Journal of Artificial Intelligence Research 75 (2022) 913-951 Submitted 03/2022; published 10/2022

Fair in the Eyes of Others

Parham Shams PARHAM.SHAMS@LIP6.FR
LIP6, Sorbonne Université, CNRS, F-75005 Paris, France

Aurélie Beynier AURELIE.BEYNIER@LIP6.FR
LIP6, Sorbonne Université, CNRS, F-75005 Paris, France

Sylvain Bouveret SYLVAIN.BOUVERET@IMAG.FR
LIG, Université Grenoble Alpes, CNRS, Grenoble INP, Grenoble, France

Nicolas Maudet NICOLAS.MAUDET@LIP6.FR
LIP6, Sorbonne Université, CNRS, F-75005 Paris, France

Abstract

Envy-freeness is a widely studied notion in resource allocation, capturing some aspects of fairness. The notion of envy being inherently subjective though, it might be the case that an agent envies another agent, but that from the other agents' point of view, she has no reason to do so. The difficulty here is to define the notion of objectivity, since no ground-truth can properly serve as a basis of this definition. A natural approach is to consider the judgement of the other agents as a proxy for objectivity. Building on previous work by Parijs (who introduced "unanimous envy") we propose the notion of approval envy: an agent a_i experiences approval envy towards a_j if she is envious of a_j, and sufficiently many agents agree that this should be the case, from their own perspectives. Another thoroughly studied notion in resource allocation is proportionality. The same variant can be studied, opening natural questions regarding the links between these two notions. We exhibit several properties of these notions. Computing the minimal threshold guaranteeing approval envy and approval non-proportionality clearly inherits well-known intractable results from envy-freeness and proportionality, but (i) we identify some tractable cases such as house allocation; and (ii) we provide a general method based on a mixed integer programming encoding of the problem, which proves to be efficient in practice. This allows us in particular to show experimentally that existence of such allocations, with a rather small threshold, is very often observed.

1. Introduction

Fair division is an ubiquitous problem in multiagent systems or economics (Steinhaus, 1948; Moulin, 2003; Young, 1994), with applications ranging from allocation of schools, courses or rooms to students (Abraham et al., 2005; Othman et al., 2010), to division of goods in inheritance or divorce settlement (Brams & Taylor, 1996). Envy-freeness (EF), is one of the prominent notions studied in fair division (Foley, 1967; Brams & Fishburn, 2002; Lipton et al., 2004; de Keijzer et al., 2009; Segal-Halevi & Suksompong, 2019). An allocation of items among a set of agents is said to be envy-free if no agent prefers the share of another agent to her own share. Unfortunately, in the indivisible setting, envy-freeness is a pretty demanding notion and an envy-free allocation may not exist. That explains why recent literature has proposed a lot of relaxations of envy-freeness, such as for instance envy-freeness up to one good, EF1 (Budish, 2011), or envy-freeness up to any good, EFX (Gourvès et al., 2014; Caragiannis et al., 2016). When agents interact along a social network, local notions of fairness have been investigated where an agent can only make comparisons with her neighbors in the network. In the divisible setting, local envy-freeness and local proportionality have been studied for instance by Abebe et al. (2017) and Bei et al. (2017). In the indivisible setting, similar notions have been studied by Aziz et al. (2018) and Chevaleyre et al. (2017), while complexity issues related to local envy-freness have been investigated in oriented graphs (Bredereck et al., 2018) and non-oriented graphs in house allocation problems (Beynier et al., 2019b).

We propose a slightly different relaxation of envy. To illustrate the notion we introduce, consider a given instance where no envy-free allocation exists. Now suppose that in this instance there exist two allocations π and π' that make a single agent (say, a_i) envious of some other agent a_j (for simplicity). Furthermore, assume that in allocation π, no agent but a_i thinks that a_j's bundle is better than a_i's, while in allocation π' all the other agents concur with the assessment that a_i envies a_j. According to Parijs (1997), π' exhibits *unanimous envy*, and it seems difficult to justify that π' should be returned in place of π. Inspired by this notion, we define in this paper the notion of K-approval envy, as a way to introduce a continuum between envy-freeness and unanimous envy. As may be clear from the name, the idea is simply to ask agents to express their own view about envy relations expressed by other agents. The objective will thus be to seek allocations minimizing social support for the expressed envy relations, *i.e.* minimizing the number of agents K approving the envy. Of course, this approach may be controversial: after all, the notion of preference is inherently subjective. Introducing this flavour of objectivity may lead to undesirable consequences. At the extreme, one may simply replace individual preferences by some unanimous "mean" profile, thus profoundly changing the very nature of the notion. We believe though that there are several justifications to this approach:

- First, note that we only seek the approval of other agents in the case the agent herself explicitly expresses envy: absence of envy thus remains completely subjective. While a symmetrical treatment may also be justifiable in some situations, there is an obvious reason which motivates us to start with the proposed definition, namely the fact that the notion would no longer be a relaxation of envy-freeness.

- Secondly, all other things being equal, we believe that an allocation minimizing K is socially more desirable. We do not necessarily regard this notion as a compelling choice, but we think this can enrich the picture of fallback allocations when no envy-free allocation exists, as other relaxations do (Amanatidis et al., 2018). In particular, in repeated settings, the fact that agents perceive outcomes as globally fair (not only for themselves, but also for others) may be important as an incentive for participation.

- Finally, one further motivation of our work is that our approach can be seen as providing guidance regarding agents and more specifically agents' preferences, in order to progress towards envy-freeness by helping them revise their utilities for example. In particular, if we envision systems integrating deliberative phases in the collective decision-making process, our model could be used to set the agenda of such deliberations. If a vast majority of agents contradict an agent on her envy towards another agent, it may indicate for instance that she lacks information regarding the actual value of (some items of) her share. Initiating a discussion might help to solve such "objectively unjustified" envies when they occur.

While envy-freeness is a widely studied notion in fair division of indivisible goods, another prominent notion in the literature is *proportionality*. This notion is based on the proportional share: the proportional share of an agent is equal to one of the n^{th} of the utility this agent gives to the whole set of objects (with n the number of agents). An allocation is proportional if and only if each agent receives at least her proportional share. Note that there are strong links between proportionality and envy-freeness, namely, any envy-free allocation is also proportional, whereas, on the contrary, there are instances for which a proportional allocation exists but no envy-free one (Bouveret & Lemaître, 2016). As is the case for envy-freeness, a proportional allocation is not guaranteed to exist. As a result, there has been a lot of work in recent literature about a relaxation of proportionality called proportionality up to one item (PROP1) (Conitzer et al., 2017). In the same spirit as EF1, PROP1 requires each agent to get her proportional share by obtaining the object of some other agent that she values the most. In light of these remarks and given the strong relationships between envy-freeness and proportionality, we also explore an approval version of this latter notion.

Outline of the paper. The remainder of this paper is as follows. Section 2 recalls some basic notions in fair division. Our notion of K-approval envy is presented in Section 3. Some properties of this notion are then studied in Section 4: it is shown in particular, that if the hypothetical situation of allocation π described at the beginning of the introduction occurs, then an EF allocation must also exist. We also show that our notion inherits from the complexity of related problems. After introducing the approval notion around proportionality in Section 5, some properties are put forward in Section 6 and some links between our two approval notions are studied in Section 7. As we did for approval envy, we show that the problem inherits from the hardness of the classical notion. This hardness results motivate the MIP formulations that we detail in Section 8. We next turn to the *house allocation* setting and we show that if each agent exactly holds a single item, then we can define an efficient algorithm returning an allocation minimizing the value of K for both our approval notions. One caveat of our notions is that (unlike other relaxations) it is not guaranteed to exist, as intuitively observed in the case of unanimous envy and unanimous non-proportionality. In Section 10, we provide empirical evidences showing that allocations with reasonable values of K exist under synthetic cultures as well as in real datasets.

2. Model and Definitions

We consider MultiAgent Resource Allocation problems (MARA) where we aim at fairly dividing a set of indivisible objects (also called items or goods) among a set of agents. A MARA instance I is defined as a finite set of *objects* $\mathcal{O} = \{o_1, \ldots, o_m\}$, a finite set of *agents* $\mathcal{N} = \{a_1, \ldots, a_n\}$ and a preference profile $\mathcal{P}$ representing the interest of each agent $a_i \in \mathcal{N}$ towards the objects. An allocation π is a mapping of the objects in $\mathcal{O}$ to the agents in $\mathcal{N}$. In the following, π_i will denote the set of indivisible objects (the share) held by agent a_i. An allocation is such that $\forall a_i, \forall a_j$ with $i \neq j : \pi_i \cap \pi_j = \emptyset$ (a given object cannot be allocated to more than one agent) and $\bigcup_{a_i \in \mathcal{N}} \pi_i = \mathcal{O}$ (all the objects from $\mathcal{O}$ are allocated).

In this paper, we consider cardinal preference profiles so, the preferences of an agent a_i over bundles of objects are defined by a *utility function* $u_i : 2^{\mathcal{O}} \rightarrow \mathbb{Q}^+$ measuring her satisfaction $u_i(\pi_i)$ when she obtains share π_i. We make the assumption that utility functions are additive *i.e.* the utility of an agent a_i over a share π_i is defined as the sum of the utilities over the objects forming π_i:

$$u_i(\pi_i) \stackrel{def}{=} \sum_{o_k \in \pi_i} u(i, k),$$

where $u(i, k)$ is the utility given by agent a_i to object o_k. This assumption is commonly considered in MARA (Lipton et al., 2004; Procaccia & Wang, 2014; Dickerson et al., 2014; Caragiannis et al., 2016, for instance) as additive utility functions provide a compact but yet expressive way to represent the preferences of the agents. MARA instances with additive utility functions are called add-MARA instances for short.

Different notions have been proposed in the literature to evaluate the fairness of an allocation. When the agents can compare their shares, the absence of envy (Foley, 1967; Lipton et al., 2004; Chevaleyre et al., 2017) is a particularly relevant notion of fairness. An agent a_i would envy another agent a_j if she prefers the share of a_j over her own share. More formally, an agent a_i envies an agent a_j iff

$$u_i(\pi_j) > u_i(\pi_i).$$

A completely fair allocation would thus be an *envy-free allocation i.e.* an allocation where no agent envies another agent. Formally:

$$\forall a_i, a_j \in \mathcal{N}, u_i(\pi_i) \geq u_i(\pi_j).$$

The notion of envy-freeness conveys a natural concept of fairness viewed as social stability: agents are happy with their bundle and hence do not want to swap it with any other agent's (regarding their own preferences). However, as soon as it is required to allocate all the objects in $\mathcal{O}$, an envy-free allocation may not exist. An alternative objective may be to minimize a degree of envy of the society (Lipton et al., 2004; Nguyen & Rothe, 2014; Chevaleyre et al., 2017), based on the notion of *pairwise envy*.

Definition 1 (Pairwise envy)**.** *Let π be an allocation. The pairwise envy $pe(i, j, \pi)$ of an agent a_i towards an agent a_j in π is defined as follows:*

$$pe(i, j, \pi) \stackrel{def}{=} \max\{0, u_i(\pi_j) - u_i(\pi_i)\}.$$

The pairwise envy can be interpreted as how much agent a_i envies agent a_j's share (this envy being 0 if a_i does not envy a_j). We can derive from this notion a collective measure of envy similar to the one used by Aleksandrov *et al.* (2019):

Definition 2 (Degree of envy of the society)**.** *The degree of envy $de(\pi)$ of the society for an allocation π is defined as follows:*

$$de(\pi) \stackrel{def}{=} \sum_{a_i \in \mathcal{N}} \sum_{a_j \in \mathcal{N}} pe(i, j, \pi).$$

Note that an allocation π is envy-free if and only if $de(\pi) = 0$.

To cope with the possible non-existence of an envy-free allocation, another approach is to alleviate the requirements of the fairness notion. Recently, several relaxations of envy-freeness have been proposed such as envy-freeness up to one good (EF1) (Budish, 2011) or envy-freeness up to any good (EFX) (Caragiannis et al., 2016). An allocation is said to be envy-free up to one good (resp. up to any good) if no agent a_i envies the share π_j of another agent a_j after removing from π_j *one* (resp. any) item. Existence for EF1 is guaranteed, but this is still to the best of our knowledge an open question for EFX in the general case. However, the existence guarantee of an EFX solutions has been proved for few agents (at most 3 agents) and specific utility functions. For instance it has been proved that an EFX allocation always exists for instances with identical valuations and for instances involving two agents with general and possibly distinct valuations (Plaut & Roughgarden, 2018), as well as for three agents with additive valuations (Chaudhury et al., 2020). When the objects have only two possible valuations, Amanatidis et al. (2020) proved that any allocation maximizing the Nash Social Welfare is EFX. This result provides a polynomial algorithm for computing EFX allocations in the two-agent setting. Amanitidis et al. (2018) studied four fairness notions – envy-freeness up to one good (EF1), envy-freeness up to any good (EFX), maximin share fairness (MMS), and pairwise maximin share fairness (PMMS) – and investigated the relations between these notions and their relaxations. Although PMMS is a stronger notion than EFX, Amanatidis et al. (2018) proved that both notions provide the same worst-case guarantee for MMS. In the same vein, they showed that EFX and EF1 both provide similar approximation for PMMS.

As discussed in Section 1, another widely studied fairness notion is proportionality:

Definition 3 (Proportional share)**.** *Let I be an add-MARA instance. The proportional share of an agent a_i is defined as follows:*

$$Prop_i \stackrel{def}{=} \frac{\sum_{j=0}^{m} u(i,j)}{n}.$$

An allocation π is said to be proportional if and only if every agent gets her proportional share:

$$\forall a_i \in \mathcal{N}, u_i(\pi_i) \geq Prop_i.$$

As mentioned in Section 1, even though proportionality is a less demanding fairness criterion than envy-freeness (Bouveret & Lemaître, 2016), the existence of a proportional allocation is not guaranteed. For that reason, relaxations of this notion such as proportionality up to one item (PROP1) has been proposed (Conitzer et al., 2017). An allocation satisfies PROP1 if every agent gets at least her proportional share when one item is added to her current bundle. For example, Conitzer et al. (2017) proved the existence of PROP1 allocations for a public decision setting where a decision has to be made on several public issues. Each issue has several possible alternatives and each agent has a utility for each alternative. The decision problem consists in choosing one alternative for each issue. Barman and Krishnamurthy (2019) presented a strongly polynomial-time algorithm to find PROP1 allocations for positive utilities. Note that some of these papers deal with fair division with chores (Brânzei & Sandomirskiy, 2019) or with mixed utilities (Aziz et al., 2019, 2020). It can be noticed that there is a similar kind of link between EF1 and PROP1 as the one that exists between EF and proportionality. Namely, any EF1 allocation is also PROP1 (which we could write EF1 $\implies$ PROP1 for short).

3. Approval Envy

The notion of envy being inherently subjective, it might be the case that an agent envies another agent, but that she has no reason to do so from the point of view of the other agents. The difficulty here is to define the notion of objectivity, since no ground-truth can properly serve as a basis of this definition. In her book, Guibet-Lafaye (2006) recalls the notion of *unanimous envy*, that was initially discussed by Parijs (1997), and that can be defined as follows: an agent a_i unanimously envies another agent a_j, if all the agents think that bundle π_j is strictly preferred to π_i. Here, unanimity is used as a proxy for objectivity.

As we can easily imagine, looking for allocations that are free of unanimous envy will be too weak to be interesting: as soon as one agent disagrees with the fact that a_i envies a_j, this potential envy will not be taken into account. Here, we propose an intermediate notion between envy-freeness and (unanimous envy)-freeness, based on the notion of approval: an agent a_k approves of agent a_i

envying agent a_j if a_k thinks that bundle π_i is strictly worse than π_j. This allows to define a notion of K-approval envy :

Definition 4 (K-approval envy)**.** *Let π be an allocation, a_i, a_j be two different agents, and $1 \leq K \leq n$ be an integer. We say that a_i* K-approval envies (K-app envies for short) a_j *if there is a subset $\mathcal{N}_K$ of K agents including a_i such that:*

$$\forall a_k \in \mathcal{N}_K, u_k(\pi_i) < u_k(\pi_j).$$

In other words, at least $K-1$ agents amongst $\mathcal{N} \setminus \{a_i\}$ agree with a_i on the fact that π_i is actually strictly worse than π_j.

Example 1. *Let us consider the following add-MARA instance with 3 agents and 6 objects:*

	o_1	o_2	o_3	o_4	o_5	o_6
a_1	0	[3]	3	1	3	[2]
a_2	[2]	0	7	[2]	[1]	0
a_3	0	3	[5]	0	1	3

Note that there is no envy-free allocation for this instance. In the squared allocation, a_1 is not envious, a_2 envies a_3 and a_3 envies a_1. Concerning the envy of a_2 towards a_3, a_1 disagrees with a_2 being envious of a_3 whereas agent a_3 agrees. Hence, agent a_2 2-app envies agent a_3. Concerning the envy of a_3 towards a_1, agent a_1 agrees with a_3 being envious of a_1 whereas agent a_2 does not. Hence, a_3 2-app envies a_1.

Note that in the definition, as soon as a_i does not envy a_j, then, a_i does not K-app envy a_j, no matter what the value of K is or how many agents think that π_i is actually worse than π_j. Doing so, we ensure that our approval notion is a relaxation of envy-freeness.

Let us start with an easy observation:

Observation 1. *Given an allocation π of an add-MARA instance, if a_i K-app envies a_j in π, then a_i $(K-1)$-app envies a_j in π.*

Moreover, if a_i n-app envies a_j, we will say that a_i *unanimously* envies a_j. Finally, we can observe that a_i 1-app envies a_j if and only if a_i envies a_j.

We can naturally derive from Definition 4 the counterpart of envy-freeness:

Definition 5 ((K-approval envy)-free allocation)**.** *An allocation π is said to be (K-app envy)-free if and only if a_i does not K-app envy a_j in π for all pairs of agents (a_i, a_j).*

Definition 6 ((K-approval envy)-free instance)**.** *An add-MARA instance I is said to be (K-app envy)-free if and only if it accepts a (K-app envy)-free allocation.*

Example 2. *Going back to Example 1, the squared allocation is (3-app envy)-free so the instance is (3-app envy)-free.*

A threshold of special interest is obviously $\lfloor n/2 \rfloor + 1$, since it requires a strict majority to approve the envy under inspection. A Strict Majority approval envy-free (SM-app-EF) allocation is a (K-app envy)-free allocation such that $K \leq \lfloor n/2 \rfloor + 1$, translating the fact that every time envy occurs, there is a strict majority of agents that do not agree with that envy.

Going further, it is important to notice that (K-app envy)-freeness is not guaranteed to exist. Indeed, for all number of agents n and all number of objects m, there exist instances for which no (K-app envy)-free allocation exists, no matter what K is. Suppose for instance that all the agents rank the same object (say o_1) first, and that for all a_i, $u(i,1) > \sum_{k=2}^{m} u(i,k)$. Then obviously, everyone agrees that all the agents envy the one that will receive o_1. Such instances will be called *unanimous envy instances*:

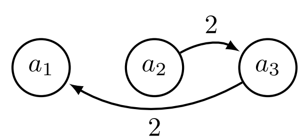
Definition 7 (Unanimous envy instance)**.** *An add-MARA instance I is said to exhibit unanimous envy if I is not (K-app envy)-free for any value of K.*

Observe that for an allocation to be (K-app envy)-free, for all pairs of agents (a_i, a_j), either a_i or at least $n - K + 1$ agents have to think that a_i does not envy a_j. Notice that it is different from requiring that at least K agents think that this allocation is envy-free. This explains the parenthesis around (K-app envy): this notion means "free of K-app envy", which is different from "K-app-(envy-free)".

A useful representation, for a given allocation, is the induced envy graph (Lipton et al., 2004): vertices are agents, and there is a directed edge from a_i to a_j if and only if a_i envies a_j. An allocation is envy-free if and only if the envy graph has no arc. In our context, we can define a weighted notion of the envy graph.

Definition 8 (**Weighted envy graph**)**.** *The* weighted envy graph *of an allocation π is defined as the weighted graph $(\mathcal{N}, E)$ where nodes are agents, such that there is an edge $(a_i, a_j) \in E$ if a_i envies a_j, with the weight $w(a_i, a_j)$ corresponding to the number of agents (including a_i) approving this pairwise envy in π.*

Example 3. *The induced weighted envy graph of Example 1 is as follows:*



Our notion of K-approval envy can be interpreted as a vote on envy, that works as follows. For each pair of agents (a_i, a_j), if a_i declares to envy a_j, we ask the rest of the agents to vote on whether they think that a_i indeed envies a_j. Then, a voting procedure is used to determine whether a_i envies a_j according to the society of agents. Several voting procedures can be used. However, since there are only two candidates (yes / no), the most reasonable voting rules are based on quotas: a_i envies a_j if and only if there is a minimum quota of agents that think so.[1] This makes a connection with a related work of Segal-Halevi and Suksompong (2019) which uses voting to decide upon envy-freeness, but in the context of fair division of resources *jointly* owned by *groups* of agents.

Finally, we want to emphasize that our notion of K-approval envy is based on *pairwise* envy. Namely, if agent a_i envies a_j, we will try to evaluate how many other agents think that this envy is justified. Another possibility[2] would be, for each envious agent a_i, to evaluate how many other

1. More precisely, these rules exactly characterize the set of anonymous and monotonic voting rules (Perry & Powers, 2010).
2. We warmly thank an anonymous reviewer for pointing out this alternative notion to us.

agents think that a_i has indeed reasons to be envious, no matter which agent a_i envies. The difference is subtle. To illustrate this, suppose that a_i envies a_j and another agent a_k disagrees with this particular envy, but thinks that a_i has indeed reasons to envy $a_l(\neq a_j)$. With the first notion (our notion), a_k's opinion will be discarded, whereas in the second one, it will be counted.

In practice, we believe that this alternative notion of approval envy will be much less discriminating than ours. The intuition can be explained as follows. Suppose that for some allocation π there is a bundle π_i that is not the top one for any agent. In that case, not only the agent a_i receiving π_i will envy another agent (since π_i is not a_i's top bundle), but every other agent will also agree that a_i should be envious. Hence, the envy of a_i in π will be unanimously approved under the alternative notion. Now suppose in the contrary that every bundle of π is the top one for some agent. If preferences are strict on bundles, then by the pigeon hole principle, the top bundle of each agent has to be a different bundle of π, meaning that there is an envy-free allocation in that case. Said otherwise, if preferences are strict on bundles, an instance can only either be envy-free or exhibit unanimous envy under the alternative definition. The only edge case happens when agents can have several tied top bundles, which does not happen very often in practice.

The experiments we ran on the Spliddit instances (see Section 10) tend to confirm this intuition. This is why we decided not to investigate further this alternative notion of approval envy.

4. Some Properties of Approval Envy

There are natural relations between the properties of (K-app envy)-freeness for different values of K. The following observation is a direct consequence of Observation 1.

Observation 2. *Let π be an allocation, and $K \leq N$ be an integer. If π is (K-app envy)-free, then π is also ((K+1)-app envy)-free.*

However, the converse does not hold. More precisely, the following proposition shows that the implication stated in Observation 2 is strict.

Proposition 1. *Let π be an allocation, and $3 \leq K \leq n$ be an integer. If π is (K-app envy)-free, π is not necessarily ((K−1)-app envy)-free.*

Proof. Let $h \in \{2, \dots, n-1\}$ be an integer, and let us consider the instance with n agents and n objects defined as follows:

- $u(1,1) = 1-(n-1)\varepsilon$;
- $u(i,1) = u(i,i) = \frac{1-(n-2)\varepsilon}{2}$ for $i \in \{2, \dots, h-1\}$;
- $u(i,i) = 1-(n-1)\varepsilon$ for $i \in \{h, n-1\}$;
- $u(n,1) = \frac{2}{n+1}$ and $u(n,j) = \frac{1}{n+1}$ for $j > 1$;

and $u(i,j) = \varepsilon$ for other pairs with $\varepsilon < \frac{1}{n+1}$.

This construction is illustrated in the general case in Figure 1. Moreover, one instance with $n = 4$ agents, $m = 4$ objects and $h = 3$ is shown in Example 4.

Consider the allocation π where each agent a_i gets item o_i. Obviously, the only envy in this allocation concerns a_n towards a_1. Moreover, only $a_1, \dots, a_{h-1}$ agree on this envy. Therefore, a_n h-app envies a_1, but does not (h+1)-app envy her. Moreover, π is ((h+1)-app envy)-free, but not (h-app envy)-free. □

	o_1	o_2	o_3	$\dots$	o_h	$\dots$	o_n
a_1	$\boxed{1-(n-1)\varepsilon}$	ε	ε	$\dots$	ε	$\dots$	ε
a_2	$\frac{1-(n-2)\varepsilon}{2}$	$\boxed{\frac{1-(n-2)\varepsilon}{2}}$	ε	$\dots$	ε	$\dots$	ε
a_3	$\frac{1-(n-2)\varepsilon}{2}$	ε	$\boxed{\frac{1-(n-2)\varepsilon}{2}}$	$\dots$	ε	$\dots$	ε
$\vdots$							
a_h	ε	ε	ε	$\dots$	$\boxed{1-(n-1)\varepsilon}$	$\dots$	ε
$\vdots$							
a_n	$\frac{2}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\dots$	$\frac{1}{n+1}$	$\dots$	$\boxed{\frac{1}{n+1}}$

Figure 1: Instance used in the proof of Proposition 1

Example 4. *In order to illustrate the previous proof, let us consider the following instance with 4 agents, 4 objects (and h=3) and the squared allocation π:*

	o_1	o_2	o_3	o_4
a_1	$\boxed{1-3\varepsilon}$	ε	ε	ε
a_2	$\frac{1}{2}-\varepsilon$	$\boxed{\frac{1}{2}-\varepsilon}$	ε	ε
a_3	ε	ε	$\boxed{1-3\varepsilon}$	ε
a_4	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\boxed{\frac{1}{5}}$

In this allocation, the only envy concerns a_4 towards a_1. Moreover, only a_1 and a_2 agree with a_4 on her envy. Hence, π is (4-app envy)-free but is obviously not (3-app envy)-free as we can find 3 agents (a_1, a_2 and a_4) agreeing on the envy of a_4 towards a_1 (in other words a_4 3-app envies a_1).

Proposition 2. *For any $K \geq 3$, there exist instances which are (K-app envy)-free but not (($K-1$)-app envy)-free.*

Proof. Consider the same instance as in Proposition 1. We have already shown that we have an allocation π that is (($h+1$)-app envy)-free which means that the instance is (($h+1$)-app envy)-free. We just have to show that there is no (h-app envy)-free allocation in order to conclude. For that purpose, we first note that each agent has to get one and exactly one object. Indeed, if it is not the case at least one agent a_i will have no object and will thus be envious of any agent a_j that has an object. Moreover, as all agents value the empty bundle with utility 0 and every object is valued with a strictly positive utility, this envy will be unanimous. Hence, each agent has to get one and exactly one object in order to minimize the (K-app envy)-freeness. Now consider objects o_j for $j \in \{h, n\}$. The agents a_j that receive an object o_j and that are envious will h-app envy the agent that received o_1. Indeed, agents a_i for $i \in \{h, n-1\}$ value objects o_j with a utility higher than (or equal to) the one of o_1 (and thus do not approve the envy) while it is the opposite for the other agents who are exactly h hence the h-app envy. So if we want to avoid that envy, we have to give the objects o_j to agents so that they do not experience envy at all but it is not possible as such agents are agents a_p for $p \in \{h, n-1\}$. It means that we have $n-1-h+1$ agents that have to receive the $n-h+1$

objects which is obviously impossible. This means that we cannot avoid h-app envy which implies that no allocation can be (h-app envy)-free. □

Proposition 2 proves that the hierarchy of (K-app) envy-free instances is strict for $K \geq 3$. Rather surprisingly, we will see that it is not the case for $K = 2$.

In order to show this result, we will resort to a tool that has been proved to be useful and powerful in many contexts dealing with envy (Biswas & Barman, 2018; Amanatidis et al., 2020; Beynier et al., 2019a): the *bundle reallocation cycle technique*. This technique, originating from the seminal work of Lipton *et al.* (2004), consists in performing a cyclic reallocation of *bundles* so that every agent is strictly better in the new allocation. Thus, such a reallocation corresponds to a cycle in the opposite direction of the edges in the — weighted — envy graph introduced in Definition 8. It is known that performing a reallocation cycle decreases the degree of envy (Lipton et al., 2004). Unfortunately, our first remark is that it does not necessarily decrease the level of K-app envy. Worse than that, it can actually increase it:

Proposition 3. *Let π be a (K-app envy)-free allocation, for $3 \leq K \leq n - 1$. After performing an improving bundle reallocation cycle (even between two agents), there can be an integer $K' > K$ such that the resulting allocation is (K'-app envy)-free (and not (K-app envy)-free).*

Proof. Let $h \in \{0, \dots, n - 4\}$ be an integer, and let us consider the instance with n agents and n objects defined by the following utility functions:

- a_1: $u(1, 1) = \varepsilon$, $u(1, 2) = 2\varepsilon$, $u(1, 3) = 1 - 3\varepsilon$;
- a_2: $u(2, 1) = 1 - \varepsilon$, $u(2, 2) = \varepsilon$;
- a_3: $u(3, 3) = 1$;
- a_l for $l \in \{4, h + 3\}$: $u(l, 1) = u(l, 3) = \varepsilon$, $u(l, j) = \frac{1-2\varepsilon}{n-3}$ for $j \geq 4$;
- a_m for $m \in \{h + 4, n\}$: $u(m, 2) = \varepsilon$, $u(m, 3) = 2\varepsilon$, $u(m, i) = \frac{1-3\epsilon}{n-3}$ for $i \geq 4$;

and $u(i, j) = 0$ for other pairs. We assume in this construction that $\varepsilon \leq \frac{1}{2n-1}$.

This construction is illustrated in Figure 2.

Consider the allocation π where each agent a_i gets item o_i (corresponding to the squared allocation in Figure 2). Obviously, in this allocation, there is no envy, except:

- a_1 envying a_2 (agents $a_{h+4} \dots a_n$ agree on that);
- a_1 envying a_3 (agents a_3 and $a_{h+4} \dots a_n$ agree on that);
- a_2 envying a_1 (agents $a_4 \dots a_{h+3}$ agree on that).

Hence the allocation is (($\max\{n - h, h + 2\}$)-app envy)-free. We now consider the allocation π' resulting from the improving bundle reallocation cycle between a_1 and a_2 (circled allocation in Figure 2). Observe that the only envy in π' is the one of a_1 towards a_3, which is approved by everyone except a_2. This allocation is thus (n-app envy)-free and not (($n - 1$)-app envy)-free. If $h > 0$, then $\max\{n - h, h + 2\}) < n$, which proves the proposition. □

	o_1	o_2	o_3	o_4	$\cdots$	o_{h+3}	o_{h+4}	$\cdots$	o_n
a_1	$\boxed{\varepsilon}$	$\enclose{circle}{2\varepsilon}$	$1-3\varepsilon$	0	$\cdots$	0	0	$\cdots$	0
a_2	$\enclose{circle}{1-\varepsilon}$	$\boxed{\varepsilon}$	0	0	$\cdots$	0	0	$\cdots$	0
a_3	0	0	$\boxed{1}$	0	$\cdots$	0	0	$\cdots$	0
a_4	ε	0	ε	$\boxed{\frac{1-2\varepsilon}{n-3}}$	$\cdots$	$\frac{1-2\varepsilon}{n-3}$	$\frac{1-2\varepsilon}{n-3}$	$\cdots$	$\frac{1-2\varepsilon}{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{h+3}	ε	0	ε	$\frac{1-2\varepsilon}{n-3}$	$\cdots$	$\boxed{\frac{1-2\varepsilon}{n-3}}$	$\frac{1-2\varepsilon}{n-3}$	$\cdots$	$\frac{1-2\varepsilon}{n-3}$
a_{h+4}	0	ε	2ε	$\frac{1-3\epsilon}{n-3}$	$\cdots$	$\frac{1-3\epsilon}{n-3}$	$\boxed{\frac{1-3\epsilon}{n-3}}$	$\cdots$	$\frac{1-3\epsilon}{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	0	ε	2ε	$\frac{1-3\epsilon}{n-3}$	$\cdots$	$\frac{1-3\epsilon}{n-3}$	$\frac{1-3\epsilon}{n-3}$	$\cdots$	$\boxed{\frac{1-3\epsilon}{n-3}}$

Figure 2: The instance used in the proof of Proposition 3.

Now consider a slight generalization of Lipton's cycles, *weakly improving cycles* (WIC), that correspond to a reallocation of bundles where all the agents in the cycle receive a bundle they like at least as much as the one they held, with one agent at least being strictly happier. Of course, our example of Proposition 3 still applies. On the other hand, this notion suffices to guarantee the decrease of the degree of envy (note that *identifying* the cycles themselves may not be easy any longer, but this is irrelevant for our purpose). The proof follows directly from the arguments of Lipton *et al.* (2004) (proof of Lemma 2).

Observation 3. *Let π be an allocation, and π' the allocation obtained after performing a* weakly improving cycle*. It holds that $de(\pi') < de(\pi)$.*

Proof. Let us consider an allocation π' obtained after performing a WIC on an allocation π. First note that the envies agents who are not involved in the WIC stay unchanged. By definition of a WIC, all the agents get at least as much in π' as they had in π. Thus basically $de(\pi') \leq de(\pi)$. Moreover, at least one agent gets a strictly better bundle so her envy strictly decreases. We finally get that $de(\pi') < de(\pi)$. □

We now show that (2-app envy)-freeness exhibits a special behaviour: in contrast with Proposition 3, improving cycles (in fact, even weakly improving cycles) enjoy the property of preserving the (2-app envy)-freeness level of an allocation. We provide this result for *swaps* (cycles involving two agents only) as this is sufficient to establish our main result.

Lemma 1. *Let π be a (2-app envy)-free allocation that is not EF. There always exists a WIC (that we can identify) between two agents such that the resulting allocation is (K'-app envy)-free, with $K' \leq 2$.*

Proof. Let a_i be an envious agent (there is at least one). We identify the agent that a_i envies the most and call her a_j (if there are several agents that a_i envies the most, we can pick randomly one of them). As a_i envies a_j and a_j necessarily does not agree on this envy because otherwise it would contradict (2-app envy)-freeness of π, swapping the bundle of a_i and a_j is a WIC. Let us call π' this new allocation. We will now show that π' is a (K'-app envy)-free allocation with $K' \leq 2$.

In π', all the agents except a_i and a_j have the same approval envy. Moreover, a_i is now EF in π' as she has received her preferred bundle. Suppose for contradiction that π' is (K'-app envy)-free with $K' > 2$. Then necessarily, this is due to a_j 2-app envying (at least) some other agent a_h (that can obviously not be a_i). For this to be the case, a_j has to envy a_h and another agent a_l has to approve this envy: (1) $u_j(\pi'_j) < u_j(\pi'_h)$, (2) $u_l(\pi'_j) < u_l(\pi'_h)$. However, as a_i envies a_j in π then (3) $u_i(\pi_i) < u_i(\pi_j)$ and as π is (2-app envy)-free and (3) holds, every agent a_l (except a_i of course) verifies (4) $u_l(\pi_i) \geq u_l(\pi_j)$.

Besides, π' is obtained after swapping the bundles of a_i and a_j in π so $\pi'_j = \pi_i$, $\pi'_i = \pi_j$ and $\pi'_h = \pi_h$; and from (2) we get: (5) $u_l(\pi_i) < u_l(\pi_h)$. By transitivity with (5) and (4), we obtain: (6) $u_l(\pi_j) < u_l(\pi_h)$. However, we know that a_j has the same utility in π and π' so $u_j(\pi'_j) = u_j(\pi_j)$. The latter combined with (1) (and the fact that $\pi'_h = \pi_h$) gives: (7) $u_j(\pi_j) < u_j(\pi_h)$. Finally, note that (6) and (7) translate the fact that a_j 2-app envies a_h in π which contradicts the fact that π is (2-app envy)-free. □

Putting Lemma 1 and Observation 3 together allows us to prove that (2-app envy)-freeness is essentially a vacuous notion, in the sense that any instance enjoying an allocation with this property will have an EF allocation as well.

Proposition 4. *If an add-MARA instance is (2-app envy)-free then it is also envy-free.*

Proof. Take π as being an arbitrary (2-app envy)-free allocation. First note that if there is no envious agent in π then, by definition, π is envy-free and the proposition holds. We perform a WIC leading to π' that is still (2-app envy)-free (see Lemma 1). If π' is envy-free then we are done. Otherwise, from Observation 3 we know the degree of envy has strictly decreased and that the resulting allocation is still (2-app envy)-free by Lemma 1. Hence we can repeat this process until the current allocation is EF. The process is guaranteed to stop because the degree of envy of the society is bounded below by zero and the degree of envy of the society strictly decreases at each step until it reaches zero (which corresponds to an envy-free allocation). □

Another consequence is that, for two agents, instances fall either in the envy-free or unanimous envy category:

Corollary 1. *Let I be an add-MARA instance with $n = 2$, if there is no envy-free allocation in I then I is a unanimous envy instance.*

Complexity We conclude this section with a few considerations on the computational complexity of the problems mentioned so far. First of all, as envy-freeness is (1-app envy)-freeness, the problem of finding the minimum K for which there exists a (K-app envy)-free allocation is at least as hard as determining whether an envy-free allocation exists.

One may also wonder how hard the problem is to determine whether a given instance exhibits unanimous envy or not, *i.e.* whether a (K-app envy)-free allocation exists for *some* value of K. For this question, instances where agents all have the same preferences provide insights.

Proposition 5. *For any add-MARA instance, if all the agents have the same preferences then the notions of (1-app envy)-freeness and (n-app envy)-freeness coincide.*

Proof. We already know from Observation 2 that (1-app envy)-freeness implies (n-app envy)-freeness for any add-MARA instance. So we just have to prove that if all the agents have the same preferences then (n-app envy)-freeness implies (1-app envy)-freeness. If an allocation π is (n-app envy)-free then it means that for any pair (a_i, a_j) of agents, a_i does not envy a_j or there is at least one agent a_h that disagrees on the envy of a_i towards a_j. Obviously, if for every pair of agents (a_i, a_j) we have a_i envy-free towards a_j then the allocation π is envy-free and the proof concludes. Besides, for every pair of envious/envied agents there is at least one agent disagreeing on the envy. But all the agents have the same preferences so it means that every agent should agree with each other. Hence, no envied agent can exist and we have (1-app envy)-freeness of allocation π. □

From Proposition 5 we get that the problem of deciding the existence of unanimous envy is at least as hard as deciding the existence of an EF allocation when agents have similar preferences which is known to be NP-hard (Lipton et al., 2004). As membership in NP is direct, we thus get as a corollary that:

Corollary 2. *Deciding whether an allocation exhibits unanimous envy is NP-Complete.*

5. Approval Non-Proportionality

As there is a clear hierarchy in the notions of fairness deriving from envy-freeness, it can be natural to consider how the different notions of this hierarchy would behave in an approval setting as we studied in Sections 3 and 4. Indeed, it has been shown (Bouveret & Lemaître, 2016) that envy-freeness implies proportionality. Moreover, some relaxations of proportionality have been studied such as PROP1 in very recent works (Aziz et al., 2020; Barman & Krishnamurthy, 2019; Brânzei & Sandomirskiy, 2019; Conitzer et al., 2017). This motivates us to investigate how we can derive an approval notion of proportionality.

In this section, we will introduce the approval version of proportionality. Observe first that our approval version of envy-freeness was based on a pairwise notion that we do not have in proportionality. This is why we slightly adapt the approval notion to this property.

Definition 9 (K-approval non-proportionality)**.** *Let π be an allocation, a_i be an agent, and $1 \leq K \leq n$ be an integer. We say that π_i* is K-approval non-proportional (K-app non-prop for short) *in π if there is a subset $\mathcal{N}_K$ of K agents including a_i such that:*

$$\forall a_k \in \mathcal{N}_K, u_k(\pi_i) < Prop_k.$$

In other words, at least $K-1$ agents amongst $\mathcal{N} \setminus \{a_i\}$ agree with a_i on the fact that she does not have her proportional share. We emphasize that we chose to focus on non-proportionality rather than on proportionality, to be consistent with our definition of K-app envy. The other related notions are defined accordingly as follows.

Definition 10 ((K-approval non-proportional)-free allocation)**.** *An allocation π is said to be (K-app non-proportional)-free if and only if no π_i is K-app non-proportional.*

Once again, observe that the interpretation of this property is that an allocation is free of K-app non-prop: each agent a_i either thinks she receives a proportional share, or, if it is not the case, no more than $K-2$ agents agree with a_i.

Definition 11 ((K-approval non-proportional)-free instance)**.** *An add-MARA instance I is said to be (K-approval non-proportional)-free if and only if it accepts a (K-approval non-proportional)-free allocation.*

Definition 12 (Unanimous non-proportional allocation)**.** *An add-MARA allocation π is said to exhibit unanimous non-proportionality if π is not (K-approval non-proportional)-free for any value of K.*

Definition 13 (Unanimous non-proportional instance)**.** *An add-MARA instance I is said to exhibit unanimous non-proportionality if I is not (K-approval non-proportional)-free for any value of K.*

Example 5. *Let us consider the add-MARA instance introduced in Example 1:*

	o_1	o_2	o_3	o_4	o_5	o_6
a_1	0	3^*	3	$\boxed{1}$	3	$\boxed{2}^*$
a_2	$\boxed{2}^*$	$\boxed{0}$	7	2^*	$\boxed{1}^*$	0
a_3	0	3	$\boxed{5}^*$	0	1	3

It is easy to notice that the proportional share of all the agents is the same and is worth 4. In the squared allocation, π_1 is not proportional as she values her bundle 3. Moreover a_2 and a_3 agree on the non-proportionality of her bundle. Hence π_1 is unanimous non-proportional. Besides π_2 is not proportional either. However neither a_1 nor a_3 agree on this non-proportionality as they value a_2's bundle with respective utilities 6 and 4. So π_2 is 1-app non-prop. Finally, π_3 is proportional. Consequently, we can say that the squared allocation is unanimous non-proportional because of a_1. However, the instance itself is not unanimous non-proportional as we can easily notice that the star allocation is proportional and hence (1-app non-prop)-free.

6. Some Properties of Approval Non-Proportionality

In this section, we will present some properties about the notion of (K-app non-prop)-freeness for different values of K. We will also present some complexity results.

We start with an easy observation which is the counterpart for (K-app non-prop)-freeness of Observation 2:

Observation 4. *Given an allocation π of an add-MARA instance, if π_i is K-app non-proportional in π, then π_i is $(K{-}1)$-app non-proportional in π.*

The following observation is a direct consequence of Observation 4.

Observation 5. *Let π be an allocation, and $K \leq N$ be an integer. If π is (K-app non-prop)-free, then π is also ($(K{+}1)$-app non-prop)-free.*

However, the converse does not hold. More precisely, the following proposition shows that the implication stated in Observation 5 is strict.

Proposition 6. *Let π be an allocation, and $3 \leq K \leq n$ be an integer. If π is (K-app non-prop)-free, π is not necessarily ($(K{-}1)$-app non-prop)-free.*

Proof. Let us consider the following instance with 3 agents and 3 objects and the squared allocation π. Recall that $Prop_i$ denotes the proportional share of a_i as stated in Definition 3:

	o_1	o_2	o_3
a_1	$\boxed{Prop_1 + 1}$	$Prop_1 + 1$	$Prop_1 - 2$
a_2	$Prop_2 - 1$	$\boxed{Prop_2 - 1}$	$Prop_2 + 2$
a_3	$Prop_3 - 1$	$Prop_3 - 1$	$\boxed{Prop_3 + 2}$

In this allocation, the only agent that does not hold her proportional share is a_2. Moreover, we can easily see that a_3 agrees with this non-proportionality whereas a_1 does not. So a_2 experiences 2-app non-prop and thus π is a (3-app non-prop)-free allocation but not (2-app non-prop)-free.

□

Proposition 7. *For any $K \geq 3$, there exists instances which are (K-app non-prop)-free but not ((K−1)-app non-prop)-free.*

Proof. Consider the same instance as in Proposition 6. We have already shown that we have an allocation π that is (3-app non-prop)-free which means that the instance is (3-app non-prop)-free. We just have to show that there is no (2-app non-prop)-free allocation in order to conclude. For that purpose, we first note that each agent has to get one and exactly one object. Indeed, if it is not the case at least one agent a_i will have no object and will thus not obtain her proportional share. Moreover, as all agents value the empty bundle with utility 0 this non-proportionality will be unanimous. Hence, each agent has to get one and exactly one object in order to minimize the (K-app non-prop)-freeness. Moreover, as a_2 and a_3 have the same preferences and only o_3 fulfils their proportional share then there is obviously no proportional allocation. Finally, this means that one of them will get either o_1 or o_2, and the non-proportionality of their bundles will be approved by the other, leading to a 2-app non-prop. Thus there is no (2-app non-prop)-free allocation and so the instance is (3-app non-prop)-free and not (2-app non-prop)-free. □

Proposition 7 proves that the hierarchy of K-app non-prop instances is strict for $K \geq 3$. As it was the case for the approval notion derived from envy-freeness we will see that it is not the case for $K = 2$ by show that (2-app non-prop)-freeness exhibits a special behaviour. For that, we start with a simple result.

Lemma 2. *Let π be an allocation. For each agent a_i, there is at least one bundle π_j such that $u_i(\pi_j) \geq Prop_i$.*

Proof. Let us consider for the sake of contradiction that there exists one allocation π in which an agent a_i cannot find any bundle that fulfils her proportional share. This means that every bundle is valued strictly less than $Prop_i = \frac{\sum_{j=0}^{m} u(i,j)}{n}$. By adding all the bundles (there are by definition n bundles in any allocation) we get that a_i values all the bundles strictly less than $n \times Prop_i = n \times \frac{\sum_{j=0}^{m} u(i,j)}{n} = \sum_{j=0}^{m} u(i,j)$ which is an obvious contradiction. □

We now establish a result similar to Lemma 1:

Lemma 3. *Let π be a (2-app non-prop)-free allocation that is not proportional. There always exists a bundle exchange between two agents (swap), that is not necessarily improving, such that the resulting allocation is (K'-app non-prop)-free (with $K' \leq 2$) and such that the number of agents with a non-proportional bundle has strictly decreased.*

Proof. Let π be a (2-app non-prop)-free allocation that is not proportional. Let a_i be an agent whose π_i is non-proportional in π (there is at least one). According to Lemma 2, there is (at least) one share π_j such that $u_i(\pi_j) \geq Prop_i$. Let π' be the allocation resulting from swapping a_i's and a_j's bundles in π. In π', all the agents except a_i and a_j have bundles with the same approval non-proportionality. Moreover, π'_i is now proportional in π' by definition of the swap we chose. Finally, π'_j is also proportional: suppose for contradiction that it is not the case. Then it would mean that $u_j(\pi'_j) < Prop_j$, which in turns implies $u_j(\pi_i) < Prop_j$. In other words, in π, π_i was not proportional and a_j agreed, which contradicts the fact that π was (2-app non-prop)-free. Hence, π'_j is proportional, and as a result, π' is still (2-app non-prop)-free, and the number of agents with a non-proportional bundle has increased by at least 1 (a_i is the new agent with a proportional bundle). □

Putting together Lemma 2 and Lemma 3 allows us to prove that (2-app non-prop)-freeness is essentially a vacuous notion, in the same sense as it is for (2-app envy)-freeness (Proposition 4):

Proposition 8. *If an add-MARA instance is (2-app non-prop)-free then it is also proportional.*

Proof. Let π be an arbitrary (2-app non-prop)-free allocation. First note that if all the agents have proportional bundles in π then, by definition, π is proportional and the proposition holds. Otherwise, we perform a swap leading to π' that is still (2-app non-prop)-free (see Lemma 3). If π' is proportional then we are done. Otherwise, thanks to the second part of Lemma 3 we know the number of agents with a non-proportional bundle has strictly decreased. We can repeat this process until the current allocation is proportional. The process is guaranteed to stop because the number of agents with a non-proportional bundle is bounded below by zero and decreases at each step until it equals zero (which corresponds to a proportional allocation). □

Another consequence is that, for two agents, instances are either proportional or unanimous non-proportional:

Corollary 3. *Let I be an add-MARA instance with $n = 2$, if there is no proportional allocation in I then I is an unanimous non-proportional instance.*

Proof. For any add-MARA instance involving exactly 2 agents, we can (by definition) only find (1-app non-prop)-free allocations or (2-app non-prop)-free allocations (as $1 \leq K \leq n$ for any add-MARA instance). By the contraposition of Proposition 8 we conclude the proof. □

We also note that, as it was the case for K-app envy, performing a reallocation cycle can increase the level of K-app non-prop:

Proposition 9. *Let π be a (K-app non-prop)-free allocation, for $3 \leq K \leq n-1$. After performing an improving bundle reallocation cycle (even between two agents), the resulting allocation may be (K'-app non-prop)-free (and not (K-app non-prop)-free) such that $K' > K$.*

Proof. Let us consider the following instance with 3 agents and 3 objects:

	o_1	o_2	o_3
a_1	$\boxed{Prop_1 - 1}$	$\underline{Prop_1 - 2}$	$Prop_1 + 3$
a_2	$\underline{Prop_2}$	$\boxed{Prop_2 + 3}$	$Prop_2 - 3$
a_3	$Prop_3 - 1$	$Prop_3$	$\boxed{\underline{Prop_3 + 1}}$

First consider the squared allocation that is (2-app non-prop)-free as only a_1 does not hold her proportional share and that it is not approved by any other agent. Let us now consider the underlined allocation π that is the result of the improving bundle reallocation between a_1 and a_2. We can see that only a_1 does not hold her proportional share and that this time a_3 approves it leading to a 2-app non-prop and thus a (3-app non-prop)-free allocation. □

Complexity We conclude this section with a few considerations on the computational complexity of the problems mentioned so far around the approval notion of proportionality. First of all, as proportionality is equivalent to (1-app non-prop)-freeness, the problem of finding the minimum K for which there exists a (K-app non-prop)-free allocation is at least as hard as determining whether a proportional allocation exists which is known to be NP-complete.

One may also wonder how hard the problem of determining whether a given instance exhibits unanimous non proportionality or not is, *i.e.* whether a (K-app non-prop)-free allocation exists for *some* value of K. For this question, as in Proposition 5, instances where agents all have the same preferences provide the answer.

Proposition 10. *For any add-MARA instance, if all the agents have the same preferences then the notions of (*1*-app non-prop)-freeness and (*n*-app non-prop)-freeness coincide.*

Proof. We already know from Observation 5 that (1-app non-prop)-freeness implies (n-app non-prop)-freeness for any add-MARA instance. So we just have to prove that if all the agents have the same preferences then (n-app non-prop)-freeness implies (1-app non-prop)-freeness. Let π be an (n-app non-prop)-free allocation. Then for any agent a_i, either $u_i(\pi_i) \geq Prop_i$, or there exists an agent a_j such that $u_j(\pi_i) \geq Prop_j$. Since all the agents have identical preferences, the last inequality reduces to $u_i(\pi_i) \geq Prop_i$, showing that a_i receives her proportional share. Hence in this case, π is proportional. □

From Proposition 10 we get that the problem of deciding the existence of a unanimous non-proportional allocation is at least as hard as deciding the existence of a proportional allocation when agents have similar preferences which is known to be NP-hard (see for instance Bouveret and Lemaître (2016)). As membership in NP is direct, we thus get as a corollary that:

Corollary 4. *Deciding whether an allocation exhibits unanimous non-proportionality is NP-Complete.*

7. Link Between Approval Envy-Freeness and Approval Non-Proportionality

After having introduced some properties of approval non-proportionality, we will now investigate the relationships between this notion and approval envy-freeness introduced earlier.

We first recall that envy-freeness implies proportionality and that this implication is still valid for EF1 and PROP1. It is thus natural to wonder whether it is also the case for our approval notions. As we will see, the answer is negative.

Proposition 11. *A unanimous envy instance can be proportional.*

Proof. Let us consider the following generic add-MARA instance (here, $\varepsilon \leq 1/n$) :

	o_1	o_2	$\dots$	o_{n-1}	o_n
a_1	$\boxed{1/n}$	$1/n-\varepsilon$	$\dots$	$1/n-\varepsilon$	$1/n+(n-2)\varepsilon$
a_2	$1/n-\varepsilon$	$\boxed{1/n}$	$\dots$	$1/n-\varepsilon$	$1/n+(n-2)\varepsilon$
$\vdots$					
a_{n-1}	$1/n-\varepsilon$	$1/n-\varepsilon$	$\dots$	$\boxed{1/n}$	$1/n+(n-2)\varepsilon$
a_n	$1/n-\varepsilon$	$1/n-\varepsilon$	$\dots$	$1/n$	$\boxed{1/n+(n-2)\varepsilon}$

In this instance, the squared allocation is proportional (and so (1-app non-prop)-free) whereas it is easy to see that the instance is a unanimous envy one as o_n is the top object of every agent. Hence the agent that gets o_n will be envied and this envy will be approved by everyone. □

From this result, we can generalize the statement to any level of K-app envy and any level of (L-app non-prop)-freeness. First of all, from Observation 5, it is clear that the counter-example of Proposition 11 establishes that a unanimous envy instance can be (L-app non-prop)-free, for any $L \geq 1$. But note also that if (counterfactually) it was the case that proportionality (or indeed any level of (L-app non-prop)-freeness) would imply some level of (K-app)-envy freeness, then by invoking Observation 2 this would also imply (unanimous envy)-freeness, a contradiction with Proposition 11. Putting all these remarks together allows us to state the following result.

Corollary 5. *For any $K \geq 1$ and any $L \geq 1$, an allocation exhibiting K-app envy can be (L-app non-prop)-free.*

Since proportionality is a weaker notion than envy-freeness, the previous result may not come as a surprise. It seems much more likely to obtain a positive result in the other direction, that is, that some level of (K-app envy)-freeness actually implies some level of (L-app non-prop)-freeness. It turns out that this is not the case.

Proposition 12. *An instance that exhibits unanimous non-proportionality can be (3-app envy)-free.*

Proof. Let us consider the following add-MARA instance for which $Prop_i = \frac{1}{n}$ for all i:

	o_1	o_2	o_3	$\dots$	o_{n-1}	o_n
a_1	$\boxed{\varepsilon}$	ε	ε	$\dots$	ε	$1-(n-1)\varepsilon$
a_2	ε	ε	ε	$\dots$	ε	$\boxed{1-(n-1)\varepsilon}$
a_3	ε	ε	$\boxed{1-(n-1)\varepsilon}$	$\dots$	ε	ε
$\vdots$						
a_n	ε	ε	ε	$\dots$	$\boxed{1-(n-1)\varepsilon}$	ε

It is obvious that in any allocation the agent that gets o_1 will not get her proportional share and that this non-proportionality will be approved by everyone. However, the squared allocation is (3-app envy)-free since the only envy in this allocation is a_1's towards a_2, and only a_2 approves this envy. □

Again, this allows us to state a more general result. First of all, it is direct from Observation 2 that the counter-example of Proposition 12 establishes that an unanimous non-proportional instance can be (K-app envy)-free, for any $K \geq 3$. But note also that if (counterfactually) it was the case that (3-app envy)-freeness (or indeed any level $L \geq 3$ of (L-app envy)-freeness) would imply some level of (K-app)-non-prop freeness, then by invoking Observation 5 this would also imply (unanimous non-prop)-freeness, a contradiction with Proposition 12. Putting all these remarks together allows us to state the following result.

Corollary 6. *For any $K \geq 3$ and any $L \geq 1$, a (K-app envy)-free instance can exhibit (L-app) non-proportionality.*

Note that this is the best we can do, since by Observation 2, Observation 5 and the well-known implication between envy-freeness and proportionality, we have an implication from (2-app envy)-freeness and any level of (L-app non-prop)-freeness.

Now in principle, and even if counter-intuitive at first sight, it could still be that exhibiting unanimous envy could imply proportionality; or that exhibiting unanimous non-proportionality could imply (3-app)-envy-free. The following result shows that both implications do not hold.

Proposition 13. *An instance can exhibit at the same time unanimous envy and unanimous non-proportionality.*

Proof. Let us consider the following instance with n agents and commensurable utilities ($Prop_i = \frac{1}{n}$ for all i and we assume that $\varepsilon < \frac{1}{n}$) :

	o_1	o_2	$\dots$	o_n
a_1	ε	ε	$\dots$	$1-(n-1)\varepsilon$
a_2	ε	ε	$\dots$	$1-(n-1)\varepsilon$
$\vdots$				
a_n	ε	ε	$\dots$	$1-(n-1)\varepsilon$

It is obvious to see that any agent getting an object different from o_n (say w.l.o.g o_1) will not be proportional and will envy the agent receiving o_n. Moreover, since all the agents have the same preferences, they will all agree with this non-proportionality and envy. □

We have summed up the relations between approval envy notions and approval non-proportionality ones in Figure 3.

8. Computation

We have seen at the end of Section 4 (respectively Section 6) that the problem of determining, for a given instance I, the minimal value of K such that a (K-app envy)-free (respectively a (K-app non-prop)-free) allocation exists inherited from the high complexity of determining whether an envy-free (respectively a proportional) allocation exists.

To address this problem, we present in this section two Mixed Integer Linear Programs that return, for a given add-MARA instance I, a (K-app envy)-free (respectively (K-app non-prop)-free) allocation with the minimal K and no solution when I is an unanimous envy (respectively

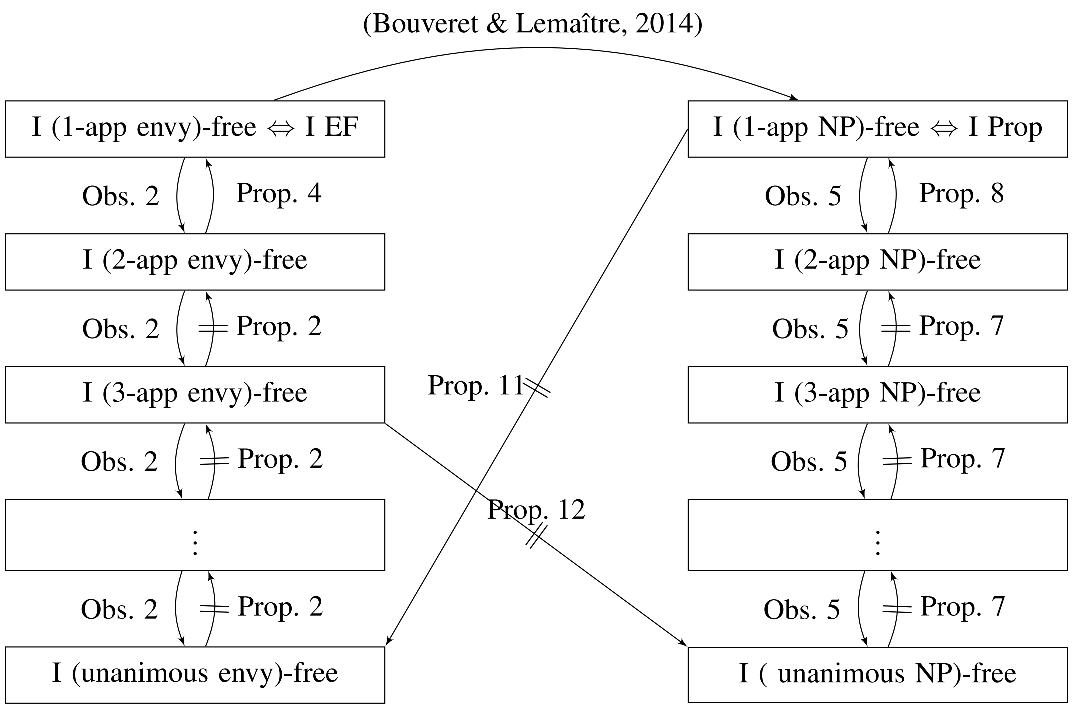


Figure 3: Hierarchy among instance properties. I stipulates that the relationships are drawn among instances. A simple edge denotes an implication relation. A striked out edge has been drawn when we have found a counter-example showing that this implication is not valid. Edges obtained by transitivity are not shown. All the remaining missing arcs are non-implication edges which can be obtained thanks to Corollaries 5 and 6.

non-prop) instance. We will first introduce and thoroughly explain the MIP for K-app envy. Then, we will show how to adapt it to K-app non-prop.

In this section, we assume that all the utilities are integers. If they are not (recall that they are assumed to be in $\mathbb{Q}^+$) we can transform the instance at stake into a new one only involving integer utilities by multiplying them by the least common multiple of their denominators.

8.1 A MILP Formulation for K-Approval Envy

In this MILP, we use $n \times m$ Boolean variables $\mathbf{z_i^j}$ (we use bold letters to denote variables) to encode an allocation: $\mathbf{z_i^j} = 1$ if and only if a_i gets item o_j. We also introduce n^3 Boolean variables $\mathbf{e_{kih}}$ such that $\mathbf{e_{kih}} = 1$ if and only if $u_k(\pi_i) < u_k(\pi_h)$. We also need to add n^2 Boolean variables $\mathbf{x_{ih}}$ used to linearize the constraints on $\mathbf{e_{kih}}$. Finally, we use an integer variable $\mathbf{K}$ corresponding to the K-app envy we seek to minimize.

We first need to write the constraints preventing an item from being allocated to several agents:

$$\sum_{i=1}^{n} \mathbf{z_i^j} = 1 \quad \forall j \in [\![1, m]\!] \tag{1}$$

By adding these constraints we also guarantee completeness of the returned allocation (all the items have to be allocated to an agent).

Secondly, we have to write the constraints that link the variables $\mathbf{e_{kih}}$ with the allocation variables $\mathbf{z_i^j}$:

$$\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) > 0 \iff \mathbf{e_{kih}} = 1 \quad \forall k,i,h \in [\![1, n]\!]$$

As the utilities are integers, we can replace > 0 by ≥ 1. In order to linearize the equivalence between the two constraints we introduce a number M that can be arbitrarily chosen such that $M > max_{a_k \in \mathcal{N}} \sum_{j=1}^{m} u(k,j)$:

$$M\mathbf{e_{kih}} \geq \sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) \quad \forall k,i,h \in [\![1, n]\!] \tag{2}$$

$$\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) \geq 1 - M(1 - \mathbf{e_{kih}}) \quad \forall k,i,h \in [\![1, n]\!] \tag{3}$$

Finally, we have to write the constraints that convey the fact that the allocation we look for is (K-app envy)-free:

$$\mathbf{e_{iih}} = 0 \vee \sum_{k=1}^{n} \mathbf{e_{kih}} \leq \mathbf{K} - 1 \quad \forall i,h \in [\![1, n]\!]$$

Since $\mathbf{e_{iih}}$ are Boolean variables, we can replace $\mathbf{e_{iih}} = 0$ by $\mathbf{e_{iih}} \leq 0$. Now, these logical constraints are linearized as follows:

$$\mathbf{e_{iih}} \leq \mathbf{x_{ih}} \quad \forall i, h \in [\![1, n]\!] \tag{4}$$

$$\sum_{k=1}^{n} \mathbf{e_{kih}} \leq \mathbf{K} - 1 + n(1 - \mathbf{x_{ih}}) \quad \forall i, h \in [\![1, n]\!] \tag{5}$$

We can now put things together. Let I be an instance. Then, we will denote by $\mathcal{M}_1(I)$ the MIP defined as:

$$\begin{array}{lll} \text{minimize} & \mathbf{K} & \\ \text{such that} & \mathbf{z_i^j}, \mathbf{e_{kih}}, \mathbf{x_{ih}} \in \{0, 1\} & \forall k, i, h \in [\![1, n]\!], j \in [\![1, m]\!] \\ & \mathbf{K} \in [\![1, N]\!] & \\ & + \text{Constraints (1, 2, 3, 4, 5)} & \end{array}$$

Proposition 14. *Let I be an add-MARA instance. Then, there is an optimal solution with $\mathbf{K} = L$ to $\mathcal{M}_1(I)$ if and only if I is an (L-app envy)-free instance and not an (($L-1$)-app envy)-free one. Moreover, $\mathcal{M}_1(I)$ does not admit any solution if and only if I is an unanimous envy instance.*

The proof of this result can be found in the Appendix.

8.2 A MILP Formulation for K-Approval Non-Proportionality

In the previous subsection, we have introduced a Mixed Integer Linear Program that returns a (K-app envy)-free allocation with the minimal K and no solution when I is an unanimous envy (respectively non-prop) instance. We will now explain how to adapt it to K-app non-proportionality.

In this adapted MIP, we use the same Boolean variables $\mathbf{z_i^j}$. We also introduce n^2 Boolean variables $\mathbf{p_{ki}}$ such that $\mathbf{p_{ki}} = 1$ if and only if according to a_k's preferences a_i's bundle is worth strictly less than the proportional share of a_k. We also need to add n Boolean variables $\mathbf{x_i}$ used to linearize the constraints on $\mathbf{p_{ki}}$. Finally, we use an integer variable $\mathbf{K}$ corresponding to the K-app non-proportionality we seek to minimize.

Recall that we assume in this section that the utilities are integers. We will further assume that $Prop_k = \sum_{j=1}^{m} u(k,j)/n$ is also an integer for each k. If it is not the case, they all the utilities can be multiplied by n without changing the result.

We first need Constraint (1) to ensure the correctness of the allocation.

Secondly, we have to write the constraints that link variables $\mathbf{p_{ki}}$ with the allocation variables $\mathbf{z_i^j}$:

$$\sum_{j=1}^{m} u(k,j) \cdot \mathbf{z_i^j} < \frac{\sum_{j=1}^{m} u(k,j)}{n} (= Prop_k) \iff \mathbf{p_{ki}} = 1 \quad \forall k, i \in [\![1, n]\!]$$

As the utilities are integers, we can replace > 0 by ≥ 1. In order to linearize the equivalence between the two constraints we introduce a number M that can be once again arbitrarily chosen such that $M > max_{a_k \in \mathcal{N}} \sum_{j=1}^{m} u(k,j)$:

$$M\mathbf{p_{ki}} \geq Prop_k - \sum_{j=1}^{m} u(k,j)\mathbf{z_i^j} \quad \forall k, i \in [\![1, n]\!] \tag{6}$$

$$Prop_k - \sum_{j=1}^{m} u(k,j)\mathbf{z_i^j} \geq 1 - M(1 - \mathbf{p_{ki}}) \quad \forall k, i \in [\![1, n]\!] \tag{7}$$

Finally, we have to write the constraints that convey the fact that the allocation we look for is (K-app non-prop)-free:

$$\mathbf{p_{ii}} = 0 \vee \sum_{k=1}^{n} \mathbf{p_{ki}} \leq \mathbf{K} - 1 \qquad \forall i \in [\![1, n]\!]$$

Since $\mathbf{p_{ii}}$ is a Boolean variable for each i, we can replace $\mathbf{p_{ii}} = 0$ by $\mathbf{p_{ii}} \leq 0$. Now, this logical constraint is linearized as follows:

$$\mathbf{p_{ii}} \leq \mathbf{x_i} \quad \forall i \in [\![1, n]\!] \tag{8}$$

$$\sum_{k=1}^{n} \mathbf{p_{ki}} \leq \mathbf{K} - 1 + n(1 - \mathbf{x_i}) \quad \forall i \in [\![1, n]\!] \tag{9}$$

We can now put things together. Let I be an instance. Then, we will denote by $\mathcal{M}_2(I)$ the MIP defined as:

$$\begin{array}{lll} \text{minimize} & \mathbf{K} & \\ \text{such that} & \mathbf{z_i^j}, \mathbf{p_{ki}}, \mathbf{x_i} \in \{0, 1\} & \forall k, i \in [\![1, n]\!], j \in [\![1, m]\!] \\ & \mathbf{K} \in [\![1, N]\!] & \\ & + \text{ Constraints } (1, 6, 7, 8, 9) & \end{array}$$

Proposition 15. *Let I be an instance. Then, there is an optimal solution with $\mathbf{K} = L$ to $\mathcal{M}_2(I)$ if and only if I is an (L-app non-prop)-free instance and not an (($L-1$)-app non-prop)-free one. Moreover, $\mathcal{M}_2(I)$ does not admit any solution if and only if I is an unanimous non-proportional instance.*

The proof of this result can be found in the Appendix.

9. House Allocation Setting

We have seen in Corollaries 2 and 4 the problems of finding the minimal level K for which there exists a (K-app envy)-free or a (K-app non-prop)-free allocation are difficult in the general case. A natural way to tackle this difficulty is to look for particular restrictions where these problems can be solved efficiently. In this section, we will deal with the House Allocation setting.

The House Allocation Problem (HAP for short) is a standard setting where there are exactly as many items as agents, and each agent receives exactly one item. This setting is relevant in many situations and has been extensively studied (Shapley & Scarf, 1974; Roth & Sotomayor, 1992; Abraham et al., 2005, to cite a few of them). In House Allocation Problems, computing an envy-free allocation and a proportional allocation reduces to the problem of finding a matching in a bipartite graph, which can be done in $O(n^3)$ (Gondran & Minoux, 1984). Indeed, an envy-free allocation exists if and only if all the agents get (one of) their top item(s) and a proportional allocation exists if and only if each agent a_i gets an item whose value is greater than $Prop_i$. It is therefore natural to wonder whether an allocation minimizing K-app envy or K-app non-proportionality could also be computed efficiently.

Our first observation hints in that direction. Indeed, characterizing unanimous envy becomes easy in house allocation problems.

Proposition 16. *Let I be an instance of HAP. I is an unanimous envy instance if and only if there exists at least one pair of items (o_i, o_j) such that all the agents unanimously strictly prefer o_i to o_j.*

Proof. ($\Rightarrow$) Suppose that for any pair of items (o_i, o_j), there are two agents (a_k, a_l) such that $u(k, i) \geq u(k, j)$ and $u(l, i) \leq u(l, j)$. Let π be an allocation, and suppose w.l.o.g that $\pi_i = \{o_i\}$. Then for any pair of agents (a_i, a_j), either (i) $u(i, i) \geq u(i, j)$, in which case a_i does not envy a_j, or (ii) $u(i, i) < u(i, j)$, in which case a_i envies a_j, but there is another agent a_k such that $u(k, i) \geq u(k, j)$. In the latter case, a_k disagrees with a_i's envy towards a_j. Hence a_i does not unanimously envy a_j. Therefore I is not an unanimous envy instance.

($\Leftarrow$) Suppose now that there is a pair of items (o_i, o_j) such that $u(k, i) > u(k, j)$ for all agents a_k. In any allocation one agent (say a_i) holds o_i while another agent (say a_j) holds o_j: a_j envies a_i and all the agents approve this envy. Therefore I is an unanimous envy instance. □

Incidentally, we get as a corollary:

Corollary 7. *One can check in $O(n^3)$ whether an instance I of HAP is a unanimous envy instance or not.*

From this characterization we can also derive a result on the likelihood that unanimous envy exists when the utilities are uniformly distributed (that is, for each agent a_i and object o_j, utilities are drawn i.i.d. following the uniform distribution on some interval $[x, y]$). The interested reader can find this result in the Appendix.

We will now investigate the case of approval non-proportionality in the context of HAP. Interestingly, it is also possible to exactly characterize the set of unanimous non-proportional instances.

Proposition 17. *Let I be an HAP instance. I is an unanimous non-proportional instance if and only if there exists at least an item o_p such that $u(k, p) < Prop_k$ for all agents a_k.*

Proof. ($\Rightarrow$) Suppose no such item o_p exists. Let π be any allocation giving to each agent a_i an item (say o_i w.l.o.g). Then either $u(i, i) \geq Prop_i$, in which case a_i receives her proportional share, or $u(i, i) < Prop_i$, in which case there is another agent a_k such that $u(k, i) > Prop_k$. a_k thus disagrees with π_i being non-proportional. Hence the instance is not unanimous non-proportional.

($\Leftarrow$) Now suppose that there is an item o_p such that $u(k, p) < Prop_k$ for all agents a_k. In any allocation one agent (say a_p) holds o_p. By definition, a_p does not get her proportional share, and all the agents agree with that. Therefore, the instance is unanimous non-proportional. □

As for approval envy-freeness, this result yields an efficient way of checking whether an instance is unanimous non-proportional or not:

Corollary 8. *One can check in $O(n^2)$ whether an instance I of HAP is an unanimous non-proportional instance or not.*

Like in the approval envy case, we can derive from this characterization an upper bound on the probability for an instance to be unanimous non-proportional (see Appendix).

We will now show that finding an allocation minimizing (K-app envy)-freeness can be done in polynomial time. Before introducing the idea, we need an additional notation. For any pair of objects $(o_j, o_{j'})$, let $\#_{\prec}(o_j, o_{j'})$ denote the number of agents strictly preferring $o_{j'}$ to o_j. For any agent a_i and object o_j, we will also define $maxEnvy(i, j)$ as follows:

$$maxEnvy(i, j) = \max_{o_{j'} \text{ s.t. } u(i,j')>u(i,j)} \#_{\prec}(o_j, o'_j)$$

In other words, $maxEnvy(i, j)$ denotes the maximal value of $\#_{\prec}(o_j, o'_j)$ among the objects that are strictly preferred to o_j by a_i. As we can imagine, this will exactly be the value of the K-app envy experienced by a_i if she gets item o_j (note that if o_j is among a_i's top objects, this value will be 0).

The key to the algorithm is to see that for a given K, determining whether a (K-app envy)-free allocation exists can be done in polynomial time by solving a matching problem. Namely, for each K, we build the following bipartite graph: $\mathcal{N} \cup \mathcal{O}$ is the set of nodes, and we add an edge $(a_i, o_j) \in \mathcal{N} \times \mathcal{O}$ if and only if $maxEnvy(i, j)$ is lower than or equal to K. We can observe that any perfect matching in this graph corresponds to a ($(K + 1)$-app envy)-free allocation. More precisely, if there exists a perfect matching, that means that the allocation π resulting from the perfect matching is ($(K+1)$-app envy)-free but there could exist another allocation with lower (approval envy)-freeness. If there is no perfect matching, then there could exist a (h-app envy)-free allocation with $h > K+1$. The only thing that remains to do is to run through all possible values of K, which can be done by dichotomous search between 0 and n. This is formalized in Algorithm 9.1.

Proposition 18. *For any HAP instance, we can find (one of) its optimal (K-app envy)-free allocations in $O(n^3 \log(n))$.*

Proof. First, the computation of the matrix *maxEnvy* runs in $O(n^3)$. Indeed, to compute *maxEnvy*(i, j) we first need to compute $\#_{\prec}(o_j, o_{j'})$ which already runs in $O(n^3)$ as we have to ask for each couple of objects (n^2 in total) the point of view of all the agents (n in total). From that, since

$$maxEnvy(i, j) = \max_{o_{j'} \text{ s.t. } u(i,j')>u(i,j)} \#_{\prec}(o_j, o_{j'})$$

we can compute $maxEnvy(i, j)$ in $O(n)$. As there are n^2 different pairs (a_i, o_j) we have the final $O(n^3)$ complexity of computing *maxEnvy*.

Due to the dichotomous search, the algorithm needs to solve $\log(n)$ perfect matching problems, that can be solved in $O(n^3)$(Gondran & Minoux, 1984). The overall complexity of Algorithm 9.1 is thus $O(n^3 \log(n))$. □

Following the same idea, we can propose an algorithm that returns an allocation minimizing (K-app non-prop)-freeness in polynomial time. For this case, we no longer need the matrix $maxEnvy$, but we have to replace it by some vector $\#nonProp$ that tells for each object o_j how many agents think this object is not worth their proportional share:

Algorithm 9.1: Minimizing (K-app envy)-freeness in the HAP

```
   input  : I = ⟨N, O, w⟩ a HAP instance
   output: Allocation π and its level minimizing the (K-app envy)-freeness or None if I is a
           unanimous envy instance
1  maxEnvy ← computeMaxEnvy();
2  res ← None;
3  low ← 0, high ← n;
4  while low ≤ high do
5      K ← ⌊(low + high)/2⌋;
6      G ← buildBipartiteGraph(maxEnvy, K);
7      π ← perfectMatching(G);
8      if π is not None then
9          res ← π, K + 1;
10         high ← K − 1;
11     else
12         low ← K + 1;
13 return res
```

$$\#nonProp(j) = |\{a_i \text{ s.t. } u(i, j) < Prop_i\}|$$

In Algorithm 9.1 we then replace Line 9.1 by an instruction computing $\#nonProp$ for each o_j. Then, we replace the bipartite graph computed at Line 5 by the graph defined as follows: $\mathcal{N} \cup \mathcal{O}$ is still the set of nodes, and we add an edge $(a_i, o_j) \in \mathcal{N} \times \mathcal{O}$ if and only if $u(i, j) < Prop_i$ or $\#nonProp(j)$ is lower than or equal to K.

Proposition 19. *For any HAP instance, we can find (one of) its optimal K-app non-prop-free allocations in $O(n^3 \log(n))$.*

Proof. We know from the proof of Proposition 18 that the algorithm runs in at least $O(n^3 \log(n))$ due to the dichotomous search associated with the perfect matching problem resolutions. But the complexity could be worse because of the computation of $\#nonProp$ and the construction of the bipartite graph. To compute $\#nonProp$, it is enough for each object o_j to run through all the agents and count how many of them think o_j is not worth their proportional share. This can be done in $O(n^2)$ steps, provided that we have pre-computed the values $Prop_i$ first (which can be done in $O(n)$ for each agent, that is, $O(n^2)$ in total). Computing the bipartite graph does not take longer than before, since we just have to check for each pair (a_i, o_j) whether $u(i, j) < Prop_i$ or $\#nonProp(j) \leq K$ (which can be made in constant time if the values $Prop_i$ and $\#nonProp(j)$ have been pre-computed). Thus in total, the adaptation of the algorithm does not cause any added complexity, so the global complexity is still $O(n^3 \log(n))$. □

10. Experimental Results

We conducted an experimental evaluation of our approval notions and solving methods. These experiments serve two purposes: (i) evaluate the behaviour of the MIPs we presented in Section 8 and of the polynomial algorithms described in Section 9 for the HAP setting, and (ii) observe the relevance of our two approval notions when varying the number of agents, of items, and the type of preferences. All the tests presented in this section have been run on an Intel(R) Core(TM) i7-2600K CPU with 16GB of RAM and using the Gurobi solver to solve the Mixed Integer Program.

We have tested our methods on three types of instances: Spliddit instances (Goldman & Procaccia, 2015), instances under uniformly distributed preferences and instances under an adaptation of Mallows distributions to cardinal utilities (Durand et al., 2016).

10.1 Spliddit Instances

We have first experimented our MIPs on real-world data from the fair division website Spliddit (Goldman & Procaccia, 2015). There is a total of 3535 instances from 2 agents to 15 agents and up to 96 items. Note that 1849 of these instances involve 3 agents and 6 objects. The program we ran for Spliddit instances proceeds as follows. It first checks whether the instance is HAP. If it is the case, it runs Algorithm 9.1 to compute the optimal level of approval envy. If this level is 1, it means that the instance is EF, and hence proportional (Bouveret & Lemaître, 2016). We stop there in this case. Otherwise, we run the adaptation of Algorithm 9.1 to compute the level of approval non-proportionality. If the instance is not HAP, we proceed the same way, replacing Algorithm 9.1 and its adaptation by MIPs $\mathcal{M}_1$ and $\mathcal{M}_2$.

Approval Envy Concerning approval envy, by setting a timeout of 1 minute, the program was able to solve all but 6 instances optimally. By extending the timeout to 10 minutes, we were able to solve 4 additional instances. We were however unable to solve the last 2 remaining instances optimally within 5 hours. Those instances respectively concern 6 agents and 15 objects, and 4 agents and 29 objects. However, by examining this latter instance, we could notice that all the agents had the same preferences. Running MIP $\mathcal{M}_2$ on this instance lead us to find an allocation that is proportional, meaning that this allocation is also envy-free in that case. Hence, in the end, only one instance still resists to our attempts. Among the 3534 instances that have been solved optimally, 63.8% admit an EF allocation, while 24.6% exhibit unanimous envy. Moreover, 29% of the 83 instances with more than 5 agents are Strict Majority-app EF (SM-app EF).

We have also implemented the alternative notion of (K-app envy)-freeness mentioned at the end of Section 3 and computed the optimal K for the 3469 easiest Spliddit instances (we removed those that timed out after 20 seconds). Among these instances, only 47 were found to be neither EF nor unanimous-envy, that is, about 1.4%, which confirms our intuition that this alternative notion is much less discriminating that the notion of K-app envy we use in this paper.

Approval Non-Proportionality Concerning approval non-proportionality, all Spliddit instances have been solved optimally within 1 minute. 69.3% of the instances turn out to be proportional, while 25.4% exhibit unanimous non-proportionality. Note that since we know (Bouveret & Lemaître, 2016) that envy-freeness implies proportionality, we knew from the previous experiments that the percentage of proportional instances would be greater than 63.8%. So we can notice that around 5.3% of the instances actually are neither proportional nor unanimous non-proportional against the 11.7% we had for the approval envy notion.

(n,m)	(2,3)	(3,4)	(4,5)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,13)
% EF	86	59	42	58	36	29	11	9	14

Table 1: Percentage of envy-free instances as a function of the number of (agents,objects).

10.2 Uniformly Distributed Preferences: General Setting

We also ran tests on instances under uniformly distributed preferences, with n varying from 2 to 10 and m such that we produce appropriate settings to study our notions of approval envy-freeness and approval non-proportionality. Under Impartial Culture, all preference profiles are equally likely. It is a commonly studied in computational social choice (Black et al., 1958; Gehrlein & Fishburn, 1976) as a limit case, also providing an easy way to get syntactic instances without knowledge on preference characteristics from a particular concrete problem.

Approval Envy We first studied the notion of approval envy and thus considered settings where few EF allocations exist (Dickerson et al., 2014). More precisely we took m almost equal to n, for example 2 agents with 3 objects, 5 agents with 7 objects and 10 agents with 13 objects. As shown by (Dickerson et al., 2014), the percentage of EF instances is tightly related to the ratio between the number of agents and the number of objects. The probability of EF instances is small when the number of objects is not much larger than the number of agents. For each couple (n, m), Table 1 reports the percentage of envy-free instances obtained over 1000 randomly generated instances. It can be noticed that the number of EF instances decreases as the numbers of agents and objects increase. The worst-case in Table 1 is obtained for 9 agents and 11 objects where only 90 over 1000 instances are envy-free. For each couple (n, m), we randomly picked 60 instances over the instances not EF that were randomly generated. Indeed, we wanted to investigate the behavior of our notion when no EF allocation exists (we know that if an EF allocation exists it will be returned by our methods). As we are in the general setting we solved the instances via the MIP $\mathcal{M}_1$ with a timeout of 10 minutes. Experimental results are depicted in Table 2.

The first three rows of Table 2 respectively report the percentage of instances that have been solved to optimal (a solution has been returned before the timeout), the percentage of unanimous envy instances and the percentage of Strict Majority-app-EF instances (SM-app-EF instances). The mean value of K/n gives a good insight on how many agents agree on the fairness notion (in Table 2, on the envy of an agent). Moreover, as it is a normalised measure it allows us to compare the level of approval non-proportionality and envy for instances with different number of agents. Finally, we store the mean computation time (in seconds) of the instances (solved to optimal).

First note that considering 2 agents is a special case as shown in Corollary 1. Indeed, as we have removed the EF instances, all the remaining instances are unanimous envy ones (denoted by - in the tables). Moreover, we observe that the percentage of SM-app-EF allocations is zero for up to 4 agents, which can be easily explained. Indeed, for 3 or 4 agents, being SM-app-EF means that there exists a (K-app envy)-free allocation with $K \leq 2$, which comes down (by Proposition 4) to say that there exists an envy-free allocation. Since all the EF instances have been removed, we cannot find an SM-app-EF allocation for $n \leq 4$.

Besides, without any surprise, the computation time rapidly increases while the percentage of instances solved to optimal (under a timeout of 10 minutes) starts decreasing for 7 agents.

n	2	3	4	5	6	7	8	9	10
% optimal	100	100	100	100	100	68.3	1.7	1.7	0
% UE I	100	21.7	5	0	0	0	0	0	0
% SM-app EF I	0	0	0	50	50	75	40	33.3	6.7
mean(K/n)	-	1	0.85	0.72	0.61	0.57	0.59	0.63	0.66
time(s)	ε	0.008	0.04	0.21	1.97	21.29	50.09	56.16	-

Table 2: Performances of MIP $\mathcal{M}_1$ on randomly generated no EF instances

n	2	3	4	5	6	7	8	9	10
% optimal	100	100	100	100	100	100	100	100	100
% proportional	86.94	83.77	98.41	99.98	99.76	99.95	100	100	100
% UNPI	100	79.2	69.8	100	70.8	100	-	-	-
mean(K/n)	-	1.0	0.96	-	0.95	-	-	-	-
time(s)	0.002	0.03	0.1	1.8	1.5	8.6	-	-	-

Table 3: Performances of MIP $\mathcal{M}_2$ on randomly generated instances.

Finally, positive results can be pinpointed. The percentage of unanimous envy instances is very low. This highlights the relevance of the K-approval envy-free notion. Indeed, in most instances, there exists allocation where we can find a subset of the agents supporting the absence of envy. Minimizing the number of agents approving the envy is thus relevant in almost all instances. Moreover, the experiments show that the percentage of SM-app-EF instances is higher than 30$\%$ except for 10 agents. Such instances are desirable as it means that the absence of envy is supported by more than half the agents: from the point of view of the social acceptance, it is thus possible to find an allocation where the fairness is supported by a majority of agents.

Approval Non-Proportionality First note that a proportional allocation is likely to exist as soon as $m \geq n$ (Suksompong, 2016). As we do not want to be in the House Allocation setting yet, we considered instances for which $m = n + 1$. We have tested our MIP $\mathcal{M}_2$ described in Section 8.2 on such instances with a timeout of 10 minutes. For each couple (n, m), we generated 10 000 instances.

The first four rows of Table 3 respectively represent the percentage of instances that have been solved to optimal (a solution has been returned before the timeout), the percentage of proportional instances, the percentage of unanimous non-proportional instances (among the ones that are not proportional) and the mean value of K/n that gives a insight on how many agents agree on the non-proportionality of an agent. Finally, we store the mean computation time (in seconds) of the instances that are not proportional.

We can first notice that all the instances have been solved to optimal and the number of proportional instances remains very high even if we considered a favourable context with $m = n + 1$. Notably, for more than 8 agents, all the instances were proportional leaving no space for our relaxation to be useful.

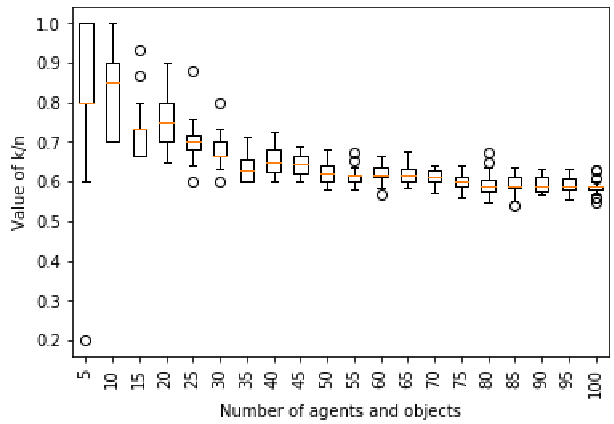


Figure 4: Optimal K/n (envy) in the HAP as a function of n

Note that considering 2 agents is a special case as shown in Corollary 3. Indeed, as we do not consider the proportional instances, all the remaining instances are unanimous non-proportional ones. For more than 2 agents, we can see that the percentage of unanimous non-proportional instances is almost 70% among non-proportional instances. Besides, we can notice that when it is relevant to look at the mean K/n metric, it tells us that the level of approval is very high. In light of these results, we could conclude that while proportionality is a much less demanding notion, it turns out that when it is not satisfied it is extremely often unanimously not satisfied.

10.3 Uniformly Distributed Preferences: House Allocation Problems

We have also tested our polynomial algorithms on HAP instances under uniformly distributed preferences. We have generated 20 instances for each number of agents from 5 to 100 agents (and objects) by steps of 5.

Approval Envy Figure 4 shows the evolution of K/n as a function of the number of agents n (and hence also as a function of m as $n = m$) when minimizing the K-approval envy. First, note that we have only found 5 unanimous envy instances and all of them involved 5 agents. Indeed the probability of unanimous envy instance can be shown to quickly converge to 0 –see Proposition 22 in Appendix. In HAP, agents are very likely to be envious as an agent envies someone as soon as she does not obtain her most preferred object. Let consider an agent a_j that holds o_j and that envies another agent a_k holding o_k. This envy is approved by all the agents that rank o_k over o_j. This envy is likely to be approved but it is also unlikely that all agents agree on this envy. In such contexts where the agents are likely to have mixed opinions, the K-approval envy-free notion and our related algorithm allow for computing allocations where the envy is supported by the smallest subset of agents. As shown in Figure 4, even if the optimal K/n value is high for small problems, it slightly decreases as the size of the instances increases.

Note that the algorithm runs, without any surprise (in light of Proposition 18) much faster than our MIP $\mathcal{M}_1$. Indeed, the mean runtime for 100 objects and agents is still around 2 seconds whereas we already observed that our MIP cannot solve easier problems within 10 minutes.

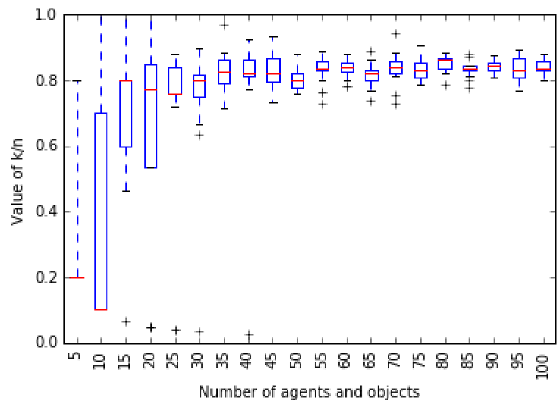


Figure 5: Optimal K/n (non-proportionality) in the HAP as a function of $n(= m)$

Approval Non-Proportionality We have also tested our polynomial algorithm to find an optimal K-approval non-proportional-free allocation. Although the algorithm was running very fast even for 100 agents and objects (confirming what we showed in Proposition 19), we almost only obtained proportional instances. We thus decided to test other instances: by using Borda utilities for each agent and randomly choosing one object per agent whose utility has been multiplied by the number of agents, we built a instances where only one object per agent fulfills the proportional share. We can see in Figure 5 that the value of K/n is stabilising around 0.8 meaning that around 80% of the agents agree with (at least) one agent's non-proportionality.

10.4 Correlated Preferences

As Impartial Culture may not reflect realistic preference profiles, we also generated instances where the preferences of the different agents may have similarities. In strict ordinal settings, a classical way to capture correlated preferences is to use Mallows distributions (Mallows, 1957) allowing us to measure the impact of the similarity of the preferences between agents. In these experiments, we used a generalization of the Mallows distribution to cardinal preferences based on Von Mises–Fisher distributions (Durand et al., 2016). Like the dispersion parameter in Mallows distributions, the similarity between the preferences of the agents is tuned by a *concentration* parameter: when the concentration is zero the agents' preferences are uniformly distributed, whereas when the concentration is infinite all the agents have the same preferences. The concentration can be viewed as the degree of conflicts among the resources. High concentration values lead to similar preferences among the agents for a given item.

We expected that the more similar the preferences between the agents are, the higher the degrees of K-app envy and non-proportionality would get and the more likely unanimous envy and non proportionality would occur. The results of our experiments both in the general setting and in HAP support this: the number of envy-free and proportional instances is decreasing along with the concentration value, and from a given threshold, all the instances exhibit unanimous envy and unanimous non-proportionality. We can see it for example through Figure 6.

Even though at the extreme (when all agents have the same preferences) all notions become unanimous, one may still wonder whether some degree of correlation among preferences may help to find large majorities of agents that contradict the envy of an agent. We thus studied how the

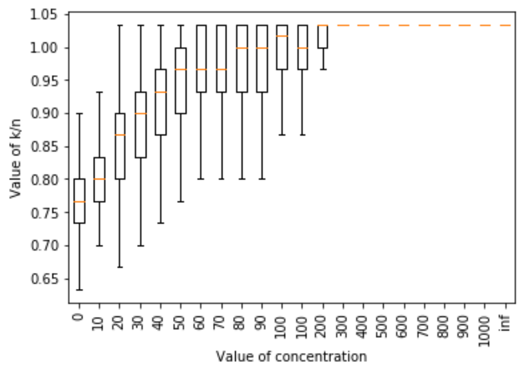


Figure 6: Optimal K/n (approval envy)-freeness in the HAP as a function of the value of concentration for $n = 30$ and $m = 30$

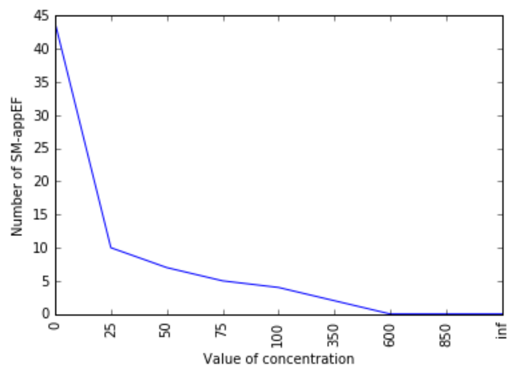


Figure 7: Number of SM-app EF instances as a function of the value of concentration for $n = 7$ and $m = 9$

number of SM-app EF instances varies as a function of the concentration. We considered instances involving 7 agents and 9 objects as we had previously noticed that under uniformly distributed preferences (which is equivalent to a value of concentration of 0), it was very likely to find SM-app EF instances. We then varied the concentration value. For each value, we generated 100 instances and counted the number of SM-app EF instances. As shown in Figure 7, the higher the concentration (and hence the more similar the preferences), the less SM-app EF instances are found, contradicting our hypothesis that correlation might make large majorities of agents contradicting envy more likely to occur.

11. Conclusion

In this paper, we have introduced a new relaxation of envy-freeness and proportionality. These relaxations use a consensus notion, approval envy or non-proportionality, as a proxy for an idealized notion of envy between pairs of agents or proportionality of an agent. We have proposed algorithms to compute an allocation minimizing the approval envy or non-proportionality, and we have experimentally studied how these notions behave on real world data, as well as on instances with uniformly distributed or correlated preferences; more particularly in situations where no envy-free allocation exists and where no proportional allocation exists. We have shown that our notion of approval envy (less so for approval non-proportionality) strikes an interesting balance allowing to discriminate in practice among instances depending on the social support envy relations experience. In comparison, using consensus to determine whether a given agent should be envious or not in general proves to be of limited interest: except in rare cases, instances will either be envy-free or unanimous envy.

This work also opens up to a more general study of consensus-based notions of envy. One could for instance look for allocations that are judged envy-free by a given quota of agents. Restrictions of the approval notions such as an underlying social graph constraining the agents that can approve or disapprove –those agents you deem legitimate to express her view about a specific envy relation– could also be of interest for future work. Other domain restrictions, beyond house allocation, could be studied. For instance, the domain of binary additive preferences, with a cap on the number of items that an agent can like, may offer other tractable cases for our problem. Besides, the approval notions introduced in this article also call for a study of the manipulation that could arise from it. Indeed, asking the opinion of the agents gives birth to new ways of manipulating. More generally, an axiomatic study of the notions proposed here could nicely complement the results obtained.

Finally, it could also be interesting to propose extensions to the case where some items can be shared. Indeed, the approval concepts are a way to mix voting concepts with fair division and shared items is another way of building a continuum between voting and fair division. There may exist a potential link between both approaches. We leave the study of these notions for future work.

Appendix

Proposition 20. *Let I be an add-MARA instance. Then, there is an optimal solution with $\mathbf{K} = L$ to $\mathcal{M}_1(I)$ if and only if I is an (L-app envy)-free instance and not an (($L-1$)-app envy)-free one. Moreover, $\mathcal{M}_1(I)$ does not admit any solution if and only if I is an unanimous envy instance.*

Proof. To prove the proposition, we show that there is an (L-app envy)-free allocation in I if and only if there is a solution to the MIP $\mathcal{M}_1(I)$ such that $\mathbf{K} = L$.

($\Rightarrow$) Let I be an instance, and π be an (L-app envy)-free allocation. Then, consider the partial instantiation of the variables such that $\mathbf{z_i^j} = 1$ if and only if $o_j \in \pi_i$. We prove that this partial instantiation extends to a solution of $\mathcal{M}_1(I)$ such that $\mathbf{K} = L$.

First observe that Constraint 1 is directly satisfied.

Now, consider any triple of agents (a_k, a_i, a_h). Suppose that agent a_k thinks a_i should envy a_h. Then in this case, we have $\sum_{j\in\pi_h} u(k,j) > \sum_{j\in\pi_i} u(k,j)$. In other words, $\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) > 0$ which is in turn equivalent to $\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) \geq 1$ since all utilities are integers. By Constraint 2, we thus have that $\mathbf{e_{kih}} = 1$ which implies that Constraint 3 is satisfied as well.

Conversely, suppose that agent a_k thinks a_i should not envy a_h. Then, we have $\sum_{j\in\pi_h} u(k,j) \leq \sum_{j\in\pi_i} u(k,j)$. In other words, $\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) \leq 0$. By Constraint 3, we thus have that $\mathbf{e_{kih}} = 0$ in this case, which in turns implies that Constraint 2 is satisfied as well. Hence, we have that $\mathbf{e_{kih}} = 1$ if and only if a_k thinks a_i should envy a_h in π.

Finally, consider any pair of agents (a_i, a_h). If a_i does not envy a_h then $\mathbf{e_{iih}} = 0$. As a consequence, $\mathbf{x_{ih}}$ can be null and still satisfy Constraints 4 and 5 (no matter the value of $\mathbf{K}$ is).

Now suppose that a_i does envy a_h (hence $\mathbf{e_{iih}} = 1$). Then, we should have $\mathbf{x_{ih}} = 1$ to satisfy Constraint 4. Since π is (L-app envy)-free, then at most $L - 1$ agents (including a_i herself) think that a_i should indeed envy a_h, which means that $\sum_{k=1}^{n} \mathbf{e_{kih}} \leq L - 1$. Instantiating $\mathbf{K}$ to L is hence enough to satisfy Constraint 5.

($\Leftarrow$) Now suppose that there is a solution to $\mathcal{M}_1(I)$ such that $\mathbf{K} = L$. Then we will prove that the allocation π such that $o_j \in \pi_i$ if and only if $\mathbf{z_{ij}} = 1$ is a valid (L-app envy)-free allocation.

First, according to Constraints 1, π is indeed a valid allocation.

Secondly, Constraint 2 ensures that if $\mathbf{e_{kih}} = 0$ then $\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) \leq 0$, in turn meaning that agent a_k thinks that a_i should not envy a_h. Conversely, Constraint 3 ensures that if $\mathbf{e_{kih}} = 1$ then $\sum_{j=1}^{m} u(k,j)(\mathbf{z_h^j} - \mathbf{z_i^j}) > 0$, in turn meaning that agent a_k thinks that a_i should envy a_h. It also obviously implies that $\mathbf{e_{iih}} = 1$ if and only if a_i envies a_h.

Now consider any pair of agents (a_i, a_h) such that a_i envies a_h. From what precedes, $\mathbf{e_{iih}} = 1$. By Constraint 4, $\mathbf{x_{ih}} = 1$. Hence, by Constraint 5, $\sum_{k=1}^{h} \mathbf{e_{kih}} \leq L - 1$. This implies that the total number of agents agreeing with the fact that a_i envies a_h is strictly lower than L. In other words, π is (L-app envy)-free. □

Proposition 21. *Let I be an instance. Then, there is an optimal solution with $\mathbf{K} = L$ to $\mathcal{M}_2(I)$ if and only if I is an (L-app non-prop)-free instance and not an (($L - 1$)-app non-prop)-free one. Moreover, $\mathcal{M}_2(I)$ does not admit any solution if and only if I is an unanimous non-proportional instance.*

Proof. The key here is to show that there is a solution to the MIP $\mathcal{M}_2(I)$ such that $\mathbf{K} = L$ iff the corresponding allocation π such that $\mathbf{z_i^j} = 1$ if and only if $o_j \in \pi_i$ is (L-app non-prop)-free. However this is done in the proof of Proposition 20. We also have to show that Constraints 6 and 7 are indeed a valid translation of the logical equivalence, and that Constraints 8 and 9 correctly encode the logical OR. The same type of linearization is also done in the proof of Proposition 20. □

Proposition 22. *The probability for an instance being randomly generated under uniformly distributed preferences to exhibit unanimous envy is upper bounded by $n(n-1)/2^n$.*

Proof. The probability of the event o_i is strictly preferred to o_j by one agent is $1/2$ if preferences are strict. As preferences are not strict, this probability becomes an upper bound (think for instance if the agent values all the objects the same then the probability to have strict preference between two objects is zero). Hence, the probability of the event o_i is strictly preferred to o_j by all agents is upper bounded by $1/2^{n-1}$ as the preferences between the agents are independent. Assuming, for all pairs of items, these events to be independent (which is not the case, hence an upper bound of the upper bound), we derive our result by summing up over the $n(n-1)/2$ possible pairs. □

Note that this value quickly tends towards 0: for instance, for 10 agents, the probability for an instance to exhibit unanimous envy is upper-bounded by 0.088.

Proposition 23. *The probability for an instance being randomly generated under uniformly distributed preferences to exhibit unanimous non-proportionality is upper bounded by $n/2^n$.*

To prove this property, we will need a small lemma:[3]

Lemma 4. *Let $U_1, \ldots, U_n$ be n independent random variables having a uniform distribution over real interval $[a, b]$, and let us denote by $\bar{U}_n$ the empiric mean of $U_1, \ldots, U_n$: $\bar{U}_n = \frac{1}{n}\sum_{i=1}^{n} U_i$.*

Then we have:

$$P(U_1 < \bar{U}_n) = \frac{1}{2}.$$

Proof. The probability we seek can be reformulated as follows:

$$P(U_1 < \bar{U}_n) = P\left(U_1 < \frac{1}{n}\sum_{i=1}^{n} U_i\right) = P\left(U_1 < \frac{1}{n-1}\sum_{i=2}^{n} U_i\right) = P(U_1 < \bar{U}_{n-1}). \quad (10)$$

We can notice that the two latter variables, U_1 and $\bar{U}_{n-1}$ are independent. For any two independent variables X and Y, we have that $P(X < Y) = \mathbb{E}[F_X(Y)]$, where F_X is the cumulative distribution function of X. To see this, we can consider the following steps:

$$\begin{aligned} P(X < Y) = \int P(X < Y | Y = y) f_Y(y) dy &= \int P(X < y) f_Y(y) dy \\ &= \int F_X(y) f_Y(y) dy \\ &= \mathbb{E}[F_X(Y)], \end{aligned} \quad (11)$$

where the first step is obtained using the law of total probability, and the last step is obtained using the law of unconscious statistician.

Putting Equations (10) and (11) together, we obtain:

$$P(U_1 < \bar{U}_n) = \mathbb{E}[F_{U_1}(\bar{U}_{n-1})] = \mathbb{E}\left(\frac{\bar{U}_{n-1} - a}{b-a}\right) = \frac{\mathbb{E}(\bar{U}_{n-1}) - a}{b-a} \quad (12)$$

Observing that $\bar{U}_{n-1}$ has the same (uniform) law as U_1, we have that $\mathbb{E}(\bar{U}_{n-1}) = \frac{a+b}{2}$. Injecting this to Equation (12) yields:

$$P(U_1 < \bar{U}_n) = \frac{1}{2} \quad (13)$$

as expected. □

We are now ready to prove Proposition 23.

Proof (Proposition 23). Let I be a random instance generated under uniformly distributed preferences. According to Proposition 17, I is unanimous non-proportional if and only if there exists at least an item o_p such that $u(k, p) < Prop_k$ for all agents a_k. In what follows, we will denote by $U_{k,p}$ the random variable corresponding to $u(k, p)$.

For any item o_p and any agent a_k, $P(U_{k,p} < \frac{1}{n}\sum_{j=1}^{n} U_{k,j}) = \frac{1}{2}$ by Lemma 4. All the variables $U_{k,p}$ being independent, we have that:

$$P\left(\bigcap_{k=1}^{n}\left(U_{k,p} < \frac{1}{n}\sum_{j=1}^{n} U_{k,j}\right)\right) = \frac{1}{2^n} \text{ for all } o_p.$$

3. We warmly thank Olivier François for this result.

The events $U_{k,p} < \frac{1}{n}\sum_{j=1}^{n} U_{k,j}$ not being independent, we can only derive an upper bound on the probability for I to have at least one object o_k such that $u(k,p) < Prop_k$ for all agents a_k. Namely:

$$P\left(\exists k \in [\![1,n]\!] | \bigcap_{k=1}^{n}\left(U_{k,p} < \frac{1}{n}\sum_{j=1}^{n} U_{k,j}\right)\right) \leq \frac{n}{2^n},$$

which concludes the proof. □

Note that once again this value quickly tends towards 0: for instance, for 10 agents, the probability for an instance to exhibit unanimous non-proportionality is upper-bounded by 0.00977.

References

Abebe, R., Kleinberg, J., & Parkes, D. C. (2017). Fair division via social comparison. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems*, AAMAS-17, p. 281–289, Richland, SC. International Foundation for Autonomous Agents and Multiagent Systems.

Abraham, D. J., Cechlárová, K., Manlove, D. F., & Mehlhorn, K. (2005). Pareto optimality in house allocation problems. In Fleischer, R., & Trippen, G. (Eds.), *Algorithms and Computation: 15th International Symposiums*, pp. 3–15. Springer Berlin Heidelberg, Berlin, Heidelberg.

Aleksandrov, M., Ge, C., & Walsh, T. (2019). Fair division minimizing inequality. In Oliveira, P. M., Novais, P., & Reis, L. P. (Eds.), *Progress in Artificial Intelligence, 19th EPIA Conference on Artificial Intelligence, EPIA 2019, Vila Real, Portugal, September 3-6, 2019, Proceedings, Part II*, Vol. 11805 of *Lecture Notes in Computer Science*, pp. 593–605. Springer.

Amanatidis, G., Birmpas, G., Filos-Ratsikas, A., Hollender, A., & Voudouris, A. A. (2020). Maximum nash welfare and other stories about EFX. In Bessiere, C. (Ed.), *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020*, pp. 24–30, Yokohama, Japan (relocated and postponed). ijcai.org.

Amanatidis, G., Birmpas, G., & Markakis, V. (2018). Comparing approximate relaxations of envy-freeness. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18*, pp. 42–48. International Joint Conferences on Artificial Intelligence Organization.

Amanatidis, G., Markakis, E., & Ntokos, A. (2020). Multiple birds with one stone: Beating $1/2$ for EFX and GMMS via envy cycle elimination. In *Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence (AAAI-20)*, pp. 1790–1797.

Aziz, H., Bouveret, S., Caragiannis, I., Giagkousi, I., & Lang, J. (2018). Knowledge, fairness, and social constraints. In *Proceedings of the 32nd AAAI conference on Artificial Intelligence (AAAI'18)*, New Orleans, Louisiana, USA. AAAI Press.

Aziz, H., Caragiannis, I., Igarashi, A., & Walsh, T. (2019). Fair allocation of indivisible goods and chores. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*, pp. 53–59. International Joint Conferences on Artificial Intelligence Organization.

Aziz, H., Moulin, H., & Sandomirskiy, F. (2020). A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation. *Operations Research Letters*, *48*(5), 573 – 578.

Barman, S., & Krishnamurthy, S. (2019). On the proximity of markets with integral equilibria. In *Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence, AAAI-19*, pp. 1748–1755.

Bei, X., Qiao, Y., & Zhang, S. (2017). Networked fairness in cake cutting. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, pp. 3632–3638.

Beynier, A., Bouveret, S., Lemaître, M., Maudet, N., Rey, S., & Shams, P. (2019a). Efficiency, Sequenceability and Deal-Optimality in Fair Division of Indivisible Goods. In *18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019)*, pp. 900–908, Montreal, Canada.

Beynier, A., Chevaleyre, Y., Gourvès, L., Harutyunyan, A., Lesca, J., Maudet, N., & Wilczynski, A. (2019b). Local Envy-Freeness in House Allocation Problems. *Autonomous Agents and Multi-Agent Systems*, *33*(5), 591–627.

Biswas, A., & Barman, S. (2018). Fair division under cardinality constraints. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18*, pp. 91–97. International Joint Conferences on Artificial Intelligence Organization.

Black, D., Newing, R. A., McLean, I., McMillan, A., & Monroe, B. L. (1958). *The theory of committees and elections*. Springer.

Bouveret, S., & Lemaître, M. (2014). Characterizing conflicts in fair division of indivisible goods using a scale of criteria. In Bazzan, A. L. C., Huhns, M. N., Lomuscio, A., & Scerri, P. (Eds.), *Proceedings of the 13th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS-14)*, pp. 1321–1328, Paris, France. IFAAMAS.

Bouveret, S., & Lemaître, M. (2016). Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems*, *30*(2), 259–290.

Brams, S. J., & Fishburn, P. C. (2002). Fair division of indivisible items between two people with identical preferences: Envy-freeness, pareto-optimality, and equity. *Social Choice and Welfare*, *17*, 247–267.

Brams, S. J., & Taylor, A. D. (1996). *Fair Division — From Cake-cutting to Dispute Resolution*. Cambridge University Press.

Brânzei, S., & Sandomirskiy, F. (2019). Algorithms for competitive division of chores. *CoRR*, *abs/1907.01766*.

Bredereck, R., Kaczmarczyk, A., & Niedermeier, R. (2018). Envy-free allocations respecting social networks. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '18, p. 283–291, Richland, SC. International Foundation for Autonomous Agents and Multiagent Systems.

Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, *119*(6), 1061–1103.

Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A. D., Shah, N., & Wang, J. (2016). The unreasonable fairness of maximum nash welfare. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, EC '16, pp. 305–322, New York, NY, USA. ACM.

Chaudhury, B. R., Garg, J., & Mehlhorn, K. (2020). Efx exists for three agents. In *Proceedings of the 21st ACM Conference on Economics and Computation*, EC '20, p. 1–19, New York, NY, USA. Association for Computing Machinery.

Chevaleyre, Y., Endriss, U., & Maudet, N. (2017). Distributed Fair Allocation of Indivisible Goods. *Artificial Intelligence*, *242*, 1–22.

Conitzer, V., Freeman, R., & Shah, N. (2017). Fair public decision making. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, EC '17, p. 629–646, New York, NY, USA. Association for Computing Machinery.

de Keijzer, B., Bouveret, S., Klos, T., & Zhang, Y. (2009). On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences. In *Proceedings of the 1st International Conference on Algorithmic Decision Theory (ADT'09)*, Lecture Notes in Artificial Intelligence, pp. 98–110, Venice, Italy. Springer Verlag.

Dickerson, J. P., Goldman, J., Karp, J., Procaccia, A. D., & Sandholm, T. (2014). The computational rise and fall of fairness. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI-14)*, pp. 1405–1411, Québec City, Québec, Canada. AAAI Press.

Durand, F., Mathieu, F., & Noirie, L. (2016). SVVAMP: simulator of various voting algorithms in manipulating populations. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence, February 12-17, 2016, Phoenix, Arizona, USA.*, pp. 4353–4354.

Foley, D. K. (1967). Resource allocation and the public sector. *Yale Economic Essays*, *7*(1), 45–98.

Gehrlein, W. V., & Fishburn, P. C. (1976). The probability of the paradox of voting: A computable solution. *Journal of Economic Theory*, *13*(1), 14 – 25.

Goldman, J., & Procaccia, A. D. (2015). Spliddit: Unleashing fair division algorithms. *SIGecom Exch.*, *13*(2), 41–46.

Gondran, M., & Minoux, M. (1984). *Graphs and algorithms*. Wiley-Interscience series in discrete mathematics. Wiley.

Gourvès, L., Monnot, J., & Tlilane, L. (2014). Near fairness in matroids. In Schaub, T., Friedrich, G., & O'Sullivan, B. (Eds.), *ECAI 2014 - 21st European Conference on Artificial Intelligence, Prague, Czech Republic - Including Prestigious Applications of Intelligent Systems (PAIS 2014)*, Vol. 263 of *Frontiers in Artificial Intelligence and Applications*, pp. 393–398. IOS Press.

Guibet Lafaye, C. (2006). *Justice sociale et éthique individuelle*. Presses Université Laval.

Lipton, R., Markakis, E., Mossel, E., & Saberi, A. (2004). On approximately fair allocations of divisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC-04)*, pp. 125–131, New York, NY. ACM.

Mallows, C. L. (1957). Non-Null Ranking Models. I. *Biometrika*, *44*(1-2), 114–130.

Moulin, H. (2003). *Fair Division and Collective Welfare*. MIT Press.

Nguyen, T. T., & Rothe, J. (2014). Minimizing envy and maximizing average nash social welfare in the allocation of indivisible goods. *Discret. Appl. Math.*, *179*, 54–68.

Othman, A., Sandholm, T., & Budish, E. (2010). Finding approximate competitive equilibria: efficient and fair course allocation. In van der Hoek, W., Kaminka, G. A., Lespérance, Y., Luck, M., & Sen, S. (Eds.), *Proceedings of the 9th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS-10)*, pp. 873–880, Toronto, Canada. IFAAMAS.

Parijs, P. V. (1997). *Real Freedom for All: What (if Anything) Can Justify Capitalism?* No. 9780198293576 in OUP Catalogue. Oxford University Press.

Perry, J., & Powers, R. C. (2010). Anonymity, monotonicity, and quota pair systems. *Mathematical Social Sciences*, *60*(1), 57–60.

Plaut, B., & Roughgarden, T. (2018). Almost envy-freeness with general valuations. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '18, p. 2584–2603, USA. Society for Industrial and Applied Mathematics.

Procaccia, A. D., & Wang, J. (2014). Fair enough: Guaranteeing approximate maximin shares. In Babaioff, M., Conitzer, V., & Easley, D. (Eds.), *Proceedings of the 15th ACM Conference on Electronic Commerce (EC-14)*, pp. 675–692, Palo Alto, CA. ACM.

Roth, A., & Sotomayor, M. (1992). Two-sided matching: A study in game-theoretic modeling and analysis. *Games and Economic Behavior*, *4*(1), 161–165.

Segal-Halevi, E., & Suksompong, W. (2019). Democratic fair allocation of indivisible goods. *Artificial Intelligence*, *277*.

Shapley, L., & Scarf, H. (1974). On cores and indivisibility. *Journal of mathematical economics*, *1*(1), 23–37.

Steinhaus, H. (1948). The problem of fair division. *Econometrica*, *16*(1), 101–104.

Suksompong, W. (2016). Asymptotic existence of proportionally fair allocations. *Mathematical Social Sciences*, *81*, 62 – 65.

Young, H. P. (1994). *Equity in Theory and Practice*. Princeton University Press.