eta)$.

Proof. Given a candidate $c \in C$, let $\mathrm{plu}(c) = |\{i \in N : \mathrm{top}(i) = c\}|$ be the number of voters in P that rank c first. Let C° be the set of internal vertices of T. For each candidate $c \in C^\circ$, let $\mathrm{lvs}(c)$ denote the set of leaf candidates in $C \setminus C^\circ$ that are adjacent to c in T.

Our algorithm proceeds as follows. For each candidate $c \in C^\circ$ it guesses a pair $(x(c), \ell(c))$, where $x(c) \in \{0, 1\}$ and $0 \leqslant \ell(c) \leqslant |\mathrm{lvs}(c)|$. The component $x(c)$ indicates whether c itself is in the committee, and $\ell(c)$ indicates how many candidates in $\mathrm{lvs}(c)$ are in the committee. We require $\sum_{c \in C^\circ}(x(c) + \ell(c)) = k$. Next, the algorithm sets $W = \{c \in C^\circ : x(c) = 1\}$, and then for each $c \in C^\circ$ it orders the candidates in $\mathrm{lvs}(c)$ in non-increasing order of $\mathrm{plu}(c)$ (breaking ties according to a fixed ordering $\rhd$ over C), and adds the first $\ell(c)$ candidates in this order to W.

3. Recall that a candidate's Plurality score is the number of voters who rank this candidate first.

Each guess corresponds to a committee of size k. Guessing can be implemented deterministically: consider all options for the collection of pairs $\{(x(c), \ell(c))\}_{c \in C^\circ}$ satisfying $\sum_{c \in C^\circ}(x(c)+\ell(c)) = k$ (there are at most $2^\eta \cdot (k+1)^\eta$ possibilities), compute the Chamberlin–Courant score of the resulting committee for each option, and output the best one.

It remains to argue that this algorithm finds a committee with the maximum Chamberlin–Courant score. To see this, let $\mathcal{S}$ be the set of all size-k committees with the maximum Chamberlin–Courant score, and let S^* be the maximal committee in $\arg\max_{W \in \mathcal{S}} |W \cap C^\circ|$ with respect to the fixed tie-breaking ordering $\rhd$ (where, given two size-k committees $S, S' \in \mathcal{S}$, we write $S' \rhd S$ if $a' \rhd a$, where a' is the maximal element of $S' \setminus S$ with respect to $\rhd$ and a is the maximal element of $S \setminus S'$ with respect to $\rhd$).

For each $c \in C^\circ$, let $x^*(c) = 1$ if $c \in S^*$ and $x^*(c) = 0$ otherwise, and let $\ell^*(c) = |\text{lvs}(c) \cap S^*|$. Our algorithm will consider the collection of pairs $\{(x^*(c), \ell^*(c))\}_{c \in C^\circ}$ at some point, and construct a committee S based on this collection. We will now argue that $S = S^*$. This would show correctness of our algorithm, since it returns a committee with a total score at least as high as that of S.

Clearly, we have $C^\circ \cap S = C^\circ \cap S^*$, so it remains to argue that $\text{lvs}(c) \cap S^* = \text{lvs}(c) \cap S$ for each $c \in C^\circ$. Suppose for the sake of contradiction that this is not the case, i.e., there exists a $c \in C^\circ$ and a pair of candidates $a, b \in \text{lvs}(c)$ with $a \in S \setminus S^*$ and $b \in S^* \setminus S$. We distinguish two cases: $c \in S^*$ or $c \notin S^*$.

If $c \in S^*$, consider the committee $S' = (S^* \setminus \{b\}) \cup \{a\}$. We claim that S' has the same Chamberlin–Courant score as S^*. Note that when moving from S^* to S',

- the contribution of the $\text{plu}(b)$ voters who rank b first changes from 0 to -1,
- the contribution of the $\text{plu}(a)$ voters who rank a first changes from -1 to 0,
- the contribution of all other voters is unaffected by the change, since they prefer $c \in S^* \cap S'$ to both a and b.

We also have $\text{plu}(a) \geqslant \text{plu}(b)$ by construction of S, and so the score of S' is at least the score of S^*, and hence $\text{plu}(a) = \text{plu}(b)$. But then by construction of S we have $a \rhd b$, and this contradicts our choice of S^* from $\arg\max_{W \in \mathcal{S}} |W \cap C^\circ|$.

Now, suppose that $c \notin S^*$. Consider the committee $S' = (S^* \setminus \{b\}) \cup \{c\}$. Again, we claim that S' has the same Chamberlin–Courant score as S^*. Note that when moving from S^* to S',

- the contribution of each of the $\text{plu}(b)$ voters who rank b first decreases by 1 (as all of them rank c second),
- the contribution of each of the $\text{plu}(a)$ voters who rank a first increases by at least 1 (as all of them rank c second),
- the contribution of any other voter does not decrease (as all of them prefer c to b).

Again, we have $\text{plu}(a) \geqslant \text{plu}(b)$ by construction of S, and so the score of S' is at least the score of S^*, and hence $\text{plu}(a) = \text{plu}(b)$. Thus, the Chamberlin–Courant score of S' is optimal, and so $S' \in \mathcal{S}$. But $|S' \cap C^\circ| > |S^* \cap C^\circ|$, which contradicts our choice of S^* from $\arg\max_{W \in \mathcal{S}} |W \cap C^\circ|$. □

The reader may wonder if Theorem 6.2 can be strengthened to an FPT result with respect to η, e.g., by guessing the subset of internal nodes to be selected and then picking the leaves in a globally greedy fashion in order of their Plurality scores. However, it can be shown that this approach does not necessarily produce an optimal committee: this is because, when considering a leaf that is adjacent to an internal node that is not selected, we need to take into account its contribution to the utility of voters who do not rank it first, and this contribution may depend on what other leaves have been selected.

Further, it is clear from our proof that Theorem 6.2 holds for every positional scoring function whose score vector satisfies $s_1 = 0$, $s_2 = -1$, $s_3 \leqslant -2$. The proof does not extend to arbitrary positional scoring functions, and Theorem B.1 in the appendix shows that for some positional scoring functions Utilitarian CC is NP-hard even if preferences are single-peaked on a star. Note that, in contrast, Theorem 5.1 holds for any positional scoring function. On the other hand, the algorithm described in the proof of Theorem 6.2 is in FPT with respect to the combined parameter (k, η). In contrast, for general preferences computing the Chamberlin–Courant winners is W[2]-hard with respect to k under the Borda scoring function (Betzler et al., 2013).

7. Structure of the Set of Trees a Profile is Single-Peaked on: The Attachment Digraph

We now move on from our study of multiwinner elections and turn towards the problem of recognizing when a given preference profile is single-peaked on a tree. In particular, for each profile P, we will study the collection $\mathcal{T}(P)$ of *all* trees on which P is single-peaked. It turns out that the set $\mathcal{T}(P)$ has interesting structural properties, and admits a concise representation. In many cases, this will allow us to pick a 'nice' tree from $\mathcal{T}(P)$ i.e., a tree that satisfies certain additional requirements. For example, to use the algorithm from Section 5, we would want to pick a tree from $\mathcal{T}(P)$ with the smallest number of leaves, and to use the algorithm from Section 6, we would want to use a tree with the smallest number of internal vertices.

Trick (1989b) presents an algorithm that decides whether $\mathcal{T}(P)$ is non-empty. If so, the algorithm produces some tree T with $T \in \mathcal{T}(P)$. While building the tree, the algorithm makes various arbitrary choices. In our approach, we will store all the choices that the algorithm could take. To this end, we introduce a data structure, which we call the *attachment digraph* of profile P.

We will start by giving a description of Trick's algorithm and its proof of correctness. We follow the presentation of Trick's paper closely, but give somewhat more detailed proofs.

Trick's algorithm can be seen as taking inspiration from algorithms for recognizing preferences that are single-peaked on a line. Those typically start out by noticing that an alternative that is ranked bottom-most by some voter must be placed at one of the ends of the axis. Trick's algorithm uses the same idea; the analogue for trees is as follows.

Proposition 7.1 (Trick, 1989b)**.** *Suppose P is single-peaked on T, and suppose a occurs as a bottom-most alternative, that is,* $\text{bottom}(i) = a$ *for some $i \in N$. Then a is a leaf of T.*

Proof. The set $A \setminus \{a\}$ is a top-initial segment of the i-th vote, and hence must be connected in T. This can only be the case if a is a leaf of T. □

Suppose we have identified a bottom-ranked alternative a. We deduce that if our profile is single-peaked on any tree T, then a is a leaf of T. Now, being a leaf, a must have exactly one neighboring vertex b. Which vertex could this be? The following simple observation gives some necessary conditions.

Proposition 7.2 (Trick, 1989b)**.** *Suppose P is single-peaked on T, and suppose $a \in C$ is a leaf of T, adjacent to $b \in C$. Let $i \in N$ be a voter. Then either*

(i) $b \succ_i a$*, or*

(ii) $a = \text{top}(i)$ *and* $b = \text{second}(i)$*.*

Proof. (i) Suppose first that a is not i's top-ranked alternative, and rather $\text{top}(i) = c$. Take the unique path in T from c to a, which passes through b since b is the only neighbor of a. Since i's vote is single-peaked on T, it is single-peaked on this path, and hence i's preference decreases along it from c to a. Since b is visited before a, it follows that $b \succ_i a$.

(ii) Suppose, otherwise, that a is i's top-ranked alternative. Then $\{a, \text{second}(i)\}$ is a top-initial segment of i's vote, which by Proposition 2.2 is a connected set in T, and hence forms an edge. Thus, a is adjacent to $\text{second}(i)$, so $\text{second}(i) = b$ as required. □

Thus, in our search for a neighbor of the leaf a, we can restrict our attention to those alternatives b that satisfy either (i) or (ii) in the proposition above, for every voter $i \in N$. Let us write this down more formally: For each $a \in C$ and $i \in N$, define

$$B(i, a) = \begin{cases} \{c \in C : c \succ_i a\} & \text{if } \text{top}(i) \neq a, \\ \{\text{second}(i)\} & \text{if } \text{top}(i) = a. \end{cases}$$

Applying Proposition 7.2 to all voters $i \in N$, we see that a needs to be adjacent to an element in

$$B(a) = \bigcap_{i \in N} B(i, a).$$

Thus, we have the following corollary.

Corollary 7.3 (Trick, 1989b)**.** *Suppose a profile is single-peaked on T, and $a \in C$ is a leaf of T. Then a must be adjacent to an element of $B(a)$.*

We have established that it is necessary for leaf a to be adjacent to some alternative in $B(a)$. It turns out that if the profile is single-peaked on a tree, then for *every* alternative $b \in B(a)$ there is some tree $T \in \mathcal{T}(P)$ in which a is adjacent to b.

Proposition 7.4 (Trick, 1989b)**.** *Let P be a profile in which a occurs bottom-ranked. Suppose that $P|_{C \setminus \{a\}}$ is single-peaked on some tree T_{-a} with vertex set $C \setminus \{a\}$, and let T be a tree obtained from T_{-a} by attaching a as a leaf adjacent to some element $b \in B(a)$. Then P is single-peaked on T.*

Proof. Let T be a tree obtained as described. We show that P is single-peaked on T. Let $S \subseteq C$ be a top-initial segment of the ranking of some voter i in P. We need to show that S is connected in T.

- If $a \notin S$, then S is connected in T_{-a} because $P|_{C\setminus\{a\}}$ is single-peaked on T_{-a}. Hence S is also connected in T.
- If $S = \{a\}$, then S is trivially connected in T.
- If $a \in S$ and $S \neq \{a\}$, then $S \setminus \{a\}$ is connected in T_{-a} because $P|_{C\setminus\{a\}}$ is single-peaked on T_{-a}. Therefore, to show that S is connected in T, it suffices to show that a's neighbor b is also an element of S. Since $b \in B(a) = \bigcap_{i\in N} B(i,a)$, we have that $b \in B(i,a)$. If $\text{top}(i) = a$, then $B(i,a) = \{\text{second}(i)\}$, so $b = \text{second}(i)$. As S is a top-initial segment of i with $|S| \geqslant 2$, we have $b \in S$, as desired. Otherwise $\text{top}(i) \neq a$, and so $B(i,a) = \{c : c \succ_i a\}$, hence $b \succ_i a$. As S is a top-initial segment of i including a, we must have $b \in S$, as desired. □

Algorithm 1 Trick's algorithm to decide whether a profile is single-peaked on a tree

```
T ← (C, ∅), the empty graph on C
C_1 ← C, r ← 1
while |C_r| ⩾ 3 do
    L_r ← {bottom(i, C_r) : i ∈ N}
    for each candidate a ∈ L_r do
        B(a) ← ⋂_{i∈N} B(i, C_r, a)
        if B(a) = ∅ then
            return fail : P is not single-peaked on any tree
        else
            select b ∈ B(a) arbitrarily
            add an edge between a and b in T
    C_{i+1} ← C_r \ L_r
    r ← r + 1
if |C_r| = 2 then
    add an edge between the two candidates in C_r to T
return P is single-peaked on T
```

With these results in place, we can now see how a recognition algorithm could work. Select an alternative a that is ranked bottom-most by some voter, select an arbitrary candidate $b \in B(a)$, add an edge $\{b, a\}$ to the tree under construction, remove a from the profile, and recurse on the remaining candidates. If at any point we find that $B(a) = \varnothing$, then we can conclude from Corollary 7.3 that the profile is not single-peaked on any tree. Algorithm 1 formalizes this procedure. To avoid recursion, the algorithm uses the following notation: for every subset $S \subset C$, for each $a \in S$, and each $i \in N$, define

$$B(i,S,a) = \begin{cases} \{c \in S : c \succ_i a\} & \text{if } \text{top}(i,S) \neq a, \\ \{\text{second}(i,S)\} & \text{if } \text{top}(i,S) = a. \end{cases}$$

Theorem 7.5 (Trick, 1989b). *Algorithm 1 correctly decides whether a profile is single-peaked on a tree.*

Proof. First, note that if Algorithm 1 succeeds and returns a graph T, then T is a tree. Indeed, it is easy to see that T has $|C| - 1$ edges. Moreover, T is connected, because every vertex has a path to a vertex in the set C_r at the end of the algorithm, and C_r is either a singleton or a connected set of size 2.

We show that the algorithm is correct by induction on $|C|$. If $|C| = 1$ or $|C| = 2$, every profile is single-peaked on the unique tree on C, and Algorithm 1 correctly determines this. If $|C| \geqslant 3$, then the *while*-loop is executed at least once. If in the first iteration the algorithm claims that the profile is not single-peaked on a tree because $B(a) = \varnothing$ for some $a \in L_1$, then this statement is correct by Corollary 7.3. Otherwise, after the first iteration the algorithm behaves as if it was run on $P|_{C_2}$ (recall that $C_2 = C \setminus L_1$).

Now, if the algorithm fails in later iterations, by the inductive hypothesis, $P|_{C_2}$ is not single-peaked on a tree. But then P is not single-peaked on a tree either: Suppose it was single-peaked on T. Then, by Proposition 7.1, all candidates in L_1 are leaves of T, and therefore $T|_{C_2}$ is still a tree, and so $P|_{C_2}$ is single-peaked on $T|_{C_2}$ (by Proposition 2.2), a contradiction. Thus, in this case, the algorithm executed on P correctly determines that P is not single-peaked on a tree.

On the other hand, if the algorithm's run on P terminates and returns a tree T, then its run on $P|_{C_2}$ would have terminated and returned the tree $T|_{C_2}$. By the inductive hypothesis, $P|_{C_2}$ is single-peaked on $T|_{C_2}$. Hence, by Proposition 7.4, P is single-peaked on T, and so the algorithm is correct. □

This concludes our presentation of Trick's approach.

Trick's algorithm makes some arbitrary choices when selecting alternatives $b \in B(a)$. Our aim is to understand the set $\mathcal{T}(P)$ of *all* trees that the input profile P is single-peaked on, so a natural approach is to record all possible choices that Trick's algorithm could make at each step, as this will encode all possible outputs of the algorithm. We do this by running Algorithm 2, which has the same structure as Algorithm 1. Given a profile that is single-peaked on *some* tree, it constructs and returns a digraph D with vertex set C which contains all possible choices that Trick's algorithm can make. We call D the *attachment digraph* of the input profile.

Example 7.6. *The attachment digraphs of the following three profiles are shown in Figure 3.*

(a) Suppose $C = \{a, b, c, d, e\}$, and let P_1 be the profile with voters $N = \{1, 2\}$ such that

$$a \succ_1 b \succ_1 c \succ_1 d \succ_1 e \qquad \text{and} \qquad e \succ_2 d \succ_2 c \succ_2 b \succ_2 a,$$

so that the two votes are the reverse of each other. Running Algorithm 2, we consider the sets $L_1 = \{a, e\}$ and $L_2 = \{b, d\}$.

(b) Suppose $C = \{a, b, c, d, e\}$, and let P_2 be the profile with voters $N = \{1, 2\}$ such that

$$a \succ_1 b \succ_1 c \succ_1 d \succ_1 e \qquad \text{and} \qquad e \succ_2 b \succ_2 c \succ_2 d \succ_2 a.$$

Running Algorithm 2, we consider the sets $L_1 = \{a, e\}$ and $L_2 = \{d\}$.

Algorithm 2 Build attachment digraph $D = (C, A)$ of P

$D \leftarrow (C, A)$, $A \leftarrow \varnothing$, so D is the empty digraph on C
$C_1 \leftarrow C$, $r \leftarrow 1$
while $|C_r| \geqslant 3$ **do**
  $L_r \leftarrow \{\text{bottom}(i, C_r) : i \in N\}$
  for each candidate $a \in L_r$ **do**
    $B(a) \leftarrow \bigcap_{i \in N} B(i, C_r, a)$
    if $B(a) = \varnothing$ **then**
      return *fail : P is not single-peaked on any tree*
    else
      for each $b \in B(a)$, add an arc (a, b) to A
  $C_{i+1} \leftarrow C_r \setminus L_r$
  $r \leftarrow r + 1$
if $|C_r| = 2$ **then**
  add an arc between the two candidates in C_r to A, arbitrarily directed
return D

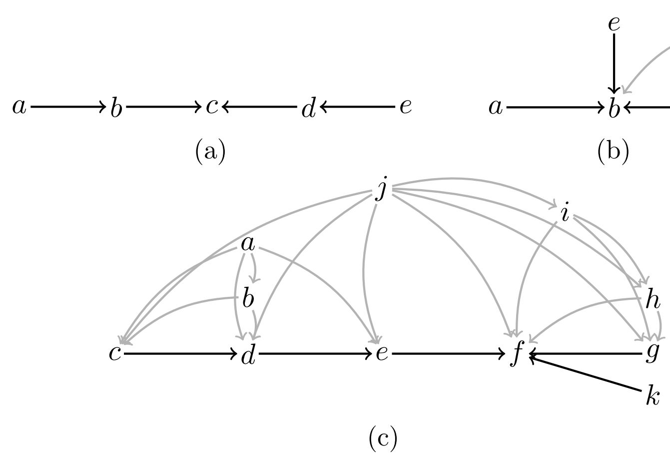


Figure 3: The attachment digraphs of the profiles in Example 7.6. If a vertex has a unique outgoing arc, the arc is drawn in black. If the vertex has at least two outgoing arcs, the arcs are drawn in gray and curved.

(c) Suppose $C = \{a, b, c, d, e, f, g, h, i, j, k\}$, and let P_3 be the profile with voters $N = \{1, 2, 3\}$ such that

$$k \succ_1 f \succ_1 e \succ_1 d \succ_1 g \succ_1 h \succ_1 c \succ_1 i \succ_1 j \succ_1 b \succ_1 a,$$
$$d \succ_2 c \succ_2 b \succ_2 e \succ_2 a \succ_2 f \succ_2 g \succ_2 h \succ_2 i \succ_2 j \succ_2 k,$$
$$g \succ_3 f \succ_3 h \succ_3 i \succ_3 e \succ_3 d \succ_3 c \succ_3 b \succ_3 a \succ_3 j \succ_3 k.$$

Running Algorithm 2, we consider the sets $L_1 = \{a, k\}$, $L_2 = \{b, j\}$, $L_3 = \{c, i\}$, $L_4 = \{d, h\}$, and $L_5 = \{e, g\}$.

Algorithm 2 runs in time $O(|N| \cdot |C|^2)$. In the rest of this section, we will analyze the structure of the attachment digraph, and its relation to the set $\mathcal{T}(P)$ of trees on which P is single-peaked. Throughout, we fix the profile P and let C_r, L_r, and $B(a)$ refer to the sets constructed while running the r-th iteration of Algorithm 2 on P. We start with a few simple properties.

Proposition 7.7. *Let $a \in C$ be a candidate with $a \in L_r$. Then $B(a) \cap L_r = \varnothing$. Hence, for every arc $(a, b) \in A$ with $a \in L_r$ and $b \in L_s$, we have that $s > r$.*

Proof. Note that the set L_r is only well-defined when $|C_r| \geqslant 3$. Assume for a contradiction that $b \in B(a)$ and $b \in L_r$. Since $b \in L_r$, there is some voter $i \in N$ such that $b = \text{bottom}(i, C_r)$. But then $b \not\succ_i a$ and $b \neq \text{second}(i, C_r)$ (as $|C_r| \geqslant 3$), so it cannot be the case that $b \in B(i, C_r, a)$, a contradiction.

For the second statement, note that if $(a, b) \in A$ is an arc, then $b \in B(a)$. Since $B(a) \subseteq C_r = C \setminus (L_1 \cup \ldots \cup L_{r-1})$, we must have $s \geqslant r$. By the previous paragraph, $s = r$ is impossible, and hence $s > r$. □

Proposition 7.8. *Every attachment digraph $D = (C, A)$ is acyclic and has exactly one sink.*

Proof. Suppose that the *while*-loop of Algorithm 2 is executed $R - 1$ times, and consider the sets $L_1, \ldots, L_{R-1}$. Set $L_R := C \setminus (L_1 \cup \ldots \cup L_{R-1})$. Then $L_1, \ldots, L_R$ is a partition of C.

For acyclicity, note that for each $a \in L_r$ with $1 \leqslant r < R$ we have $B(i, C_r, a) \subseteq C_r$ and hence $B(a) \subseteq C_r$. Together with Proposition 7.7, this implies that if $a \in L_r$ then all outgoing arcs of a point into $L_{r+1} \cup \ldots \cup L_R$. Hence, based on the partition $L_1, \ldots, L_R$, the set D can be topologically ordered and thus cannot contain a cycle.

For the number of sinks, note that there is at least one sink in D because D is acyclic. Since for every $a \in C \setminus L_R$ we have $B(a) \neq \varnothing$, at least one outgoing arc of a is added to D. Thus, no vertex in $C \setminus L_R$ is a sink. The condition of the *while*-loop implies that $|L_R| \leqslant 2$. If $|L_R| = 1$, then there is exactly one sink, and we are done. If $|L_R| = 2$, then the final **if** clause of the algorithm adds an arc between the two vertices in L_R, which ensures that only one of them is a sink. □

If we wish to extract a tree $T \in \mathcal{T}(P)$ from the attachment digraph D, Trick's algorithm tells us that we must choose, for each non-sink vertex of D, exactly one outgoing arc, and add this arc as an edge. To formalize this process, we denote the sink vertex by t, and say that a function $f : C \setminus \{t\} \to C$ is an *attachment function* for D if $(a, f(a)) \in A$ is an arc of D for every $a \in C \setminus \{t\}$. Thus, f specifies one outgoing arc for each $a \in C \setminus \{t\}$. Given an attachment function f, we write $T(f)$ for the tree on C with edge set

$$\{\{a, f(a)\} : a \in C \setminus \{t\}\}.$$

We now prove that every attachment function corresponds to a tree, and all trees in $\mathcal{T}(P)$ can be obtained in this way.

Theorem 7.9. *Let P be a profile that is single-peaked on some tree, and let D be its attachment digraph. Then $T \in \mathcal{T}(P)$ if and only if $T = T(f)$ for some attachment function f. In other words, P is single-peaked on a tree T if and only if the set of edges of T is obtained by picking exactly one outgoing arc for each non-sink vertex of D and converting it into an undirected edge.*

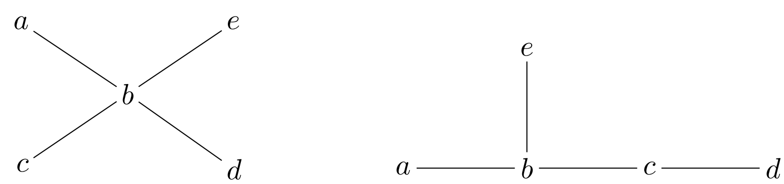


Figure 4: The set $\mathcal{T}(P_2)$ of trees on which the profile P_2 from Example 7.6 is single-peaked consists of these two trees. For the tree on the left, the attachment function has $f(d) = b$, while for the tree on the right, it has $f(d) = c$.

Proof. Suppose $T = T(f)$ for some attachment function f. Then T is a possible output of Algorithm 1, for a suitable way of making the selections from $B(a)$ for each vertex a processed in the *while*-loop. Thus, by Theorem 7.5, the profile P is single-peaked on T.

We prove the converse by induction on $|C|$. If $|C| \leqslant 2$, then P is single-peaked on the unique tree on C, which can be obtained as $T(f)$ for the unique attachment function f. So suppose that $|C| \geqslant 3$, and that $T = (C, E)$ is a tree such that P is single-peaked on T. During the first iteration of Algorithm 2, the algorithm determines the set L_1 of candidates occurring in bottom position, and sets $C_2 = C \setminus L_1$. By Proposition 7.1, each vertex in L_1 is a leaf of T. Hence, the induced subgraph $T|_{C_2}$ is also a tree, and thus $P|_{C_2}$ is single-peaked on $T|_{C_2}$. Also, by inspection of Algorithm 2, the attachment digraph of $P|_{C_2}$ is $D|_{C_2}$. By the inductive hypothesis, $T|_{C_2} = T(f')$ for some attachment function f' defined for $D|_{C_2}$. Thus, we can define an attachment function f so that for each $a \in C_2 \setminus \{t\}$ we set $f(a) = f'(a)$, and for each $a \in L_1$ we set $f(a)$ to be the unique neighbor of a in T. By Corollary 7.3, T is obtained from $T|_{C_2}$ by attaching each $a \in L_1$ to an element of $B(a)$, which implies that f is a legal attachment function. Thus, $T = T(f)$, which proves the claim. □

Using this characterization of the set $\mathcal{T}(P)$ and noting that $T(f_1) \neq T(f_2)$ whenever $f_1 \neq f_2$, we can conclude that the number of trees in $\mathcal{T}(P)$ is equal to the number of different attachment functions. This observation can be restated as follows.

Corollary 7.10. *The number of trees in $\mathcal{T}(P)$ is equal to the product of the out-degrees of the non-sink vertices of D. Hence we can compute $|\mathcal{T}(P)|$ in polynomial time.*

For the profiles in Example 7.6, we see that P_1 is single-peaked on a unique tree (a path), that P_2 is single-peaked on exactly 2 trees (shown in Figure 4), and that P_3 is single-peaked on exactly 336 different trees.

It turns out that attachment digraphs have a lot of structure beyond the results of Proposition 7.8. A key property, which will allow us to use essentially greedy algorithms, is what we call circumtransitivity.

Definition 7.11. *A directed acyclic graph $D = (C, A)$ is* circumtransitive *if its vertices can be partitioned into a set $C^{\rightarrow}$ of* forced *vertices and a set $C^{\rightrightarrows} = C \setminus C^{\rightarrow}$ of* free *vertices such that*

1. *every forced vertex $a \in C^{\rightarrow}$ has out-degree at most 1, and if $(a, b) \in A$ then also $b \in C^{\rightarrow}$, and*

2. *every free vertex* $a \in C^{\rightrightarrows}$ *has out-degree at least 2, and whenever* $a, b \in C^{\rightrightarrows}$ *and* $c \in C$ *are such that* $(a, b), (b, c) \in A$*, then* $(a, c) \in A$.

Intuitively, a circumtransitive digraph consists of an inner part (the *forced part*), and an outer part, which is transitively attached to the inner part.

Recall that every directed acyclic graph D has at least one sink. If D is also circumtransitive, then its sink must be among the forced vertices.

Theorem 7.12. *Every attachment digraph* (C, A) *is circumtransitive.*

Proof. We will argue that the partition

$$C^{\rightarrow} = \{a : d^+(a) \leqslant 1\}, \qquad C^{\rightrightarrows} = \{a : d^+(a) \geqslant 2\}.$$

satisfies the conditions in Definition 7.11. Suppose that the *while*-loop of Algorithm 2 is executed $R - 1$ times, partitioning the set C as $L_1 \cup \dots L_{R-1} \cup L_R$, where $L_R := C \setminus (L_1 \cup \dots \cup L_{R-1})$.

Forced: Let $a \in C^{\rightarrow}$. If $d^+(a) = 0$, there is nothing to prove, so assume that $d^+(a) = 1$, i.e., $(a, b) \in A$ for some $b \in C$. We will show that $d^+(b) \in \{0, 1\}$ and hence that $b \in C^{\rightarrow}$.

If b is a sink, we are done. Otherwise, there exists an arc $(b, c) \in A$ for some $c \in C$. Suppose that $a \in L_r$ and $b \in L_s$ for some $1 \leqslant r, s \leqslant R$. Note that $r < s \leqslant R - 1$ because neither a nor b are sinks, and $(a, b) \in A$ (see Proposition 7.7). We will argue that $\text{top}(i, C_s) = b$ for some $i \in N$; this shows that b has exactly one out-neighbor, as desired.

Indeed, suppose this is not the case. Then $c \in B(b)$ implies that $c \succ_i b$ for each $i \in N$. As $c \in C_r$, it cannot be the case that $\text{top}(i, C_r) = a$, $\text{second}(i, C_r) = b$ for some $i \in N$. Consequently, $B(i, C_r, a) = \{x \in C_r : x \succ_i a\}$ for each $i \in N$. As we have $B(a) = \{b\}$, it follows that $b \succ_i a$ for each $i \in N$. But then by transitivity $c \succ_i a$ for each $i \in N$, and hence $c \in B(a)$, a contradiction.

Free: Consider vertices $a, b, c \in C$ with $a, b \in C^{\rightrightarrows}$ and $(a, b), (b, c) \in A$. Since $a, b \in C^{\rightrightarrows}$, we have $a, b \notin L_R$ (indeed, recall that the out-degree of each vertex in L_R is at most 1). Thus, there exist r, s with $1 \leqslant r < s < R$ such that $a \in L_r$ and $b \in L_s$. Note that if there was a voter $i \in N$ with $\text{top}(i, C_r) = a$, then $|B(a)| = 1$, a contradiction with $d^+(a) > 1$. Hence $\text{top}(i, C_r) \neq a$ for all $i \in N$. As $(a, b) \in A$, we have $b \in B(i, C_r, a)$ and therefore $b \succ_i a$ for all $i \in N$. Similarly, since $(b, c) \in A$ and $d^+(b) > 1$, we have $c \succ_i b$ for all $i \in N$. Hence, by transitivity, $c \succ_i a$ for all $i \in N$. Therefore $c \in \bigcap_{i \in N} B(i, C_r, a) = B(a)$, and so $(a, c) \in A$, as desired. □

Suppose that f is an attachment function for D. Then for each forced vertex $a \in C^{\rightarrow} \setminus \{t\}$, the value of $f(a)$ is uniquely determined, since a has exactly one out-neighbor. Note also that $D|_{C^{\rightarrow}}$ is connected because we can reach the sink t from every forced vertex. Hence, $\mathcal{G}(D|_{C^{\rightarrow}})$ is a tree. It follows that for every $T \in \mathcal{T}(P)$, the tree $\mathcal{G}(D|_{C^{\rightarrow}})$ is a subtree of T.

We will now study the free vertices $C^{\rightrightarrows}$ in more detail. The following proposition states that for every free vertex a, we can identify a pair of forced vertices that are adjacent in D such that a can be attached to either of these vertices.

Proposition 7.13. *Suppose* $|C| \geqslant 3$*. For every free vertex* $a \in C^{\rightrightarrows}$ *of the attachment digraph* $D = (C, A)$*, there are two forced vertices* $b, c \in C^{\rightarrow}$ *with* $(a, b), (a, c), (b, c) \in A$.

Proof. Our proof proceeds in four steps. Let $a \in C^{\rightrightarrows}$ be a free vertex.

Step 1. There is a forced vertex $b \in C^{\rightarrow}$ with $(a, b) \in A$.

Proof. The directed acyclic graph D has a unique sink t, so there exists a directed path $a = c_1 \to c_2 \to \cdots \to c_p = t$ from a to t. Take such a path of minimum length. We will argue that c_2 is a forced vertex. If $p = 2$, we are done, since in this case $(a, t) \in A$, and $t \in C^{\rightarrow}$. Now, suppose that $p \geqslant 3$. Suppose for the sake of contradiction that c_2 is a free vertex. Then $c_1, c_2 \in C^{\rightrightarrows}$, and $(c_1, c_2), (c_2, c_3) \in A$. Since D is circumtransitive, we have $(c_1, c_3) \in A$. But then $c_1 \to c_3 \to \cdots \to c_p$ is a shorter path from a to t, a contradiction with our choice of the path.

Step 2. There are at least two forced vertices $b, c \in C^{\rightarrow}$ with $(a, b), (a, c) \in A$.

Proof. Let us say that a free vertex a is *semi-forced* if there is a unique forced vertex $b \in C^{\rightarrow}$ with $(a, b) \in A$; we need to argue that the set of semi-forced vertices is empty. Assume for the sake of contradiction that this is not the case, and consider the maximum value of r such that L_r contains a semi-forced vertex; let a be some semi-forced vertex in L_r. As $a \in C^{\rightrightarrows}$, we have $d^+(a) \geqslant 2$, and so there exists a free vertex $x \in C^{\rightrightarrows}$ with $(a, x) \in A$. Since $(a, x) \in A$, we have $x \in L_s$ for some $s > r$. Now, x is a free vertex, and $s > r$ implies that x is not semi-forced. Thus, there exist two distinct forced vertices $y, z \in C^{\rightarrow}$ with $(x, y), (x, z) \in A$. But then by circumtransitivity we have $(a, y) \in A$, $(a, z) \in A$, which is a contradiction with our choice of a.

Step 3. The set $\{b \in C^{\rightarrow} : (a, b) \in A\}$ induces a subtree in $\mathcal{G}(D)$.

Proof. Consider a vertex $a \in L_r$, and suppose that A contains arcs (a, b) and (a, c) where $b, c \in C^{\rightarrow}$. Since $\mathcal{G}(D|_{C^{\rightarrow}})$ is a tree, there is a unique path Q from b to c in $\mathcal{G}(D|_{C^{\rightarrow}})$; let $C_Q \subseteq C^{\rightarrow}$ be the vertex set of this path. Fix some tree $T \in \mathcal{T}(P)$. Then $\mathcal{G}(D|_{C^{\rightarrow}})$ is a subgraph of T, and so Q is a path in T. Pick a voter $i \in N$. Since $b, c \in B(a)$, we have $|B(a)| > 1$ and so $|B(i, C_r, a)| > 1$. Hence, $a \neq \mathrm{top}(i, C_r)$ and thus we have $b \succ_i a, c \succ_i a$. Consider the top-initial segment of i's vote given by $W = \{x \in C : x \succ_i a\}$. By Proposition 2.2, since P is single-peaked on T, the set W is connected in T. Since $b, c \in W$, the path Q must be contained in $T|_W$, and hence $C_Q \subseteq W$. Thus, $x \succ_i a$ for each $x \in C_Q$, and so $C_Q \subseteq B(i, C_r, a)$. As this holds for every $i \in N$, we have $C_Q \subseteq B(a)$, and so $C_Q \subseteq \{b \in C^{\rightarrow} : (a, b) \in A\}$. Hence, $\{b \in C^{\rightarrow} : (a, b) \in A\}$ is connected in $\mathcal{G}(D|_{C^{\rightarrow}})$.

Step 4. There are two forced vertices $b, c \in C^{\rightarrow}$ with $(a, b), (a, c), (b, c) \in A$.

Proof. The set $\{b \in C^{\rightarrow} : (a, b) \in A\}$ is connected (by Step 3) and contains at least two members (by Step 2). Hence, by definition of $\mathcal{G}$, it contains some vertices b and c with $(b, c) \in A$. $\square$

In the next section, we will use the properties of attachment digraphs established in this section to develop algorithms that can check whether a given profile is single-peaked on a tree that satisfies certain constraints.

8. Recognition Algorithms: Finding Nice Trees

Suppose we are given a profile P with $\mathcal{T}(P) \neq \varnothing$ and wish to find trees in $\mathcal{T}(P)$ that satisfy additional desiderata. In particular, we may want to find trees that can be used with the parameterized algorithms for the Chamberlin–Courant rule that we presented earlier. We will now show how the attachment digraph can be used to achieve this. We consider a variety of objectives, going beyond minimizing the number of internal vertices or the number of leaves. These results may be useful for applications other than the computation of the Chamberlin–Courant rule.

We assume throughout this section that $|C| \geqslant 3$, since otherwise there is a unique tree T on C, and the problem of selecting the best tree is trivial.

8.1 Minimum Number of Internal Vertices

In Section 6, we saw an algorithm that could efficiently solve Utilitarian CC with the Borda scoring function for profiles single-peaked on a tree T with few internal vertices, where T was taken as input to the algorithm. We now show how to find, given a profile P, a tree $T \in \mathcal{T}(P)$ that has the fewest internal vertices. Algorithm 3 constructs an attachment function, and tries to make every vertex a leaf, if possible. In particular, every free vertex in the attachment digraph will become a leaf. We begin by showing that Algorithm 3 is well-defined, in the sense that whenever the algorithm chooses a vertex from a set, this set is non-empty, and whenever the attachment function is assigned a value, the respective arc is present in the attachment digraph.

Algorithm 3 Find $T \in \mathcal{T}(P)$ with fewest internal vertices

```
Let D = (C, A) be the attachment digraph of P
Let C^→, C^⇉ be the sets of forced and free vertices in D
Let t be the sink vertex of D
f ← ∅, the attachment function under construction
for each a ∈ C^→ \ {t} do
    f(a) ← b where b is the unique b ∈ C with (a, b) ∈ A
if |C^→| = 2 then
    pick some c ∈ C^→
    for each a ∈ C^⇉ do
        f(a) ← c
else if |C^→| > 2 then
    for each a ∈ C^⇉ do
        find c ∈ C^→ such that (a, c) ∈ A and c is internal in G(D|_{C^→})
        f(a) ← c
return T* = T(f)
```

Proposition 8.1. *Algorithm 3 returns a tree $T^* \in \mathcal{T}(P)$.*

Proof. Our claim follows from Theorem 7.9 once we can show that the choices of the algorithm are possible. Our running assumption that $|C| \geqslant 3$, combined with Proposition 7.13, implies that $|C^{\rightarrow}| \geqslant 2$.

Suppose that $|C^{\rightarrow}| = 2$. By Proposition 7.13, each $a \in C^{\rightrightarrows}$ is adjacent to both vertices in $C^{\rightarrow}$, and thus $(a, c) \in A$ irrespective of which $c \in C^{\rightarrow}$ is chosen by the algorithm. Thus the function f is an attachment function, i.e., the algorithm returns a tree in $\mathcal{T}(P)$.

On the other hand, suppose that $|C^{\rightarrow}| > 2$. By Proposition 7.13, each free vertex $a \in C^{\rightrightarrows}$ has outgoing arcs to two forced vertices that are adjacent in $\mathcal{G}(D|_{C^{\rightarrow}})$. Since $|C^{\rightarrow}| > 2$, at most one of them can be a leaf in the tree $\mathcal{G}(D|_{C^{\rightarrow}})$. Hence, there is a $c \in C^{\rightarrow}$ with $(a, c) \in A$ such that c is internal in $\mathcal{G}(D|_{C^{\rightarrow}})$. Thus, the algorithm is well-defined in this case as well. □

Next, we show that Algorithm 3 returns an optimal tree.

Proposition 8.2. *Algorithm 3 runs in polynomial time and returns a tree* $T^* \in \mathcal{T}(P)$ *with the minimum number of internal vertices among trees in* $\mathcal{T}(P)$.

Proof. The bound on the running time is immediate from the description of the algorithm.

By Theorem 7.9 and the definition of $C^{\rightarrow}$, for every tree $T \in \mathcal{T}(P)$ we have $\mathcal{G}(D|_{C^{\rightarrow}}) \subseteq T$. Thus, if $a \in C^{\rightarrow}$ is not a leaf in the tree $\mathcal{G}(D|_{C^{\rightarrow}})$, then a cannot be a leaf in T.

Suppose that $|C^{\rightarrow}| = 2$. Since $|C| \geqslant 3$, we have $C^{\rightrightarrows} \neq \varnothing$. Since the two members of $C^{\rightarrow}$ are adjacent in any $T \in \mathcal{T}(P)$, it cannot be the case that both of them are leaves in T. Hence the number of leaves in $T \in \mathcal{T}(P)$ is at most $|C^{\rightrightarrows}| + 1$. The tree T^* has exactly $|C^{\rightrightarrows}| + 1$ leaves, and hence is optimal.

On the other hand, suppose that $|C^{\rightarrow}| > 2$. Note that every free vertex $a \in C^{\rightrightarrows}$ is a leaf in T^* because $f(a) \in C^{\rightarrow}$ for all $a \in C \setminus \{t\}$. Further, every leaf of $\mathcal{G}(D|_{C^{\rightarrow}})$ is also a leaf in T^*. By our initial observation, none of the remaining vertices can be leaves in any $T \in \mathcal{T}(P)$, so T^* has the maximum possible number of leaves, and hence the minimum number of internal vertices. □

8.2 Minimum Diameter

It turns out that the tree found by Algorithm 3 is also optimal with respect to another metric: it minimizes the diameter.

Proposition 8.3. *Algorithm 3 returns a tree* $T^* \in \mathcal{T}(P)$ *with the minimum diameter among trees in* $\mathcal{T}(P)$.

Proof. Suppose that $|C^{\rightarrow}| = 2$. Then T^* is a star with center c; no tree on three or more vertices has smaller diameter than a star.

On the other hand, suppose that $|C^{\rightarrow}| > 2$. In this case the diameter of T^* is equal to the diameter of $\mathcal{G}(D|_{C^{\rightarrow}})$. To see this, consider a longest path $(c_1, \dots, c_k)$ in T^*. If $k = 2$, then T^* is a star, which is a minimum-diameter tree when there are $|C| \geqslant 3$ vertices. So suppose that $k \geqslant 3$. On a longest path, only c_1 and c_k can be free vertices, since all free vertices are leaves in T^*. Suppose $c_1 \in C^{\rightrightarrows}$. Then by construction of T^* we have $c_2 \in C^{\rightarrow}$, and c_2 is an internal vertex of $\mathcal{G}(D|_{C^{\rightarrow}})$. Hence, c_2 has at least two neighbors in $\mathcal{G}(D)$ that are forced. Thus, we can find a neighbor c_1' of c_2 such that c_1' is forced and $c_1' \neq c_3$. Then we can replace c_1 by c_1' in the longest path (noting that c_1' cannot appear elsewhere on the path because $\mathcal{G}(D|_{C^{\rightarrow}})$ is a tree). Similarly, if $c_k \in C^{\rightrightarrows}$, we can replace c_k by a forced neighbor of c_{k-1}. Having replaced all free vertices on the path by forced vertices, we have obtained a longest path in T^* that is completely contained in $\mathcal{G}(D|_{C^{\rightarrow}})$. Hence, the diameter of T^* is equal to the diameter of $\mathcal{G}(D|_{C^{\rightarrow}})$.

As $\mathcal{G}(D|_{C^{\rightarrow}}) \subseteq T$ for every $T \in \mathcal{T}(P)$, the diameter of any $T \in \mathcal{T}(P)$ must be at least the diameter of $\mathcal{G}(D|_{C^{\rightarrow}})$. Hence T^* has the minimum diameter. □

8.3 Minimum Number of Leaves

In Section 5, we saw an algorithm for Utilitarian CC that is efficient when the input profile is single-peaked on a tree with few leaves. The algorithm assumed that the tree T is part of the input. We will now describe a procedure that, given a profile P, finds a tree $T^* \in \mathcal{T}(P)$ with the minimum number of leaves.

Algorithm 4 Find $T \in \mathcal{T}(P)$ with fewest leaves

Let $D = (C, A)$ be the attachment digraph of P
$f \leftarrow \varnothing$, the attachment function under construction
Let t be the sink vertex of D, and let $s \in C^{\nrightarrow}$ be a forced vertex with a unique outgoing arc $(s, t) \in A$ (such a vertex exists by Proposition 7.13)
$f(s) \leftarrow t$
Construct a bipartite graph H with vertex set $L \cup R$ where $L = \{\ell_a : a \in C \setminus \{s, t\}\}$ and $R = \{r_a : a \in C\}$, and edge set $E_H = \{\{\ell_a, r_c\} : (a, c) \in A\}$
Find a maximum matching $M \subseteq E_H$ in H
for each $a \in C \setminus \{s, t\}$ **do**
    if ℓ_a is matched in M, i.e. $\{\ell_a, r_c\} \in M$ for some $c \in C$ **then**
        $f(a) \leftarrow c$
    else
        take any $c \in C$ with $(a, c) \in A$
        $f(a) \leftarrow c$
return $T^* = T(f)$

Minimizing the number of leaves of a tree is equivalent to maximizing its number of internal vertices. Thus, to proceed, we first characterize the set of internal vertices of a tree $T(f)$.

Proposition 8.4. *Let f be an attachment function for the attachment digraph D. Then $a \in C \setminus \{t\}$ is an internal vertex of the tree $T(f)$ if and only if $|f^{-1}(a)| \geqslant 1$, i.e., a is in the range of f. The sink vertex t is an internal vertex of $T(f)$ if and only if $|f^{-1}(t)| \geqslant 2$, i.e., there are two distinct vertices $a, b \in C$ with $f(a) = f(b) = t$.*

Proof. A vertex is internal in a tree if and only if it has degree at least two. From the definition of $T(f)$, for $a \in C \setminus \{t\}$, the degree of a is $1 + |f^{-1}(a)|$, and the degree of t is $|f^{-1}(t)|$. The claim follows immediately. □

Using this observation, we can prove that Algorithm 4 returns an optimal tree. The algorithm is based on constructing a maximum matching.

Proposition 8.5. *Algorithm 4 runs in polynomial time and returns a tree $T^* \in \mathcal{T}(P)$ with the minimum number of leaves among trees in $\mathcal{T}(P)$.*

Proof. The bound on the running time is immediate from the description of the algorithm. The algorithm constructs an attachment function, and hence by Theorem 7.9 the output T^* of the algorithm is a member of $\mathcal{T}(P)$.

We will now argue that T^* has the maximum number of internal vertices among trees in $\mathcal{T}(P)$. By Proposition 8.4, it suffices to show that Algorithm 4 finds an attachment function f that maximizes the number of vertices a with $|f^{-1}(a)| \geqslant 1$ if $a \neq t$ or $|f^{-1}(a)| \geqslant 2$ if $a = t$.

First, we claim that under the attachment function f constructed by Algorithm 4, a vertex $c \in C$ is an internal vertex of $T(f)$ if and only if r_c is matched in a maximum matching M. We start with the 'if' direction.

- Suppose $c = t$. If r_t is matched in M to ℓ_a, then both $a \in f^{-1}(t)$ and $s \in f^{-1}(t)$, where s is the vertex chosen at the very start of the algorithm. By definition of the bipartite graph H, $a \neq s$, and so $|f^{-1}(t)| \geqslant 2$, and hence t is an internal vertex by Proposition 8.4.
- Suppose $c \neq t$. If r_c is matched in M to ℓ_a, then $a \in f^{-1}(c)$, and so c is an internal vertex by Proposition 8.4.

For the 'only if' direction, suppose that r_c is not matched in M. Then in the *for*-loop of Algorithm 4, we never set $f(a) \leftarrow c$ for any $a \in C \setminus \{t\}$, because otherwise we could add the edge $\{\ell_a, r_c\}$ to the matching M, contradicting its maximality. Hence, if $c = t$ and c is not matched, then $f^{-1}(t) = \{s\}$, and so t is not internal. If $c \neq t$ and r_c is not matched, then $f^{-1}(c) = \varnothing$, so c is not internal. It follows that the number of internal vertices of $T(f)$ is $|M|$.

Now suppose that $T(f)$ is not optimal, and that $T' \in \mathcal{T}(P)$ is a tree with $q > |M|$ internal vertices. By Theorem 7.9, since $T' \in \mathcal{T}(P)$, we have $T' = T(g)$ for some attachment function g. But then we can construct a matching M' in H of size $|M'| = q$, as follows:

- If t is an internal vertex in T', then by Proposition 8.4, we have $|g^{-1}(t)| \geqslant 2$. Select some $a \in g^{-1}(t)$ with $a \neq s$, and add $\{\ell_a, r_t\}$ to M'.
- For each $c \in C \setminus \{t\}$ that is an internal vertex of T', select some $a \in g^{-1}(c)$ (which exists by Proposition 8.4), and add $\{\ell_a, r_c\}$ to M'.

Clearly, we have added q edges to M'. As g is a function, M' is a matching. Since $|M'| > |M|$, we have a contradiction to the maximality of M. □

8.4 Minimum Max-Degree

Another measure of tree complexity is its maximum degree. To minimize this quantity, we can use the following algorithm, which accepts as input a profile P and a positive integer k, and decides whether P is single-peaked on some tree with maximum degree at most k. It is based on constructing a maximum flow network. An example network is shown in Figure 5.

Proposition 8.6. *Algorithm 5 runs in polynomial time and returns a tree $T^* \in \mathcal{T}(P)$ with maximum degree at most k if one exists.*

Proof. The bound on the running time is immediate from the description of the algorithm.

Let f be some attachment function. By definition of $T(f)$, for each $a \in C \setminus \{t\}$, the degree of a in $T(f)$ is $1 + |f^{-1}(a)|$, because there is one edge in $T(f)$ corresponding to an outgoing arc of a in D, and $|f^{-1}(a)|$ edges in $T(f)$ corresponding to incoming arcs of a in D.

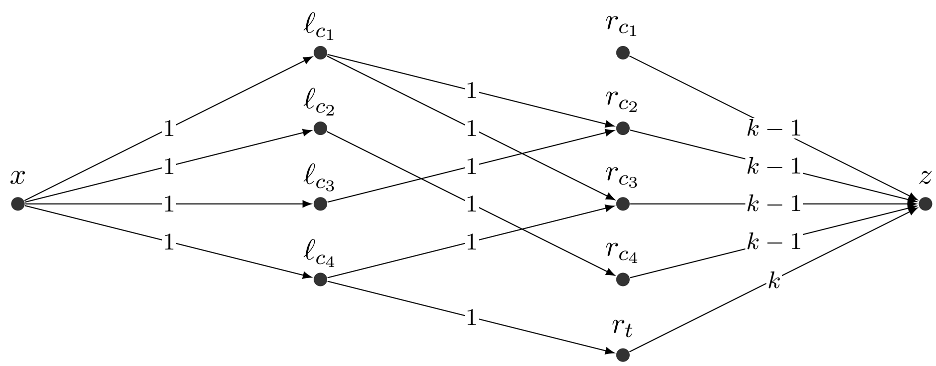


Figure 5: Flow network H constructed by Algorithm 5.

Algorithm 5 Decide whether there is $T \in \mathcal{T}(P)$ with maximum degree at most k

Let $D = (C, A)$ be the attachment digraph of P
Let t be the sink vertex of D
Let $L = \{\ell_a : a \in C \setminus \{t\}\}$ and $R = \{r_a : a \in C\}$ and construct a flow network H on vertex set $\{x, z\} \cup L \cup R$ with arc set

$$E_H = \{(x, \ell_a) : a \in C \setminus \{t\}\} \cup \{(\ell_a, r_b) : a, b \in C, (a, b) \in A\} \cup \{(r_a, z) : a \in C\},$$

and capacities $\text{cap}(x, \ell_a) = 1$ for all $a \in C \setminus \{t\}$, $\text{cap}(\ell_a, r_b) = 1$ for all $(a, b) \in A$, $\text{cap}(r_a, z) = k - 1$ for all $a \in C \setminus \{t\}$, and $\text{cap}(r_t, z) = k$
Find a maximum flow in H
$f \leftarrow \varnothing$, the attachment function under construction
if the size of the maximum flow is $|C| - 1$ **then**
  For each $(a, b) \in A$ such that the arc (ℓ_a, r_b) is saturated, set $f(a) \leftarrow b$
  return $T^* = T(f)$
else
  return there is no $T^* \in \mathcal{T}(P)$ with maximum degree at most k

Also, the degree of the sink vertex t in $T(f)$ is $|f^{-1}(t)|$. Thus, our task reduces to deciding whether there exists an attachment function f with

$$1 + |f^{-1}(a)| \leqslant k \text{ (i.e., } |f^{-1}(a)| \leqslant k - 1\text{) for each } a \in C \setminus \{t\} \text{ and } |f^{-1}(t)| \leqslant k. \qquad (1)$$

Such attachment functions are in one-to-one correspondence with (integral) flows of size $|C| - 1$ in the flow network constructed by Algorithm 5. Indeed, suppose f is an attachment function satisfying (**??**). Send one unit of flow from the super-source x along each of its $|C| - 1$ outgoing links. For each $a \in C \setminus \{t\}$, send the one unit of flow that arrives to ℓ_a towards $r_{f(a)}$. Finally, for each $b \in C$, send the flow that arrives into r_b towards the super-sink z. This flow satisfies the capacity constraints because f satisfies (**??**). Conversely, any integral flow of size $|C| - 1$ can be used to define an attachment function that satisfies (**??**): for each $a \in C \setminus \{t\}$ there must be one unit of flow leaving ℓ_a; we set $f(a) = b$, where r_b is the destination of this flow. The resulting f satisfies (**??**) due to the capacity constraints of the links between nodes in R and the super-sink z. □

8.5 Minimum Pathwidth

Here, we show how to find a tree $T \in \mathcal{T}(P)$ of minimum pathwidth. Our algorithm is based on an algorithm by Scheffler (1990), which computes a minimum-width path decomposition of a given tree in linear time.

We need a preliminary result showing that a tree always admits a minimum-width path decomposition with a certain property: one endpoint of each edge appears in a bag of the path decomposition that has some 'slack', in the sense that the bag does not have maximum cardinality.

Lemma 8.7. *For every tree $T = (C, E)$, there exists a path decomposition $S_1, \dots, S_r$ of T of minimum width w such that, for every edge $e \in E$, there is an endpoint $a \in e$ for which there exists a bag S_i with $a \in S_i$ such that $|S_i| \leqslant w$ (note that $\max_i |S_i| = w + 1$).*

Proof. We show how to transform an arbitrary path decomposition of T into a path decomposition of the same width having the desired property.

Suppose $S_1, \dots, S_r$ is a path decomposition of T with width w. For each edge $\{a, b\} \in E$, we do the following: Because $\{a, b\}$ is an edge, there exists a bag containing both a and b Consider the smallest value of $i \in \{1, \dots, r\}$ such that $a, b \in S_i$.

If $i = 1$, we set $S^* = S_i \setminus \{b\}$, and append the new bag S^* to the left of the sequence $S_1, \dots, S_r$. Then $S^*, S_1, \dots S_r$ is still a path decomposition of T, in this path decomposition a appears in S^*, and $|S^*| < |S_1| \leqslant w + 1$, so $|S^*| \leqslant w$.

If $i > 1$, then one of a or b does not appear in S_{i-1}. Assume without loss of generality that $b \notin S_{i-1}$. Again, set $S^* = S_i \setminus \{b\}$, and note that $|S^*| \leqslant w$. Place the new bag S^* in between S_{i-1} and S_i. Then $S_1, \dots, S_{i-1}, S^*, S_i, \dots, S_r$ is still a path decomposition of T, in this path decomposition a appears in S^*, and $|S^*| \leqslant w$. □

Clearly, the transformation described in the proof of Lemma 8.7 can be performed in polynomial time. Since one can find some path decomposition of a tree in polynomial time (Scheffler, 1990), one can find a path decomposition with the property stated in Lemma 8.7 in polynomial time as well.

Proposition 8.8. *Algorithm 6 returns a tree $T^* \in \mathcal{T}(P)$ with minimum pathwidth among trees in $\mathcal{T}(P)$ in polynomial time.*

Proof. Note first that the forced part $\mathcal{G}(D|_{C^{\rightarrow}})$ is a tree and hence its minimum-width path decomposition can be computed efficiently.

Next we claim that the path decomposition constructed by Algorithm 6 is in fact a path decomposition of the output tree $T(f)$. Indeed, each free vertex $a \in C^{\rightrightarrows}$ becomes a leaf in $T(f)$, and only occurs in a single bag S_a in the constructed path decomposition. Since a is a leaf, there is only one edge of T that contains it (namely, $\{a, f(a)\}$), and we have $a, f(a) \in S_a$. Also, since a only occurs in a single bag, the set of bags containing a is trivially an interval of the path decomposition sequence.

Next, observe that the path decomposition of $T(f)$ has the same width w as the pathwidth of the forced part $\mathcal{G}(D|_{C^{\rightarrow}})$, because all new bags have cardinality at most $w + 1$. Now, because $\mathcal{G}(D|_{C^{\rightarrow}})$ is a subgraph of every $T \in \mathcal{T}(P)$, no tree in $\mathcal{T}(P)$ can have a smaller pathwidth than $\mathcal{G}(D|_{C^{\rightarrow}})$. Since Algorithm 6 identifies a tree $T \in \mathcal{T}(P)$ with the same pathwidth as $\mathcal{G}(D|_{C^{\rightarrow}})$, it outputs an optimal solution. □

Algorithm 6 Find a tree $T \in \mathcal{T}(P)$ of minimum pathwidth

Let $D = (C, A)$ be the attachment digraph of P
Let $S_1, \dots, S_r$ be a path decomposition of $\mathcal{G}(D|_{C^{\nrightarrow}})$ of minimum width w that satisfies the condition of Lemma 8.7
$f \leftarrow \varnothing$, the attachment function under construction
for each $a \in C \setminus \{t\}$ **do**
  if $a \in C^{\nrightarrow}$ **then**
    $f(a) \leftarrow b$, for the unique $b \in C$ with $(a, b) \in A$
  else if $a \in C^{\rightrightarrows}$ **then**
    Let $c_1, c_2 \in C^{\nrightarrow}$ be two forced vertices such that $(c_1, c_2), (a, c_1), (a, c_2) \in A$
      (these exist by Proposition 7.13)
    Since $\{c_1, c_2\}$ is an edge of $\mathcal{G}(D|_{C^{\nrightarrow}})$, by the condition of Lemma 8.7,
      there is a bag S_j with $c_i \in S_j$ and $|S_j| \leqslant w$, for some $i \in \{1, 2\}$
    $f(a) \leftarrow c_i$
    Make a new bag $S_a = S_j \cup \{a\}$ and place it to the right of S_j
      in the sequence of the path decomposition
return $T^* = T(f)$

8.6 Other Graph Types

To conclude this section, we explain how to recognize whether $\mathcal{T}(P)$ contains trees of certain types.

Paths The literature contains several algorithms for recognizing profiles that are single-peaked on a path. The algorithms by Doignon and Falmagne (1994) and Escoffier et al. (2008) can be implemented to run in time $O(mn)$. One could also use some of the algorithms presented above. Algorithm 4 finds a tree $T \in \mathcal{T}(P)$ with a minimum number of leaves. Clearly, if $\mathcal{T}(P)$ contains a path, then this will be discovered by the algorithm. Alternatively, Algorithm 5 can be used to look for a tree $T \in \mathcal{T}(P)$ with maximum degree $k = 2$; it will succeed if and only if P is single-peaked on a path. However, both Algorithm 4 and Algorithm 5 depend on pre-computing the attachment digraph, which takes time $O(m^2n)$. Thus, an attachment digraph-based approach would necessarily be slower than the linear-time algorithms from previous work.

Stars In Proposition 6.1, we observed that a profile is single-peaked on a star graph if and only if there is a candidate $c \in C$ such that every voter ranks c in either first or second position. This condition can easily be verified in $O(n)$ time, without needing to compute the attachment digraph. Note that Algorithm 3 (minimizing the number of internal vertices) will output a star whenever $\mathcal{T}(P)$ contains a star graph.

Caterpillars Caterpillar graphs are exactly the trees of pathwidth 1 (Proskurowski & Telle, 1999), and so Algorithm 6 can check whether a profile is single-peaked on a caterpillar. In fact, one can use an even simpler algorithm: it suffices to compute $\mathcal{G}(D|_{C^{\nrightarrow}})$ and check that it is a caterpillar. Indeed, if not, then no tree in $\mathcal{T}(P)$ can be a caterpillar. On the other hand, if $\mathcal{G}(D|_{C^{\nrightarrow}})$ is a caterpillar then Algorithm 3 finds a caterpillar graph in $\mathcal{T}(P)$.

To see this, recall that this algorithm attaches every free vertex as a leaf to an internal vertex of $\mathcal{G}(D|_{C^{\nrightarrow}})$.

Subdivision of a Star A tree is a subdivision of a star if at most one vertex has degree 3 or higher. We can find a subdivision of a star in $\mathcal{T}(P)$, should one exist, by adapting Algorithm 5: we guess the center of the subdivision of the star, and then assign suitable upper bounds on the vertex degrees by appropriately setting the capacity constrains in the flow network.

9. Hardness of Recognizing Single-Peakedness on a Specific Tree

The algorithms presented in Section 8 enable us to answer a wide range of questions about the set $\mathcal{T}(P)$. However the following NP-hardness result shows that not every such question can be answered efficiently unless P = NP.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\phi : V_1 \to V_2$ such that for all $u, v \in V_1$, it holds that $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$; we write $G_1 \cong G_2$ whenever this is the case. We consider the following computational problem.

Single-Peaked Tree Labeling	
Instance:	Profile P over C, a tree T_0 on $\lvert C\rvert$ vertices
Question:	Is there a tree $T = (C, E)$ such that $T \cong T_0$ and P is single-peaked on T?

In this problem, we are given a 'template' unlabeled tree T_0, and need to decide whether we can label the vertices in this template by candidates so as to make the input profile single-peaked on the resulting labeled tree. For example, if T_0 is a path, then the problem is to decide whether the profile P is single-peaked on a path, and in this case the problem is easy to solve. However, the template T_0 occurs in the input to the decision problem, and it is not clear how to proceed if we would like to check whether T_0 'fits' into the attachment digraph. In fact, as we now show, this problem is NP-complete.

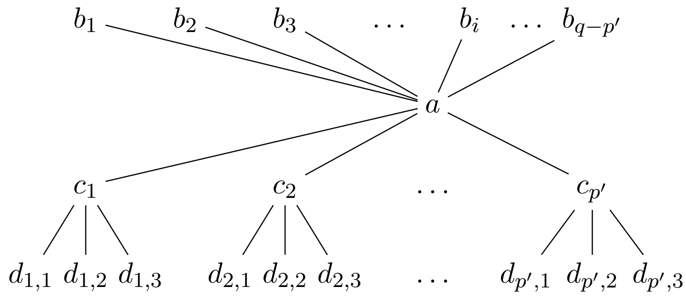
Theorem 9.1. *The problem* Single-Peaked Tree Labeling *is* NP*-complete even if* T_0 *is restricted to diameter at most four.*

Proof. The problem is in NP since, having guessed a tree T and an isomorphism ϕ, we can easily check that ϕ is an isomorphism between T and T_0 and that the input profile is single-peaked on T.

For the hardness proof, we reduce Exact Cover by 3-Sets (X3C) to our problem. An instance of X3C is given by a ground set X and a collection $\mathcal{Y}$ of size-3 subsets of X. It is a 'yes'-instance if there is a subcollection $\mathcal{Y}' \subseteq \mathcal{Y}$ of size $|X|/3$ such that each element of X appears in exactly one set in $\mathcal{Y}'$. This problem is NP-hard (Garey & Johnson, 1979).

Suppose we are given an X3C-instance with ground set $X = \{x_1, \dots, x_p\}$, where $p = 3p'$ for some positive integer p', and a collection $\mathcal{Y} = \{Y_1, \dots, Y_q\}$ of 3-element subsets of X. We then construct an instance of Single-Peaked Tree Labeling as follows. First, we construct a tree T_0 with vertex set $C_0 = \{a, b_1, \dots, b_{q-p'}, c_1, \dots, c_{p'}\} \cup \{d_{i,j} : 1 \leqslant i \leqslant p', 1 \leqslant j \leqslant 3\}$, and edge set $E_0 = \{\{a, b_i\} : 1 \leqslant i \leqslant q - p'\} \cup \{\{a, c_i\} : 1 \leqslant i \leqslant p'\} \cup \{\{c_i, d_{i,j}\} : 1 \leqslant$

$i \leqslant p', 1 \leqslant j \leqslant 3\}$. The resulting tree is drawn below. It has $3p' + p' + 1 + (q - p') = p + q + 1$ vertices and diameter 4.



Next, we construct a profile P with $|N| = p + q$ voters on the candidate set $C = \{z, x_1, \dots, x_p, y_1, \dots, y_q\}$. P will contain one vote for each object in X and one vote for each set in $\mathcal{Y}$. In the following, all indifferences can be broken arbitrarily. For each object x_i, we add a voter v_{x_i}:

$$z \succ \{y_j : Y_j \ni x_i\} \succ x_i \succ \{y_j : Y_j \not\ni x_i\} \succ X \setminus \{x_i\}.$$

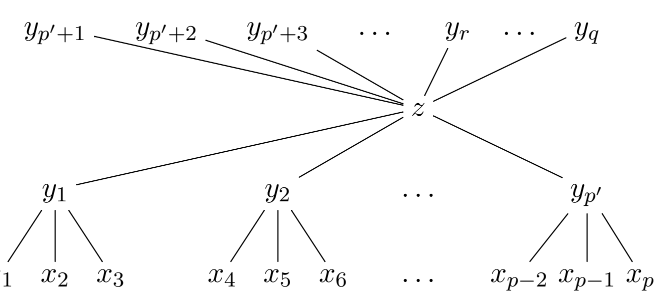
Intuitively, the presence of this voter will force x_i to be attached to z or to a candidate that corresponds to a set containing x_i. For each set Y_j, we add a voter v_{Y_j}:

$$z \succ y_j \succ \{y_\ell : 1 \leqslant \ell \leqslant q, \ell \neq j\} \succ X.$$

The presence of this voter will force an edge from z to y_j.

This completes the description of the reduction. We now prove that it is correct.

Suppose the given X3C-instance is a 'yes'-instance, and let $\mathcal{Y}'$ be a cover consisting of p' sets. Renumbering the elements and sets if necessary, we can assume that $\mathcal{Y}' = \{Y_1, \dots, Y_{p'}\}$ and $Y_j = \{x_{3j-2}, x_{3j-1}, x_{3j}\}$ for each $j = 1, \dots, p'$. Then we build a labeling isomorphism $\phi : C_0 \to C$ as follows: We set $\phi(a) = z$. For each $j = 1, \dots, p'$, we set $\phi(c_j) = y_j$, and $\phi(d_{j,k}) = x_{3(j-1)+k}$ for $k = 1, 2, 3$. Also, for each $j = 1, \dots, q - p'$, we set $\phi(b_j) = y_{p'+j}$. Note that ϕ is a bijection because $\mathcal{Y}'$ is an exact cover. The resulting labeled tree T is shown below. It is easy to check that the profile P is single-peaked on T.



Conversely, suppose that there is a tree T isomorphic to T_0 such that P is single-peaked on T. Let $\phi : C_0 \to C$ is a witnessing isomorphism. Note that the vertex z of T must have degree at least q, because for each $j \in \{1, \dots, q\}$, voter v_{Y_j} can only be single-peaked on T if z and y_j are adjacent in T. There is only one such vertex in T_0, namely a, and hence $\phi(a) = z$.

The vertex z of T has exactly q neighbors, which then must all be labeled by some y_j. Exactly p' of the q neighbors of z have degree 4. Let $\mathcal{Y}' = \{Y_j \in \mathcal{Y} : y_j = \phi(c_i) \text{ for some } 1 \leqslant i \leqslant p'\}$ be the collection of the p' sets corresponding to candidates that occupy the vertices of degree 4. We claim that $\mathcal{Y}'$ is a cover. Let $x_i \in X$. The vertex labeled with x_i must be a leaf of T because all internal vertices of T have already been labeled otherwise. Then, because v_{x_i} is single-peaked on T, the set $\{z, x_i\} \cup \{y_j : Y_j \ni x_i\}$ must be connected in T, so the neighbor of x_i must be a member of that set. But x_i cannot be a neighbor of z, so x_i is a neighbor of some y_j where $x_i \in Y_j$. This implies that y_j is the label of a degree-4 vertex. Hence $Y_j \in \mathcal{Y}'$, and so x_i is covered by $\mathcal{Y}'$. □

By creating multiple copies of the center vertex and adding some peripheral vertices, we can adjust this reduction to show that Single Peaked Tree Labeling remains hard even if each vertex of T_0 has degree at most three (we omit the somewhat tedious proof).

In the appendix, by modifying the reduction in the proof of Theorem 9.1, we show that it is also NP-complete to decide whether a given preference profile is single-peaked on a *regular* tree, i.e., a tree where all internal vertices have the same degree (Theorem C.1). This hardness result stands in contrast to the many easiness results of Section 8.

10. Conclusions

Without any restrictions on the structure of voters' preferences, winner determination under the Chamberlin–Courant rule is NP-hard. Positive results have been obtained when preferences are assumed to be single-peaked, and we studied whether these results can be extended to preferences that are single-peaked on a tree. For the egalitarian variant of the rule, we showed that this is indeed the case: a winning committee can be computed in polynomial time for any tree and any scoring function. For the utilitarian setting, we show that winner determination is hard for general preferences single-peaked on a tree, but we find positive results when imposing additional restrictions. One algorithm we present runs in polynomial time when preferences are single-peaked on a tree which has a constant number of leaves, and another runs efficiently on a tree with a small number of internal vertices. Interestingly, the two algorithms are, in some sense, incomparable. Specifically, the former algorithm works for all scoring functions, while for the latter algorithm this is not the case (though it does work for the most common scoring function, i.e., the Borda scoring function). On the other hand, the latter algorithm establishes that computing a winning committee is in FPT with respect to the combined parameter 'committee size, number of internal vertices', while the former algorithm does not establish fixed-parameter tractability with respect to the combined parameter 'committee size, number of leaves'. An open question is whether there exist FPT algorithms or W[1]-hardness results for the parameters 'number of internal vertices' and the combined parameter 'committee size, number of leaves'.

It would be interesting to see whether our easiness results for preferences that are single-peaked on a tree extend to the egalitarian version of the Monroe rule (Monroe, 1995), where each committee member has to represent approximately the same number of voters. Betzler et al. (2013) show that this rule becomes easy for preferences single-peaked on a path, but their argument for that rule is much more intricate than for egalitarian Chamberlin–Courant.

To make our parameterized winner determination algorithms applicable, we have investigated the problem of deciding whether the input profile is single-peaked on a 'nice' tree, for several notions of 'niceness'. To this end, we have proposed a new data structure, namely, the attachment digraph, which compactly encodes the set $\mathcal{T}(P)$ of all trees T such that the profile P is single-peaked on T, and showed how to use it to identify trees with desirable properties. In particular, we showed how to find a tree in $\mathcal{T}(P)$ with the minimum number of leaves or the minimum number of internal vertices, to be used with our winner determination algorithms. To demonstrate the power of our framework, we also designed efficient algorithms for several other notions of 'niceness', such as small diameter, small maximum degree and small pathwidth. However, there are also notions of 'niceness' that defy this approach: we show that it is NP-hard to decide whether an input profile is single-peaked on a regular tree. Another interesting measure of 'niceness' is vertex deletion distance to a path, i.e., the number of vertices that need to be deleted from a tree so that the remaining graph is a path. In particular, this parameter is relevant for the elicitation results of Dey and Misra (2016). Finding a tree in $\mathcal{T}(P)$ that minimizes this parameter is equivalent to finding a tree with the maximum diameter, which is closely related to the problem of finding a maximum-length path in the attachment digraph. However, we were not able to design a polynomial-time algorithm for this problem (or to show that it is computationally hard). Similarly, the complexity of finding a tree in $\mathcal{T}(P)$ that has the minimum path cover number (another parameter considered by Dey and Misra (2016)) is an open problem.

It would also be interesting to explore the parameterized complexity of the problems related to the identification of 'nice' trees. For instance, Single-Peaked Tree Labeling is trivially fixed-parameter tractable with respect to the number of candidates, as we could explore all possible labelings; however, its parameterized complexity with respect to the number of voters is an interesting open problem. In a similar vein, we can ask if we can find a tree in $\mathcal{T}(P)$ with approximately minimal vertex deletion distance to a path or approximately minimal path cover number. Indeed, constant-factor approximation algorithms for these problems can still be used in conjunction with the elicitation algorithms of Dey and Misra (2016) in order to improve over elicitation algorithms for unstructured preferences.

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A preliminary version of the results in this paper appeared in two conference papers (Yu et al., 2013; Peters & Elkind, 2016). In this paper, we unify the exposition of these papers, provide more intuition, and give pseudocode and detailed analysis of several algorithms that were only sketched in the conference versions. Yu et al. (2013) leave as an open question whether the Chamberlin–Courant rule on profiles single-peaked on a tree remains NP-hard for the Borda scoring function. We answer this question in Theorem 4.1.

We would like to thank the anonymous JAIR reviewers for their careful reading of the paper and many useful suggestions, which ranged from catching typos to simplifying some of the proofs.

Appendix A. Hardness of Utilitarian Chamberlin–Courant for Low-Degree Trees

Here we modify the reduction in the proof of Theorem 4.1 to establish that UTILITARIAN CC remains hard on trees of maximum degree 3.

Theorem A.1. *Given a profile P that is single-peaked on a tree with maximum degree 3, a target committee size k, and a target score B, it is* NP*-complete to decide whether there exists a committee of size k with score at least B under the utilitarian Chamberlin–Courant rule with the Borda scoring function.*

Proof. We will provide a reduction from the classic VERTEX COVER problem. Given an instance (G, t) of VERTEX COVER such that $G = (V, E)$, $V = \{u_1, \dots, u_p\}$ and $E = \{e_1, \dots, e_q\}$, we construct an instance of UTILITARIAN CC as follows.

Let $M = 5p^2q$; intuitively, M is a large number. We introduce three candidates a_i, y_i and z_i for each vertex $u_i \in V$, and M dummy candidates. Formally, we set $A = \{a_1, \dots, a_p\}$, $Y = \{y_1, \dots, y_p\}$, $Z = \{z_1, \dots, z_p\}$, $D = \{d_1, \dots, d_M\}$, and define the candidate set to be $C = A \cup Y \cup Z \cup D$. We set the target committee size to be $k = p + t$.

We now introduce the voters, who will come in three types: $N = N_1 \cup N_2 \cup N_3$.

N_1			N_2			N_3		
$5pq$	$\cdots$	$5pq$	1	$\cdots$	1	M	$\cdots$	M
y_1		y_p	A		A	z_1		z_p
z_1		z_p	$y_{j_{1,1}}$		$y_{j_{q,1}}$	y_1		y_p
A		A	$y_{j_{1,2}}$		$y_{j_{q,2}}$	A		A
D		D	D		D	D		D
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$

- The set N_1 consists of $5pq$ identical voters for each $u_i \in V$: they rank y_i first, z_i second, and a_i third, followed by other candidates:

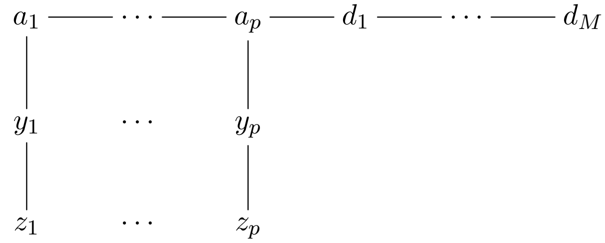
$$y_i \succ z_i \succ a_i \succ a_{i+1} \succ \cdots \succ a_p \succ a_{i-1} \succ \cdots \succ a_1 \succ d_1 \succ \cdots \succ d_M \succ \cdots$$

- The set N_2 consists of a single voter for each edge $e_j \in E$: this voter ranks candidates in A first (as $a_1 \succ \cdots \succ a_p$), followed by the two candidates from Y that correspond to the endpoints of e_j (in an arbitrary order), followed by the dummy candidates $d_1, \dots, d_M$, followed by all other candidates as specified below. The purpose of these voters is to ensure that every edge is covered by one of the vertices that correspond to a committee member, and to incur a heavy penalty of M if the edge is uncovered.

- The set N_3 is a set of M identical voters for each $u_i \in V$ who all rank z_i first and y_i second:

$$z_i \succ y_i \succ a_i \succ a_{i+1} \succ \cdots \succ a_p \succ a_{i-1} \succ \cdots \succ a_1 \succ d_1 \succ \cdots \succ d_M \succ \cdots$$

We complete the voters' preferences so that the resulting profile is single-peaked on the following tree:



This tree is obtained by starting with a path through A and D, and then attaching y_i as a leaf onto a_i and z_i as a leaf onto y_i for every $i = 1, \dots, p$. Note that the resulting tree has maximum degree 3. It remains to specify how to complete each vote in our profile to ensure that the resulting profile is single-peaked on this tree. Inspecting the tree, we see that it suffices to ensure that for each $i = 1, \dots, p$ it holds that in all votes where the positions of y_i and z_i are not given explicitly, candidate y_i is ranked above z_i.

We will again reason about costs rather than scores. We set the upper bound on cost to be $B = (5pq)(p - t) + q(p + 1)$ (note that by construction, $M > B$). This completes the description of our instance of the Utilitarian CC problem with the Borda scoring function $\mathbf{s} = (0, -1, -2, \dots)$. Intuitively, the 'correct committee' we have in mind consists of all z_i candidates (of which there are p) and a selection of y_i candidates that corresponds to a vertex cover (of which there should be t), should a vertex cover of size t exist. Now let us prove that the reduction is correct.

Suppose we have started with a 'yes'-instance of Vertex Cover, and let S be a collection of t vertices that forms a vertex cover of G. Consider the committee $W = Z \cup \{y_i : u_i \in S\}$. Note that $|W| = p + t = k$. The voters in N_3 and $5pqt$ voters in N_1 have their most-preferred candidate in W, so they contribute 0 to the cost of W. For the remaining $(5pq)(p - t)$ voters in N_1, their contribution to the cost of W is 1, since $z_i \in W$ for all i. Further, each voter in N_2 contributes at most $p + 1$ to the cost. Indeed, the candidates that correspond to the endpoints of the respective edge are ranked in positions $p + 1$ and $p + 2$ in this voter's ranking, and since S is a vertex cover for G, one of these candidates is in S. We conclude that $\text{cost}^+_\mu(P, W) \leqslant (5pq)(p - t) + q(p + 1) = B$.

Conversely, suppose there exists a committee W of size $k = p + t$ with $\text{cost}^+_\mu(P, W) \leqslant B$. Note first that W has to contain all candidates in Z: otherwise, there are M voters in N_3 with cost at least 1, and then the utilitarian Chamberlin–Courant cost of W is at least $M > B$, a contradiction. Thus $Z \subseteq W$. We will now argue that $W \setminus Z$ is a subset of Y, and that $S' = \{u_i : y_i \in W \setminus Z\}$ is a vertex cover of G. Suppose that $W \setminus Z$ contains too few candidates from Y, i.e., at most $t - 1$ candidates from Y. Then N_1 contains at least $(5pq)(p - (t - 1))$ voters who contribute at least 1 to the cost of W, so $\text{cost}^+_\mu(P, W) \geqslant (5pq)(p - t + 1) > (5pq)(p - t) + q(p + 1) = B$, a contradiction. Thus, we have $W \setminus Z \subseteq Y$. Now, suppose that S' is not a vertex cover for G. Let $e_j \in E$ be an edge that is not covered by S', and consider the voter in N_2 corresponding to e_j. Clearly, none of the candidates ranked in positions $1, \dots, p + 2 + M$ by this voter appear in W. Thus, this voter contributes more than M to the cost of W, so the total cost of W is more than $M > B$, a contradiction. Thus, a 'yes'-instance of Utilitarian CC corresponds to a 'yes'-instance of Vertex Cover. □

Appendix B. Hardness of Utilitarian Chamberlin–Courant for Stars

For the Borda scoring function, we have seen in Theorem 4.1 that Utilitarian CC is NP-complete for trees of diameter 4, but in Section 6 we have argued that this problem is easy for stars, i.e., trees of diameter 2. The algorithm that worked for stars uses specific properties of the Borda scoring function. In this section, we show that for some positional scoring functions Utilitarian CC remains hard even on stars.

Recall from Proposition 6.1 that a profile P is single-peaked on a star if and only if there is a candidate c such that for every $i \in N$, either $\text{top}(i) = c$ or $\text{second}(i) = c$.

Theorem B.1. Utilitarian CC *is* NP*-hard even for profiles that are single-peaked on a star. The hardness result holds for any family of positional scoring functions whose scoring vectors* $\mathbf{s}$ *satisfy* $s_1 = 0$, $s_2 = \cdots = s_\ell < 0$, $s_{\ell+1} < s_\ell$ *for some* $\ell \geqslant 5$.

Proof. We will reduce from the restricted version of Exact Cover by 3-Sets (X3C) to our problem. Recall that an instance of X3C is given by a ground set X and a collection $\mathcal{Y}$ of size-3 subsets of X; it is a 'yes'-instance if there is a subcollection $\mathcal{Y}' \subseteq \mathcal{Y}$ of size $|X|/3$ such that each element of X appears in exactly one set in $\mathcal{Y}'$. This problem is NP-hard. Moreover it remains NP-hard even if each element of X appears in at most three sets in $\mathcal{Y}$ (Gonzalez, 1985).

Fix a family of positional scoring functions μ that satisfy the condition in the statement of the theorem for some $\ell \geqslant 5$. For brevity, we write $s = s_2$ and $S = s_{\ell+1}$.

Given an instance $(X, \mathcal{Y})$ of X3C such that $X = \{x_1, \dots, x_p\}$, $p = 3p'$, $\mathcal{Y} = \{Y_1, \dots, Y_q\}$, and $|\{Y_j \in \mathcal{Y} : x_i \in Y_j\}| \leqslant 3$ for each $x_i \in X$, we construct an instance of our problem as follows. We set $Y = \{y_1, \dots, y_q\}$, create p sets of dummy candidates $D_1, \dots, D_p$ of size ℓ each, and let $C = \{a, z\} \cup X \cup Y \cup D_1 \cup \ldots \cup D_p$.

We now introduce the voters, who will come in three types $N = N_1 \cup N_2 \cup N_3$.

For each $Y_j \in \mathcal{Y}$ we construct $p+1$ voters who rank y_j first, a second and z third, followed by all other candidates in an arbitrary order; let N_1 denote the set of voters constructed in this way. Further, for each $x_i \in X$, we construct a voter who ranks x_i first, followed by a, followed by the candidates y_j such that $x_i \in Y_j$ (in an arbitrary order), followed by candidates in D_i, followed by all other candidates in an arbitrary order. Denote the resulting set of voters by N_2. Finally, let N_3 be a set of $(p+1)(q+1)$ voters who all rank z first and a second, followed by all other candidates in an arbitrary order. Set $B = s((p+1)(q-p')+p)$ and $k = p'+1$. This completes the description of our instance of the Utilitarian CC problem. Observe that every voter in N ranks a second, so by Proposition 6.1 the constructed profile in single-peaked on a star.

N_1			N_2			N_3
$p+1$	$\cdots$	$p+1$	1	$\cdots$	1	$(p+1)(q+1)$
y_1		y_q	x_1		x_p	z
a		a	a		a	a
z		z	$y_{j_{1,1}}$		$y_{j_{p,1}}$	$\vdots$
$\vdots$		$\vdots$	$y_{j_{1,2}}$		$y_{j_{p,2}}$	
			$y_{j_{1,3}}$		$y_{j_{p,3}}$	
			d_1		d_1	
			$\vdots$		$\vdots$	
			d_M		d_M	
			$\vdots$		$\vdots$	

Suppose we have started with a 'yes'-instance of X3C, and let $\mathcal{Y}'$ be a collection of p' subsets in $\mathcal{Y}$ that cover X. Set $W = \{z\} \cup \{y_j : Y_j \in \mathcal{Y}'\}$. Clearly, all voters in N_3 and $(p+1)p'$ voters in N_1 are perfectly represented by W and so contribute 0 to the score of W. The remaining voters in N_1 contribute $s = s_3$ to the score of W, since $z \in W$. Further, each voter in N_2 contributes at least $s = s_5$ to the score of W. Indeed, consider a voter who ranks some $x_i \in X$ first. Then he ranks the candidates y_j such that $x_i \in Y_j$ in positions 5 or higher. Since $\mathcal{Y}'$ is a cover of X, at least one of these candidates appears in W. We conclude that $\text{score}^+_\mu(P, W) \geqslant s((p+1)(q-p')+p) = B$.

Conversely, suppose there exists a committee W of size $p'+1$ such that $\text{score}^+_\mu(P, W) \geqslant B$. Note first that W has to contain z: otherwise, each voter in N_3 would contribute at most s to the score, and the total score of W would be at most $s|N_3| < B$. We will now argue that $W \setminus \{z\}$ is a subset of Y and that $\mathcal{Y}'' = \{Y_j : y_j \in W \setminus \{z\}\}$ is an exact cover of X. Suppose first that $W \setminus \{z\}$ contains at most $p'-1$ candidates from Y. Then N_1 contains at least $(p+1)(q-p'+1)$ voters who contribute at most s to the score of W, so $\text{score}^+_\mu(P, W) \leqslant s(p+1)(q-p'+1) < B$, a contradiction. Thus, we have $W \setminus \{z\} \subseteq Y$. Now, suppose that $\mathcal{Y}''$ is not an exact cover of X, let x_i be an element of X that is not covered by $\mathcal{Y}''$, and consider the voter in N_2 that ranks x_i first. Clearly, none of the candidates ranked in positions $1, \ldots, \ell$ by this voter appear in W. Thus, this voter contributes at most S to the score of W. All other voters in N_2 contribute at most s to the score of W. Further, there are $(p+1)(q-p')$ voters in N_1 who are not perfectly represented by W. We conclude that the total score of W is at most $s((p+1)(q-p')+(p-1)) + S < B$, a contradiction. Thus, a 'yes'-instance of Utilitarian CC corresponds to a 'yes'-instance of X3C. □

Appendix C. Hardness of Recognizing Preferences Single-Peaked on Regular Trees

Recall that a tree is k-regular if every non-leaf vertex has degree k.

Theorem C.1. *Given a profile P, it is* NP*-complete to decide whether P is single-peaked on a regular tree, i.e., whether there exists a positive integer k such that P is single-peaked on a k-regular tree. The problem is also hard for each fixed $k \geqslant 4$.*

Proof. The problem is in NP since for a given k-regular tree T we can easily check whether P is single-peaked on T.

We start by giving a hardness proof for fixed $k = 4$, and later explain how to modify the reduction for other fixed k and for non-fixed k.

Again, we reduce from X3C. Suppose that we are given an X3C-instance with ground set $\{x_1, \dots, x_p\}$, $p = 3p'$, and a collection of subsets $\mathcal{Y} = \{Y_1, \dots, Y_q\}$. We construct a profile over the following candidates. Let

$$X = \{x_1, \dots, x_p\}, \quad S = \{s_0, s_1, \dots, s_q, s_{q+1}\}, \quad L = \{\ell_1, \dots, \ell_q\}, \quad Y = \{y_1, \dots, y_q\}.$$

Then our candidate set is $X \cup S \cup L \cup Y$; that is, there is one candidate per element x_i, three candidates s_j, ℓ_j, y_j for each set Y_j, and two further candidates s_0 and s_{q+1}. The candidate y_j represents the set Y_j. We now introduce the voters; the reader may find it helpful to look at the tree in Figure 6 to understand the intuition behind the construction.

First, we force $(s_0, s_1, \dots, s_q, s_{q+1})$ to form a path. For this, we need to force that (s_j, s_{j+1}) is an edge, for each $j = 0, \dots, q$. To this end, for each $j = 0, \dots, q$ we create a voter who ranks the candidates as

$$s_j \succ s_{j+1} \succ s_{j+2} \succ \cdots \succ s_{q+1} \succ s_{j-1} \succ \cdots \succ s_1 \succ s_0 \succ Y \succ L \succ X.$$

We also force s_0 and s_{q+1} to be leaves. This requires introducing two more voters:

$$s_1 \succ \cdots \succ s_{q+1} \succ Y \succ L \succ X \succ s_0,$$

$$s_0 \succ \cdots \succ s_q \succ Y \succ L \succ X \succ s_{q+1}.$$

Further, we force each $\ell_j \in L$ and each $x_i \in X$ to be a leaf. That is, for each $j = 1, \dots, q$ we introduce a voter who ranks the candidates as

$$S \succ Y \succ L \setminus \{\ell_j\} \succ X \succ \ell_j,$$

and for each $i = 1, \dots, p$ we introduce a voter who ranks the candidates as

$$S \succ Y \succ L \succ X \setminus \{x_i\} \succ x_i.$$

Here (and below) we write S as shorthand for $s_0 \succ s_1 \succ \cdots \succ s_{q+1}$.

Now, for each $j = 1, \dots, q$ the vertices ℓ_j and y_j need to have an edge to s_j. Thus, for each $j = 1, \dots, q$ we introduce two voters who rank the candidates as

$$\ell_j \succ s_j \succ s_{j+1} \succ \cdots \succ s_{q+1} \succ s_{j-1} \succ \cdots \succ s_0 \succ Y \succ L \setminus \{\ell_j\} \succ X,$$

$$y_j \succ s_j \succ s_{j+1} \succ \cdots \succ s_{q+1} \succ s_{j-1} \succ \cdots \succ s_0 \succ Y \setminus \{y_j\} \succ L \succ X.$$

Finally, for each $i = 1, \dots, p$, we introduce a voter whose role is to ensure that x_i is attached to a vertex y_j such that $x_i \in Y_j$ (or to an element of S, but this will never happen):

$$S \succ \{y_j : x_i \in Y_j\} \succ x_i \succ \{y_j : x_i \notin Y_j\} \succ L \succ X \setminus \{x_i\}. \tag{2}$$

This concludes the description of the reduction. We will now prove that it is correct.

Suppose that a collection $\mathcal{Y}' = \{Y_{j_1}, \dots, Y_{j_{p'}}\}$ forms an exact cover of X. Then the constructed profile is single-peaked on the 4-regular tree on the candidate set with the following set of edges E (see Figure 6): $\{s_j, s_{j+1}\}$ for all $j = 0, \dots, q$; $\{\ell_j, s_j\}$ for all $j = 1, \dots, q$; $\{y_j, s_j\}$ for all $j = 1, \dots, q$; $\{x_i, y_k\}$ for all $x_i \in X$, where Y_k is the set in $\mathcal{Y}'$ that contains x_i. By considering top-initial segments of the votes given above, we see that this choice makes all votes single-peaked on this tree.

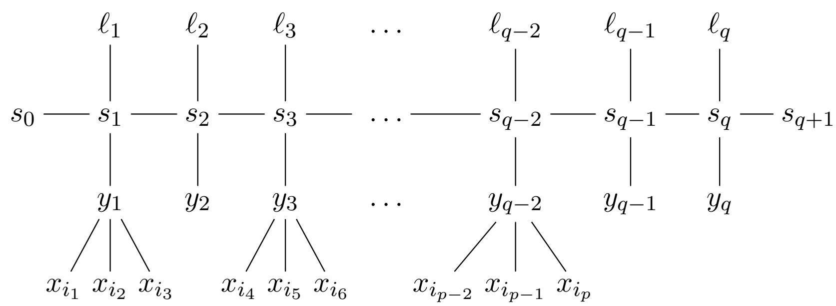


Figure 6: Tree T constructed in the proof of Theorem C.1.

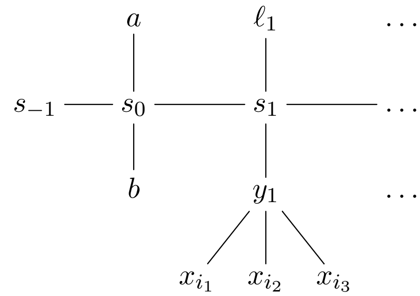
Conversely, suppose there is a 4-regular tree $T = (C, E)$ such that all votes are single-peaked on T. We show that in this case our instance of X3C admits an exact cover. When introducing the voters, we argued that for any such tree it must be the case that $\{s_j, s_{j+1}\} \in E$, for all $j = 0, \dots, q$ and $\{\ell_j, s_j\}, \{y_j, s_j\} \in E$ for all $j = 1, \dots, q$. Further, we know that each of the vertices in $L \cup X \cup \{s_0, s_{q+1}\}$ is a leaf of T. However, we do not yet know how the vertices in X are attached to the rest of the tree. As no vertex can be attached to a leaf, each vertex $x_i \in X$ is attached to some vertex in $Y \cup S \setminus \{s_0, s_{q+1}\}$. Now, each vertex s_j in $S \setminus \{s_0, s_{q+1}\}$ already has four neighbors in T (namely, $s_{j-1}, s_{j+1}, \ell_j, y_j$); thus, since T is 4-regular, x_i cannot be a neighbor of s_j. Thus x_i's neighbor is some $y_j \in Y$. Now, consider the voter whose preferences are given by (2); for this voter's preferences to be single-peaked on T, x_i must be attached to a vertex y_j that corresponds to a set $Y_j \ni x_i$. On the other hand, by 4-regularity, each y_j is connected to either 0 or 3 vertices in X. Hence this tree encodes an exact cover of X.

For other fixed values of $k \geqslant 5$, we can perform essentially the same reduction from the problem EXACT COVER BY $(k-1)$-SETS, which is also NP-hard;[4] the only modification we need to make is to use $k - 3$ copies of the set L.

If the value of k is not fixed, we can do the following (see picture): Prepend s_{-1} to the path S (where now s_{-1} is forced to be a leaf, but s_0 is not), and introduce new leaves a and b that must be attached to s_0. Modify the votes given by (2) in such a way that x_i can be attached to $s_1, \dots, s_q$ or an appropriate y_j, but not to s_0. Then any regular tree on which the profile is single-peaked will have to be 4-regular, due to s_0 having degree 4, and the

4. For example, one can reduce from X3C to X4C by taking an X3C instance with $3p$ elements, adding p dummy elements $d_1, \dots, d_p$, and replacing each 3-set Y by the p 4-sets $Y \cup \{d_1\}, \dots Y \cup \{d_p\}$.

argument above goes through.



□

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