Journal of Artificial Intelligence Research 76 (2023) 287-339 Submitted 04/2022; published 01/2023

Favoring Eagerness for Remaining Items: Designing Efficient, Fair, and Strategyproof Mechanisms

Xiaoxi Guo GUOXIAOXI@PKU.EDU.CN
Key Laboratory of High Confidence Software Technologies (MOE),
School of Computer Science, Peking University,
Beijing 100871, China

Sujoy Sikdar SSIKDAR@BINGHAMTON.EDU
Department of Computer Science, Binghamton University,
4400 Vestal Parkway East, Binghamton 13902, New York, USA

Lirong Xia XIALIRONG@GMAIL.COM
Department of Computer Science, Rensselaer Polytechnic Institute,
110 Eighth Street, Troy 610101, New York, USA

Yongzhi Cao⊠ CAOYZ@PKU.EDU.CN
Key Laboratory of High Confidence Software Technologies (MOE),
School of Computer Science, Peking University,
Beijing 100871, China

Hanpin Wang WHPXHY@PKU.EDU.CN
Key Laboratory of High Confidence Software Technologies (MOE),
School of Computer Science, Peking University,
Beijing 100871, China

Abstract

In the assignment problem, the goal is to assign indivisible items to agents who have ordinal preferences, efficiently and fairly, in a strategyproof manner. In practice, *first-choice maximality*, i.e., assigning a maximal number of agents their top items, is often identified as an important efficiency criterion and measure of agents' satisfaction. In this paper, we propose a natural and intuitive efficiency property, *favoring-eagerness-for-remaining-items* (FERI), which requires that each item is allocated to an agent who ranks it highest among remaining items, thereby implying first-choice maximality. Using FERI as a heuristic, we design mechanisms that satisfy ex-post or ex-ante variants of FERI together with combinations of other desirable properties of efficiency (Pareto-efficiency), fairness (strong equal treatment of equals and sd-weak-envy-freeness), and strategyproofness (sd-weak-strategyproofness). We also explore the limits of FERI mechanisms in providing stronger efficiency, fairness, or strategyproofness guarantees through impossibility results.

1. Introduction

In the *assignment problem* (Hylland & Zeckhauser, 1979; Zhou, 1990), n agents have unit demands and *strict ordinal preferences* for n items, each with unit supply, and the goal is to compute an *assignment* which allocates each agent with one unit of items and (approximately) maximizes agent satisfaction. This serves as a useful model for a variety of resource allocation problems involving houses (Shapley & Scarf, 1974), dormitory rooms (Chen & Sönmez, 2002), school choice without priorities (Miralles, 2009), and computational resources in cloud computing (Ghodsi et al., 2011, 2012; Grandl et al., 2014). Due to the wide applicability of the assignment problem, there is a rich literature pursuing the design of assignment mechanisms satisfying desirable properties of efficiency, fairness, and strategyproofness. However, many of these properties are incompatible with each other, and trade-offs must be made.

In several practical assignment problems, whether a maximal number of agents are allocated their respective top items, or *first-choice maximality* (FCM), is identified as an important measure of agents' satisfaction with an assignment. For example, in school choice programs, the percentage of students admitted to their most preferred school is often prominently reported in mass media as a measure of student welfare, and is therefore also an important consideration for school administrators (Dur et al., 2018). Often, additional efficiency guarantees are also desired such as *Pareto-efficiency* (PE), which requires that an assignment cannot be improved upon so that some agents are better off and no agent is worse off. In light of these considerations, we seek to address the following question in this paper: *Can we design mechanisms that satisfy important efficiency criteria (such as FCM and PE simultaneously), while also providing desirable fairness and strategyproofness guarantees?*

The desire for efficiency has motivated the design of mechanisms that (approximately) maximize total satisfaction by a natural heuristic which seeks to allocate each item to an agent who ranks it as highly as possible. A prominent example is the famous Boston mechanism, which proceeds iteratively by allocating as many items as possible to agents who rank it as their first choice, then allocating as many items as possible to agents who rank the items in the second position, and so on. In fact, Kojima and Ünver (2014) showed that the Boston mechanism is characterized by a formalization of this natural heuristic, the *favoring-higher-ranks* (FHR) efficiency property implying both FCM and PE, which requires that each item is allocated to an agent that ranks it highest unless every such agent is allocated an item she ranks higher.

However, the Boston mechanism has long been criticized for failing to provide strategyproofness (Abdulkadiroğlu et al., 2006; Pathak, 2017; Pathak & Sönmez, 2008; Roth et al., 2005) which is often considered equally important to FCM in practical applications like school choice and kidney exchange. Several works have attempted to address this failure by proposing variants of the Boston mechanism. Most notably, Mennle and Seuken (2021) showed that some members of *adaptive Boston mechanisms* (ABM, Alcalde, 1996) satisfy strategyproofness, but they do not consider the question of fairness.

When items are indivisible, even relatively basic fairness notions such as the equal treatment of agents with identical preferences can only be satisfied by a random mechanism, such as the Boston mechanism where ties between agents are broken using a lottery. Ramezanian and Feizi (2021) showed that for any lottery, the expected output of the Boston mechanism satisfies *ex-post* FHR (ep-FHR). However, they also showed that no ep-FHR mechanism can satisfy either the fairness property *sd-envy-freeness* (sd-EF) or the strategyproofness property *sd-strategyproofness* (sd-SP). Ep-FHR is also incompatible with a combination of the weaker strategyproofness property *sd-weak-strategyproofness* (sd-WSP) and the basic fairness property of *strong equal treatment of equals* (SETE). Here, sd-EF is an extension of envy-freeness (Foley, 1966; Varian, 1973) which requires that no agent considers her allocation to be dominated by that of another agent when allocations are compared using

	ex-post efficiency			ex-ante efficiency			ex-ante fairness			strategyproofness	
	ep-FERI	ep-FHR	ep-PE	ea-FERI	ea-FHR	sd-E	sd-EF	sd-WEF	SETE	sd-SP	sd-WSP
RP	N^{P12}	N^{a}	Y^{c}	N^{P12}	N^{b}	N^{c}	N^{c}	Y^{c}	Y^{d}	Y^{c}	Y^{c}
PS	N^{P13}	N^{a}	Y^{a}	N^{P13}	N^{b}	Y^{c}	Y^{c}	Y^{c}	$Y^{c,d}$	N^{c}	Y^{c}
BM*	N^{P14}	Y^{a}	Y^{a}	N^{P14}	N^{b}	N^{b}	N^{P14}	N^{P14}	Y^{P14}	N^{a}	N^{a}
ABM*	Y^{P16}	N^{P16}	Y^{e}	N^{P16}	N^{P16}	N^{P16}	N^{P16}	?	Y^{T16}	N^{e}	Y^{f}
EBM	Y^{T1}	N^{P15}	Y^{e}	N^{P15}	N^{P15}	N^{P15}	N^{P15}	Y^{T1}	Y^{T1}	N^{P15}	Y^{T1}
PR	N^{P17}	$Y^{a,b}$	Y^{b}	N^{P17}	Y^{b}	Y^{b}	N^{b}	N^{b}	Y^{P17}	N^{b}	N^{b}
UPRE	N^{P18}	N^{P18}	Y^{C2}	Y^{T4}	N^{P18}	Y^{C2}	N^{P18}	Y^{T3}	Y^{T3}	N^{P18}	N^{P18}

Table 1: Properties of RP, PS, BM, EBM, PR and PRE.

Note: A 'Y' indicates that the mechanism at that row satisfies the property at that column, and an 'N' indicates that it does not. Results annotated with 'a' follow from Ramezanian and Feizi (2021), 'b' from Chen et al. (2021), 'c' from Bogomolnaia and Moulin (2001), 'd' from Nesterov (2017), 'e' from Dur (2019), and 'f' from Mennle and Seuken (2021) respectively. A result annotated with T, P or C refers to a Theorem, Proposition or Corollary in this paper (or Appendix A), respectively. Table 2 and Table 3 summarize the mechanisms and properties we study in this paper.
*Here, we refer to the expected outputs of the BM and ABM when the priority order over agents is drawn from a uniform distribution over priority orders.

the notion of *stochastic dominance* (sd, Bogomolnaia & Moulin, 2001); sd-SP and sd-WSP require that no agent can manipulate the outcome of the mechanism to her benefit by misreporting her preferences (Bogomolnaia & Moulin, 2001); and SETE requires that agents who share a common prefix in their rankings of items are allocated the items in the shared prefix with the equal probability (Nesterov, 2017).

1.1 Our Contributions

We begin by showing that ep-FHR is not compatible with SETE and sd-WEF (a mildly weaker variant of sd-EF) in Proposition 1, which complements the impossibility results of the incompatibility of FHR with sd-EF and with SETE and sd-WSP by Ramezanian and Feizi (2021). Together, this means that mechanisms that satisfy ep-FHR do not provide an avenue to answer our question (see Section 3).

Our main conceptual contribution is a natural alternative principle for the design of assignment mechanisms: each item is allocated to an agent most *"eager"* for it, i.e., ranks it highest *among remaining items*. This forms the basis of a novel efficiency property, favoring-eagerness-for-remaining-items (FERI), which implies both FCM and PE.

We provide an affirmative answer to the question we seek to address in the paper through our main technical contributions. Using FERI as a heuristic, we design two mechanisms that satisfy desirable combinations of efficiency, fairness, and strategyproofness properties (defined formally in Section 2.3, and summarized in Table 3):

An ex-post FERI (ep-FERI), fair, and strategyproof mechanism. The *eager Boston mechanism* (EBM, Algorithm 1) we design is efficient, fair, and strategyproof. EBM satisfies ex-post FERI (ep-FERI), which implies ex-post FCM and ex-post PE, and also satisfies SETE, sd-WEF, and sd-WSP (Theorem 1). EBM bears a close resemblance to the

ABM family of mechanisms (Algorithm 2), but as we show, EBM is not a member of ABM (Remark 6), although they are closely related: Every ep-FERI assignment, including the output of EBM, can be computed by some member of ABM, and every member of ABM satisfies ep-FERI (Theorem 2).

An ex-ante FERI (ea-FERI) and fair mechanism. We identify the *uniform probabilistic respecting eagerness mechanism* (UPRE, Definition 4) and show that it satisfies SETE and sd-WEF (Theorem 3). Besides, UPRE satisfies ex-ante FERI (ea-FERI) which implies ex-post FCM, ep-PE, and sd-efficiency (the sd version of PE). This is because UPRE belongs to the family of *probabilistic respecting eagerness mechanisms* (PRE, Algorithm 3), and as we show that: Every member of PRE satisfies ea-FERI, and every ea-FERI assignment must be the output of some member of PRE (Theorem 4).

In addition, we explore if ep-FERI or ea-FERI is compatible with stronger notions of fairness (sd-EF over sd-WEF) and strategyproofness (sd-SP over sd-WSP), and find that no mechanism can satisfy the following combinations of properties: ep-FERI and sd-EF (Proposition 4); ea-FERI and sd-EF (Proposition 5); ep-FERI, SETE and sd-SP (Proposition 6); ea-FERI, SETE, and sd-WSP (Proposition 7); and ep-FERI, ea-FERI, and SETE (Proposition 8).

1.2 Related Work

Chen et al. (2021) proposed another extension of FHR, ex-ante FHR (ea-FHR)[1] and provided the probabilistic rank mechanism which satisfies ea-FHR. Since ea-FHR implies ep-FHR, it suffers the same incompatibility with fairness and strategyproofness as ep-FHR does. Apart from FHR, *rank-maximality* (See Definition 6 in Appendix A.2) is a well-known efficiency property that implies FCM (Irving et al., 2006; Paluch, 2013), which has been widely studied for assigning schools to students (Abraham, 2009), assigning papers to referees (Garg et al., 2010), and rental items to customers (Abraham et al., 2006). However, since rank-maximality is stronger than FHR (Belahcene et al., 2021), once again, the incompatibility with fairness and strategyproofness extends to rank-maximality. Popularity (See Definition 5 in Appendix A.1) is another well-studied efficiency property that implies FCM (Abraham et al., 2007). However, an assignment satisfying popularity does not always exist for every instance, and therefore its existence together with other properties cannot be guaranteed either.

Looking beyond mechanisms that attempt to allocate items to agents who rank them highest, random priority (RP) mechanism (Abdulkadiroğlu & Sönmez, 1998) and probabilistic serial (PS) mechanism (Bogomolnaia & Moulin, 2001) are famous mechanisms widely studied in the literature due to their fairness and strategyproofness guarantees (see Table 1). However, both RP and PS fail to satisfy FCM (See Appendix A.3), and therefore they do not provide a positive answer to the question we study in the paper.

Table 1 compares the properties of EBM and UPRE to the properties of RP, PS, Boston mechanism (BM, Abdulkadiroğlu & Sönmez, 2003; Kojima & Ünver, 2014), adaptive Boston mechanism (ABM, Alcalde, 1996; Dur, 2019), and probabilistic rank mechanism (PR, Chen

1. Chen et al. (2021) named this property sd-rank-fairness. We rename it here to emphasize its connection with FHR.

et al., 2021). Figure 5 in Appendix A.1 shows the relationship between efficiency properties based on FERI to extensions of PE or FHR.

We note that Harless (2018) proposed the immediate division$^+$ mechanism and proved that it satisfies sd-WEF. This mechanism appears similar to UPRE, although we are unable to prove or disprove their equivalence. In our paper, we define the family of PRE mechanisms (of which UPRE is a member) and prove that it is characterized by the newly-proposed property ea-FERI, which has not been considered earlier to the best of our knowledge. In addition, with the impossibility results we proved for ea-FERI, we show the limit of the family of PRE, including UPRE, on guarantees of efficiency, fairness, and strategyproofness.

Abbr.	full names
ABM	adaptive Boston mechanism (Alcalde, 1996; Dur, 2019)
EBM	eager Boston mechanism
BM	Boston mechanism (Kojima & Ünver, 2014)
PR	probabilistic rank (Chen et al., 2021)
PRE	probabilistic respecting eagerness
PS	probabilistic serial (Bogomolnaia & Moulin, 2001)
RP	random priority (Abdulkadiroğlu & Sönmez, 1998)
UPRE	uniform probabilistic respecting eagerness

Table 2: Acronyms for mechanisms studied in this paper.

2. Preliminaries

An instance of the *assignment problem* is given by a tuple (N, M) and a *preference profile* R, where $N = \{1, \dots, n\}$ is a set of n *agents*, and $M = \{o_1, \dots, o_n\}$ is a set of n *items* with a single unit of *supply* of each item.

Preferences. A *preference profile* $R = (\succ_j)_{j\in N}$ specifies the ordinal preference of each agent $j \in N$ as a strict linear order over M, and $\succ_{-j}$ denotes the collection of preferences of agents in $N \setminus \{j\}$. Let $\mathcal{R}$ be the set of all the preference profiles. For any $j \in N$, we use $rk(\succ_j, o)$ to denote the rank of item o in $\succ_j$, and $top(\succ_j, S)$ to denote the item ranked highest in $\succ_j$ among $S \subseteq M$. We also use $rk(j, o)$ and $top(j, S)$ for short if it is clear in the context. For any linear order $\succ$ over M and item o, $U(\succ, o) = \{o' \in M \mid o' \succ o\} \cup \{o\}$ represents the items weakly preferred to o. For any pair of agents $j, k \in N$, the common prefix of their preferences $\succ_{j,k}$ is the preference over the first several items which have the same upper contour set in $\succ_j$ and $\succ_k$. Formally, $\succ_{j,k}$ is a strict linear preference over $M' \subseteq M$ such that (i) for any $o \in M'$, $rk(j, o) = rk(k, o) = rk(\succ_{j,k}, o) \leq |M'|$, and (ii) $top(j, M \setminus M') \neq top(k, M \setminus M')$.

Allocations, Assignments, and Mechanisms. A *random allocation* is a stochastic n-vector $p = [p_o]_{o\in M}$ describing the probabilistic share of each item. Let Π be the set of all the possible random allocations. A *random assignment* is a bistochastic $n \times n$ matrix $P = [p_{j,o}]_{j\in N, o\in M}$. For each agent $j \in N$, the j-th row of P, denoted P_j, is agent j's random allocation, and for each item $o \in M$, $p_{j,o}$ is j's probabilistic share of o. We use $\mathcal{P}$ to denote the set of all possible random assignments. A *deterministic assignment* $A : N \to M$ is a one-to-one mapping from agents to items, represented by a binary bistochastic $n \times n$ matrix. For each agent $j \in N$, we use $A(j)$ to denote the item allocated to j, and for each item $o \in M$, $A^{-1}(o)$ to denote the agent allocated o. Let $\mathcal{A}$ denote the set of all the deterministic assignment matrices. By the Birkhoff-Von Neumann theorem, every random assignment $P \in \mathcal{P}$ describes at least one probability distribution over $\mathcal{A}$.

A *mechanism* $f : \mathcal{R} \to \mathcal{P}$ is a mapping from preference profiles to random assignments. For any profile $R \in \mathcal{R}$, we use $f(R)$ to refer to the random assignment output by f. For every agent $j \in N$, we use $f(R)_j$ to denote agent j's random allocation, and for every item $o \in M$, we use $f(R)_{j,o}$ to denote j's share of o.

2.1 Economic Efficiency for Deterministic Assignments

We first introduce some notions of efficiency that are commonly used in evaluating deterministic assignments.

Pareto-efficiency (PE). A deterministic assignment A satisfies PE if no agent can be assigned a better item without assigning any other agent a worse item, i.e., there does not exist another A' and a set $N' \subseteq N$ with $N' \neq \emptyset$ such that $A'(j) \succ_j A(j)$ for any $j \in N'$ and $A'(k) = A(k)$ for $k \in N \setminus N'$.

First-choice maximality (FCM). A deterministic assignment A satisfies FCM if it assigns a maximal number of agents their top ranked items, i.e., there does not exist another A' such that $|\{j \in N \mid rk(j, A'(j)) = 1\}| > |\{j \in N \mid rk(j, A(j)) = 1\}|$.

Favoring-higher-ranks (FHR). A deterministic assignment A satisfies FHR if every item is allocated to an agent that ranks it highest unless every such agent is allocated an item she ranks higher. Formally, A satisfies FHR if for any agents j and $k \in N$, $rk(j, A(j)) \leq rk(k, A(j))$ or $rk(k, A(k)) < rk(k, A(j))$.

Example 1. Consider the preference profile R in Figure 1.

$$\begin{aligned}
\succ_1 &: a \succ_1 b \succ_1 c \succ_1 d \succ_1 e \succ_1 f \\
\succ_2 &: b \succ_2 a \succ_2 c \succ_2 d \succ_2 e \succ_2 f \\
\succ_3 &: c \succ_3 e \succ_3 d \succ_3 f \succ_3 a \succ_3 b \\
\succ_4 &: c \succ_4 e \succ_4 d \succ_4 f \succ_4 a \succ_4 b \\
\succ_5 &: c \succ_5 e \succ_5 d \succ_5 f \succ_5 a \succ_5 b \\
\succ_6 &: c \succ_6 a \succ_6 b \succ_6 d \succ_6 e \succ_6 f
\end{aligned}$$

Figure 1: A linear preference profile R.

In any assignment that satisfies FHR, by definition, each item must be assigned to one of the agents who ranks it on the top if such agents exist. Therefore, a and b go to agents 1 and 2, respectively. Notice that agents 3-6 all rank c on top. If c is allocated to agents 3-5, then by FHR, agent 6 cannot be assigned either item d or item e, since for any $j \in \{3, 4, 5\}$, $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$. The items circled in red represent one such deterministic assignment that satisfies FHR. $\square$

Remark 1. FHR implies FCM (Dur et al., 2018; Kojima & Ünver, 2014) and PE (Ramezanian & Feizi, 2021). FCM and PE do not imply each other.

2.2 Economic Efficiency for Random Assignments

By the Birkhoff-Von Neumann theorem, all of the properties for deterministic assignments can naturally be extended to random assignments: For any property $X \in \{\text{PE, FCM, FHR}, \dots\}$ for deterministic assignments, a random assignment satisfies *ex-post* X if it is a convex combination of deterministic assignments, each of which satisfies the property X. In the paper, we also say that a mechanism f satisfies a property Y if $f(R)$ satisfies Y for every profile $R \in \mathcal{R}$.

Besides the efficiency notions above, we also introduce ex-ante notions for random assignments. One of the notions is based on stochastic dominance (sd), which extends an agent's preference over the single items to the lotteries over items (Segal-Halevi et al., 2020) and is used in comparing random allocations and assignments.

Definition 1. (Bogomolnaia & Moulin, 2001) *Given a preference relation $\succ$ over M, the* **stochastic dominance** *relation associated with $\succ$, denoted by $\succeq^{sd}$, is a partial ordering over Π such that for any pair of random allocations $p, q \in \Pi$, p (weakly)* stochastically dominates *q, denoted by $p \succeq^{sd} q$, if for any $o \in M$, $\sum_{o' \in U(\succ, o)} p_{o'} \geq \sum_{o' \in U(\succ, o)} q_{o'}$.*

Sd-efficiency (sd-E). A random assignment P satisfies sd-E if P is not stochastically dominated by other random assignments, i.e., there does not exist a random assignment $Q \neq P$ such that $Q_j \succeq_j^{sd} P_j$ for every $j \in N$.

Ex-ante FHR (ea-FHR) A random assignment P satisfies ea-FHR, if the shares of every item are allocated to agents that rank it highest unless every such agent's demand is satisfied. Formally, P satisfies ea-FHR if for every agent $j \in N$ and every $o \in M$ such that $p_{j,o} > 0$, it holds that for every $k \in N$ such that $rk(k, o) < rk(j, o)$, $\sum_{o' \in U(k,o)} p_{k,o'} = 1$.

Remark 2. Ea-FHR implies sd-E and ep-FHR (Chen et al., 2021), while both sd-E and ep-FHR implies ep-PE (Bogomolnaia & Moulin, 2001; Ramezanian & Feizi, 2021).

2.3 Fairness and Strategyproofness

Apart from efficiency, fairness and strategyproofness are of great concern in mechanism. In the following, we introduce fairness properties for random assignments and strategyproofness properties for mechanisms.

Strong equal treatment of equals (SETE). A random assignment P satisfies SETE if any two agents have the same allocation over items appearing in the common prefix of their preferences. Formally, for every pair of j and $k \in N$, $p_{j,o} = p_{k,o}$ for any o appearing in $\succ_{j,k}$.

Sd-envy-freeness (sd-EF). A random assignment P is sd-EF, if every agent's allocation weakly stochastically dominates the others', i.e., $P_j \succeq_j^{sd} P_k$ for every pair of j and $k \in N$.

Sd-weak-envy-freeness (sd-WEF). A random assignment P is sd-WEF, if no agent's allocation is dominated by others', i.e., $P_k \succeq_j^{sd} P_j \implies P_j = P_k$ for every pair of j and $k \in N$.

Remark 3. Sd-EF implies sd-WEF (Bogomolnaia & Moulin, 2001) and SETE (Nesterov, 2017), while sd-WEF and SETE do not imply each other.

Sd-strategyproofness (sd-SP). When an agent reports the true preference, a mechanism f satisfying sd-SP always outputs an allocation which weakly dominates the ones when she misreports. Formally, for every $R \in \mathcal{R}$, it holds that $f(R) \succeq_j^{sd} f(R')$ for every $j \in N$ and $R' = (\succ_j', \succ_{-j})$,

Sd-weak-strategyproofness (sd-WSP). A mechanism f satisfying sd-WSP guarantees that when an agent misreports her preference, she would not receive an allocation dominating the one when she truly reports. Formally, for every $R \in \mathcal{R}$, it holds that $f(R') \succeq_j^{sd} f(R) \implies f(R')_j = f(R)_j$ for every $j \in N$, and $R' = (\succ_j', \succ_{-j})$.

Remark 4. Sd-SP implies sd-WSP (Bogomolnaia & Moulin, 2001).

Abbr.	full names	category
ea-FERI	ex-ante favoring-eagerness-for-remaining-items	ex-ante efficiency
ea-FHR	ex-ante favoring-higher-ranks	ex-ante efficiency
ep-FERI	ex-post favoring-eagerness-for-remaining-items	ex-post efficiency
ep-FHR	ex-post favoring-higher-ranks	ex-post efficiency
ep-PE	ex-post Pareto-efficiency	ex-post efficiency
FCM	first-choice maximality	efficiency*
FERI	favoring-eagerness-for-remaining-items	efficiency*
FHR	favoring-higher-ranks	efficiency*
PE	Pareto-efficiency	efficiency*
sd-EF	sd-envy-freeness	ex-ante fairness
sd-E	sd-efficiency	ex-ante efficiency
sd-SP	sd-strategyproofness	strategyproofness
sd-WEF	sd-weak-envy-freeness	ex-ante fairness
sd-WSP	sd-weak-strategyproofness	strategyproofness
SETE	strong equal treatment of equals	ex-ante fairness

Table 3: Acronyms for properties used in this paper.

Note: Properties annotated with * are for deterministic assignments

3. Incompatibility of FHR with Fairness

In this section, we show that FHR mechanisms are unable to satisfy desirable properties of fairness. In Proposition 1, we show that requiring ep-FHR together with SETE leads to a violation of sd-WEF, meaning that no FHR mechanisms can satisfy all of these properties simultaneously. This complements the results by Ramezanian and Feizi (2021) which showed that ep-FHR is not compatible with either sd-EF or sd-SP, and that no mechanism satisfies ep-FHR, SETE, and sd-WSP. Together these negative results demonstrate that ep-FHR mechanisms cannot provide an answer to the question proposed in Section 1.

Proposition 1. *No mechanism simultaneously satisfies ex-post favoring-higher-ranks (ep-FHR), sd-weak-envy-freeness (sd-WEF), and strong equal treatment of equals (SETE).*

Proof. We prove it using the instance with preference R in Figure 1. Let P be the random assignment satisfying ep-FHR and SETE.

First, we look into the deterministic assignments satisfying FHR. By FHR implying FCM, if an item is ranked top by some agents, then it should be assigned to one of them. Therefore, agents 1 and 2 get a and b respectively, which means that $p_{1,a} = p_{2,b} = 1$, and one of agents 3-6 gets c. Since agents 3-5 share the same preference, there are two kinds of assignments satisfying FHR:

(i) if agent 6 gets c, then $\{d, e, f\}$ can be assigned arbitrarily among agents 3-5;

(ii) if agent 6 does not gets c, then she does not get e or d since $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$ with $j \in \{3, 4, 5\}$, which also means that $p_{6,d} = p_{6,e} = 0$.

Then, by SETE, agents 3-5 have the same allocation, and $p_{j,c} = p_{k,c} = 1/4$ for any $j, k \in \{3, 4, 5, 6\}$. From the observation above, P must be the following assignment.

Assignment P

	a	b	c	d	e	f
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3-5	0	0	1/4	1/3	1/3	1/12
6	0	0	1/4	0	0	3/4

Assignment P is not sd-WEF because $\sum_{o' \in U(\succ_6, o)} p_{6,o'} \leq \sum_{o' \in U(\succ_6, o)} p_{1,o'}$ holds for any $o \in M$, and it is strict when $o \in \{e, d\}$. □

Since ea-FHR implies ep-FHR, we can extend Proposition 1 to ea-FHR (Corollary 1). We also discuss in Appendix A.2 the compatibility of rank-maximality, which also implies FCM, with fairness, but the result is still negative.

Corollary 1. *No mechanism simultaneously satisfies ex-ante favoring-higher-ranks (ea-FHR), strong equal treatment of equals (SETE), and sd-weak-envy-freeness (sd-WEF).*

4. Ex-post Favoring Eagerness for Remaining Items

Motivated by the desire for FCM mechanisms that are also fair and strategyproof, we propose **favoring-eagerness-for-remaining-items (FERI)**, an efficiency property which implies both FCM (Remark 5) and PE (Proposition 2). As we will show in Section 4.1, the ex-post variant of FERI is compatible simultaneously with fairness (sd-WEF and SETE) and strategyproofness (sd-WSP).

Informally, a deterministic assignment satisfies FERI (Definition 2) if it can be decomposed in a manner that every item ranked highest by some agents is allocated to one such agent, subject to which, every remaining item is allocated to a remaining agent who ranks it highest among remaining items if such an agent exists, and so on.

Definition 2 (**FERI**). *Given any deterministic assignment A, we define for each $r \in \{1, 2, \dots\}$ a set of items $T_{A,r} = \{o \in M : o = top(j, M \setminus \bigcup_{r'<r} T_{A,r'})$ for some $j \in N$ with $A(j) \notin \bigcup_{r'<r} T_{A,r'}\}$.*

The assignment A satisfies **favoring-eagerness-for-remaining-items** *if for every $r \in \{1, 2, \dots\}$ and every item $o \in T_{A,r}$, it holds that the item o is assigned to an agent most eager for it, i.e, $o = top(A^{-1}(o), M \setminus \bigcup_{r'<r} T_{A,r'})$.*

Definition 2 suggests the following heuristic for designing an FERI mechanism: In each iteration, first remove all the agents who have already been allocated an item; Then eliminate the allocated items from the preference lists of every remaining agent; Now allocate each remaining item to an agent who ranks it as the top remaining item according to their preferences over *remaining* items, if such an agent exists, using a tie-breaking rule if there are multiple such agents. In this way, every remaining agent has a chance of being allocated her most preferred remaining item in each iteration. In contrast, in the r-th iteration of the Boston mechanism (which characterizes FHR assignments), each agent only has a chance of being allocated an item if that item has not been allocated yet and is ranked at the r-th position by the agent; if there is no such item, she cannot be allocated any item in iteration r. This notion of iteratively making decisions based on preferences over remaining alternatives is similar in spirit to that of single transferable voting rules (Hare, 1861) that are resistant to strategic manipulation (Bartholdi & Orlin, 1991) in social choice, and the iterated elimination of dominated strategies for solving strategic games in game theory (Osborne, 2004). In this vein, FERI is a natural alternative to FHR since it also implies FCM and PE.

Remark 5. *FERI implies FCM. Specifically, in any FERI assignment A, when $r = 1$, it requires that for every item o ranked on the top by some agents, i.e., $o \in T_{A,1} = \{o \in M : o = top(j, M)$ for some $j \in N\}$, item o is allocated to one such agent, i.e., $o = top(A^{-1}(o), M)$.*

Although FHR and FERI both imply FCM, they do not imply each other as we show in Example 2.

Example 2. [FHR$\not\Rightarrow$FERI, FERI$\not\Rightarrow$FHR] Consider again the profile in Figure 1. Let A be the FHR assignment indicated by the circled items, and A^* be the following assignment, where $j \leftarrow o$ means agent j is allocated item o:

$$A^* : 1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow c, 4 \leftarrow e, 5 \leftarrow f, 6 \leftarrow d.$$

It is easy to see that A violates FERI because item $d \in T_{A,2}$ due to the fact that $T_{A,1} = \{a, b, c\}$, $d = top(6, M \setminus T_{A,1})$, and $A(6) \notin T_{A,1}$; but $A^{-1}(d) = 5$ and $d \neq top(5, M \setminus T_{A,1}) = e$.

Besides, we show that A^* satisfies FERI:

- For $r = 1$, it is easy to see that for every $o \in T_{A^*,1} = \{a, b, c\}$, $o = top(j, M)$ for each j with $A^*(j) = o$.
- For $r = 2$, $T_{A^*,2} = \{e, d\}$. Items e and d are allocated to agents most eager for them among the remaining items $M' = M \setminus T_{A^*,1}$, i.e., $e = top(4, M') = A^*(4)$ and $d = top(6, M') = A^*(6)$.

- For $r = 3$, $T_{A^*,3} = \{f\}$. Since $M'' = M \setminus T_{A^*,1} \cup T_{A^*,2} = \{f\}$, we have that $f = top(5, M'') = A^*(5)$ trivially.

But A^* violates FHR because $A^*(6) = d$, $rk(6, d) > rk(5, d)$, and $d \succ_5 A^*(5)$. □

Proposition 2 shows that FERI is a stronger efficiency property than PE, which means that ep-FERI implies ep-PE. We also discuss in Appendix A.1 the relation of FERI to popularity (Abraham et al., 2007) which is also a famous efficiency property.

Proposition 2. [FERI⇒PE, PE⇏FERI] *A deterministic assignment satisfying favoring-eagerness-for-remaining-items (FERI) also satisfies Pareto-efficiency (PE), but not vice versa.*

Proof. **(FERI ⇒ PE)** Consider an arbitrary preference profile R, and let A be any deterministic assignment that satisfies FERI. Suppose for the sake of contradiction that A is Pareto dominated by another assignment. Then, since agents have strict preferences, there must exist an assignment A' that Pareto dominates A and can be obtained from A by agents in an *improving cycle* exchanging items along the cycle, while all other agents' allocations remain unchanged. More formally, there exists an assignment A' such that a set of $h \leq n$ agents $N' = \{j_1, j_2, \cdots, j_h\}$ are involved in an improving cycle where for any $i = 1, \ldots, h$, $A'(j_i) = A(j_{i+1 \pmod h}) \succ_i A(j_i)$, and for every agent $j \in N \setminus N'$, $A'(j) = A(j)$.

For ease of exposition, let the agents in the improving cycle be $N' = \{1, \ldots, h\}$, and for any $i = 1, \ldots, n$, let $o_i = A(i)$. Without loss of generality, let o_1 be the item that belongs to the set $T_{A,r}$ with the smallest possible value of r among $\{o_1, \ldots, o_h\}$. Then, by A satisfying FERI,

$$o_1 = top(1, M \setminus \bigcup_{r'<r} T_{A,r'}). \tag{1}$$

By our choice of r, item $o_2 \in M \setminus \bigcup_{r'<r} T_{A,r'}$, and Eq (1) implies that $o_1 \succ_1 o_2$. However, by our assumption that A' Pareto dominates A, we must have that $o_2 = A'(1) \succ_1 A(1) = o_1$, a contradiction. Therefore, any deterministic assignment satisfying FERI is also PE.

(PE⇏ FERI) For the instance with the following profile R from Ramezanian and Feizi (2021), let A be the deterministic assignment indicated by the items circled in red in the following.

$$\succ_1\colon\ a \succ_1 b \succ_1 c,$$

$$\succ_2\colon\ a \succ_2 c \succ_2 b,$$

$$\succ_3\colon\ b \succ_3 a \succ_3 c.$$

The assignment A is PE since it is an outcome of RP with the priority order $2 \rhd 1 \rhd 3$. We see that $b \in T_{A,1} = M$ since $top(3, M) = b$. However, $A^{-1}(b) = 1$ and $b \neq top(1, M) = a$, which violates FERI. □

4.1 EBM Satisfies ep-FERI, sd-WEF, SETE, and sd-WSP

In this section, we define the *eager Boston mechanism* (EBM, Algorithm 1), and prove that it is efficient (ep-FERI and therefore ex-post FCM and ep-PE), fair (sd-WEF and SETE), and strategyproof (sd-WSP). EBM proceeds in multiple rounds using FERI as a heuristic to allocate items. In each round, each unsatisfied agent j applies for the item that she is most eager for, i.e., her top remaining item o. We use N_o to refer to the set of agents who apply for o. Every agent in N_o gets o with probability $1/|N_o|$, and the winner is determined by a random lottery winner generator G: Given a set of agents $S \subseteq N$, $G(S)$ is a single agent drawn from S uniformly at random. At the end of each round, for every item o with $N_o \neq \emptyset$, both the item o and the winner $G(N_o)$ are removed. We illustrate the execution of EBM in Example 3. The output EBM(R) is a deterministic assignment, which can be computed in polynomial time if G runs in polynomial time as we show in Appendix A.4.

Algorithm 1 Eager Boston mechanism (EBM)

```
1: Input: An assignment problem (N, M), a strict linear preference profile R, and a
   lottery winner generator G.
2: M' ← M. N' ← N. A ← 0^{n×n}.
3: while M' ≠ ∅ do
4:   for each o ∈ M' do
5:     N_o ← {j ∈ N' | top(j, M') = o}.
6:     Run a lottery over N_o ≠ ∅ to pick an agent j_o = G(N_o) and allocate o, i.e., A_{j_o,o} ← 1.
7:   M' ← M' \ {o ∈ M' | N_o ≠ ∅}. N' ← N' \ ∪_{o∈M'} {j_o}.
8: return A
```

Example 3. We execute EBM on the instance in Figure 1. The table below shows for each round, which item each agent applies for, and a '/' represents the fact that an agent does not apply for any item since she has already been allocated one. The circled items represent the allocation of an item to the lottery winner.

Round \ Agent	1	2	3	4	5	6
1	a	b	c	c	c	c
2	/	/	/	e	e	d
3	/	/	/	/	f	/

- At round 1, agents 1 and 2 apply for a and b, respectively, and win them since they are the only applicants, while agents 3 - 6 apply for c and enter a lottery with equal chances of winning.
- If agent 3 wins c at round 1, then at round 2, agents 4 and 5 apply for e, while agent 6 applies for d alone and gets it.
- If agent 4 wins e at round 2, agent 5 applies for and gets f at round 3.

Then, EBM outputs the assignment A^* in Example 2. □

Notice that Algorithm 1 is a random algorithm that produces as output a deterministic assignment depending on the outcomes of the lotteries in each round. We use $\mathbb{E}(\text{EBM}(R))$ to refer to the expected outcome of EBM, which is a random assignment [2]. We prove that EBM satisfies ex-post FERI (ep-FERI), sd-WEF, and sd-WSP in Theorem 1. Here we recall that a random assignment satisfies ep-FERI (ex-post favoring-eagerness-for-remaining-items) if it is a convex combination of FERI deterministic assignments. **All the missing proofs can be found in Appendix B.**

Theorem 1. *EBM satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI), sd-weak-envy-freeness (sd-WEF), strong equal treatment of equals (SETE), and sd-weak-strategyproofness (sd-WSP).*

Proof sketch. Given any profile R, let $P = \mathbb{E}(\text{EBM}(R))$. For convenience, we refer to each possible execution of EBM, i.e., each way in which lottery winners are picked, as a possible *world* below.

(ep-FERI) Let $A = \text{EBM}(R)$. We show that the following two conditions hold for each $r \geq 1$:

(1) the set of items assigned at each round r of Algorithm 1, i.e., $\{o \in M' | N_o \neq \emptyset\}$, are exactly those in $T_{A,r}$, and

(2) the assignment A allocates every item $o \in T_{A,r}$ to an agent who ranks o as the top item in the set of remaining items M', i.e. an agent in $\{j \in N' | top(j, M') = o\}$.

When $r = 1$, we obtain condition (1) trivially with $M' = M$ and $N' = N$. Condition (2) holds because at the beginning of round 1, we have that for any $o \in T_{A,1}$, $o = top(j_o, T_{A,1})$ with $j_o = A^{-1}(o)$ by Line 6 of Algorithm 1, which means that $N_o \neq \emptyset$ and o is assigned to j_o who ranks o as the top item among M. Before round 2, by Line 7, every such item o and its winner j_o are removed from M' and N' respectively . With the updated M' and N', we can obtain the two conditions hold for $r = 2$ with a similar analysis. The proof follows by repeating a similar argument at each subsequent round. In this way we see that A is FERI, and therefore P is ep-FERI.

(sd-WEF) For any pair of agents j, k with $P_k \succeq_j^{sd} P_j$, we show that the following two conditions hold at any round r during the execution of Algorithm 1, where k has not been allocated an item yet:

(1) if j applies for o, then k also applies for o, and

(2) if j gets some item o at round $r' < r$, then k applies for the item that j ranks highest among remaining items.

Condition (1) shows that at any round where both agents are unassigned, they apply for the same item and therefore have the same chance to win it. Condition (2) shows that if j gets an item at an earlier round, then k applies to the item j would have applied to had k been allocated an item in an earlier round. The proof proceeds by comparing the probabilities of the worlds in which j and k get o respectively, and shows that they are equal for every item o, considered one by one according to the preference order $\succ_j$, from which it follows that $P_k = P_j$.

2. Here, $\mathbb{E}(\text{EBM}(R))$ is not a random variable. It is a bistochastic matrix that may correspond to more than one distribution over deterministic assignments.

(SETE). The proof involves comparing the probabilities that agent j and k get each $o \in U(\succ_{j,k}, o_m)$.

Case 1: First, for the world w where j gets o at round r while k gets $o' \in U(\succ_{j,k}, o_m)$ at round r', we can find out another world w' where only j and k swap their items. It follows that $Pr(w) = Pr(w')$ since the other lotteries keep the same as w.

Case 2: Then for the worlds W_j where j gets o at round r and k does not get items in $\succ_{j,k}$, we can also construct another set of worlds W_k such that: k gets o at round r, and j participates in lotteries instead from round $r+1$ to the last round that k applies for items in $\succ_{j,k}$. It follows that $Pr(W_j) = Pr(W_k)$.

With the fact that $Pr(w) = Pr(w')$ holds for Case 1 and $Pr(W_j) = Pr(W_k)$ for Case 2, it follows that $p_{j,o} = p_{k,o}$ for any o. This completes the proof.

(sd-WSP) Let $R' = (\succ'_j, \succ_{-j})$ be the profile when agent j misreports her preferences as $\succ'_j$, $Q = \text{EBM}(R')$, and assume that $Q_j \succeq^{sd}_j P_j$. The proof proceeds by considering each item o according to the order $\succ_j$, and shows that if j applies for o at round r in some world w for $\text{EBM}(R)$, then j also applies for o at round r in any world with the lotteries and winners before round r identical to those of w for $\text{EBM}(R')$. This means that despite misreporting, j applies for the same items as she does when truthfully reporting her preferences, and therefore j has the same probability to win each item. It follows that $p_{j,o} = q_{j,o}$ for each $o \in M$, and therefore, that if $Q_j \succeq^{sd} P_j$, then $Q_j = P_j$. □

4.2 Ep-FERI and Adaptive Boston Mechanism

We now show that not only is every member of adaptive Boston mechanism (ABM, Alcalde, 1996) guaranteed to output an ep-FERI assignment, but also that every ep-FERI assignment can be computed by some member of ABM, meaning that the output of EBM must also be the output of some member of ABM which depends on the instance of the assignment problem. As we show in Remark 6, although EBM appears similar to ABM, EBM does not belong to the family of ABM mechanisms.

Each algorithm in ABM (Algorithm 2) is specified by a probability distribution π over the priority orderings of agents, denoted ABM^π, and it computes an assignment as follows. First, a priority order $\rhd$, a strict linear order over N is picked according to the probability distribution π. Then, items are allocated to agents in multiple rounds. In each round, each unsatisfied agent applies for a remaining item that she is most eager for. Each remaining item o, if it has applicants, is assigned to the agent j_o who is ranked highest in $\rhd$ among all the applicants. At the end of each round, every such item o and the corresponding j_o are removed from M and N, respectively.

Theorem 2 shows that ep-FERI characterizes the family of ABM algorithms. Throughout, we will use $\pi(\rhd)$ to denote the probability of a priority order $\rhd$ according to the distribution π. If $\pi(\rhd) = 1$ for a certain $\rhd$, we will use $\text{ABM}^\rhd$ to refer to the corresponding algorithm for convenience. We also use $\mathbb{E}(\text{ABM}^\pi(R))$ to refer to the expected outcome of ABM^π.

Theorem 2. *Given a profile R, a random assignment P satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI) if and only if there exists a probability distribution π over all the priorities such that $P = \mathbb{E}(ABM^\pi(R))$.*

Algorithm 2 Adaptive Boston mechanism (ABM, Alcalde, 1996)

1: **Input:** An assignment problem (N, M), a strict linear preference profile R, a probability distribution π over all priority orderings of agents.
2: $M' \leftarrow M$. $N' \leftarrow N$. $A \leftarrow 0^{n\times n}$.
3: Randomly choose a priority order $\rhd$ according to π.
4: **while** $M' \neq \emptyset$ **do**
5:   **for** each $o \in M'$ **do**
6:     $N_o \leftarrow \{j \in N' \mid top(j, M') = o\}$.
7:     Allocate o to agent j_o which is ranked highest in $\rhd$ among N_o, i.e., $A_{j_o,o} \leftarrow 1$.
8:   $M' \leftarrow M' \setminus \{o \in M' \mid N_o \neq \emptyset\}$. $N' \leftarrow N' \setminus \cup_{o\in M'}\{j_o\}$.
9: **return** A

Proof. **(Satisfaction)** The proof is similar to proving EBM satisfies ep-FERI, and is provided in Appendix B for the sake of completeness.

(Uniqueness) Consider an arbitrary random assignment P satisfying ep-FERI for a preference profile R. Then, P can be decomposed into a set $\mathcal{A}' \subseteq \mathcal{A}$ of deterministic assignments satisfying FERI with positive probability, i.e., $P = \sum_{A_i\in\mathcal{A}'} \alpha_i * A_i$, where $\alpha_i > 0$.

Consider any $A \in \mathcal{A}'$. Since A satisfies FERI, by Definition 2, there exist non-empty sets $T_{A,1}, \ldots, T_{A,K}$, such that for each $r \in \{1, \ldots, K\}$, $T_{A,r} = \{o \in M : o = top(j, M \setminus \bigcup_{r'<r} T_{A,r'})\}$ for some $j \in N$ with $A(j) \notin \bigcup_{r'<r} T_{A,r'}$.

Consider any priority ordering $\rhd$ where for any $r', r \in \{1, \ldots, K\}$ with $r' < r$, and any pair of items $o' \in T_{A,r'}$ and $o \in T_{A,r}$, it holds that agent $A^{-1}(o')$ has higher priority than $A^{-1}(o)$, denoted as $A^{-1}(o') \rhd A^{-1}(o)$. It is easy to see that since A is deterministic, and every agent receives exactly one item, at least one such priority ordering always exists.

Let $B = \text{ABM}^{\rhd}(R)$. We claim that at any round r during the execution of $\text{ABM}^{\rhd}(R)$, every item in $o \in T_{A,r}$ is allocated to $A^{-1}(o)$, i.e., $B^{-1}(o) = A^{-1}(o)$. It is easy to see that the claim is true for $r = 1$. Since A satisfies FERI, for any item $o \in T_{A,1}$, which is the set of items that are ranked on top by some agent, $A^{-1}(o)$ ranks o as her top item, i.e., $A^{-1}(o) \in N_o = \{j \in N \mid top(j, M) = o\}$, and therefore she applies for o at round 1. Due to the construction of $\rhd$, $A^{-1}(o)$ must have the highest priority among N_o and obtain item o, i.e., $B^{-1}(o) = A^{-1}(o)$.

Now, assume that it holds that at any round $r' < r$, every $o' \in T_{A,r'}$ is allocated to $A^{-1}(o')$, i.e., $B^{-1}(o') = A^{-1}(o')$. We show that at round r, any $o' \in T_{A,r}$ is allocated to $A^{-1}(o)$, i.e., $B^{-1}(o) = A^{-1}(o)$. Assume for the sake of contradiction that there exists an item $o \in T_{A,r}$, such that $B^{-1}(o) = k \neq j = A^{-1}(o)$. By our assumption about rounds $r' < r$, both j and k have not been assigned an item in an earlier round by $\text{ABM}^{\rhd}(R)$. Notice that by Line 6 of Algorithm 2, $top(k, M \setminus \bigcup_{r'<r} T_{A,r'}) = o$, since every item in $\bigcup_{r'<r} T_{A,r'}$ is allocated in an earlier round by our assumption. Also, since A is FERI, we also have that $top(j, M \setminus \bigcup_{r'<r} T_{A,r'}) = o$. Therefore, both j and k apply for item o during round r of $\text{ABM}^{\rhd}(R)$. Then, it must hold that $k \rhd j$ since k is assigned o in round r of the execution of $\text{ABM}^{\rhd}(R)$.

However, by the construction of $\rhd$ and the assumption that $j = A^{-1}(o)$, $k \rhd j$ implies that there exists some $r^* \leq r$ such that k gets an item in T_{A,r^*}. Then, there must exist some item $o^* \in T_{A,r^*}$ such that $k = A^{-1}(o^*)$. It also means that $o^* \neq o = B(k)$, and therefore $B^{-1}(o^*) \neq k = A^{-1}(o^*)$, a contradiction to our assumption that $B^{-1}(o') = A^{-1}(o')$ for every $o' \in T_{A,r'}$ with $r' < r$. Thus, by induction, it holds that $B = A$.

We have shown that for every deterministic assignment $A_i \in \mathcal{A}'$, there exists a priority order $\rhd_i$ such that the output of $\text{ABM}^{\rhd_i}(R) = A_i$. Then, for the ep-FERI assignment $P = \sum_{A_i \in \mathcal{A}'} \alpha_i A_i$, is the output of a member of the family of ABM algorithms specified by the probability distribution π over priority orderings where for any $A_i \in \mathcal{A}'$, $\pi(\rhd_i) = \alpha_i$, i.e., $\text{ABM}^{\pi}(R) = P$. This completes the proof. □

Remark 6. Mennle and Seuken (2021) proved that ABM^{π} is sd-WSP if $\pi(\rhd) > 0$ for every $\rhd$. However, as we show in Appendix A.5, there is no such a distribution π that $\text{EBM}(R) = \text{ABM}^{\pi}(R)$ for every preference profile R, meaning that EBM is not a member of ABM. Therefore, the sufficient condition proposed by Mennle and Seuken (2021) for checking if a member of ABM satisfies sd-WSP is not applicable for the proof of EBM satisfying sd-WSP.

5. Ex-ante Favoring Eagerness for Remaining Items

In this section, we design a family of mechanisms that satisfy the ex-ante variant of FERI, **ex-ante favoring-eagerness-for-remaining-items (ea-FERI)**, which implies ex-post FCM (Remark 7), ep-PE, and sd-E (Proposition 3). We further prove that a member of this family of mechanisms satisfies sd-WEF and SETE.

Intuitively, a random assignment satisfies ea-FERI if the shares of every remaining item are only distributed among the agents who are most eager for it unless every such agent's demand has been satisfied by better items.

Definition 3 (ea-FERI). *Given a random assignment P, we defined $M_{P,0} = \emptyset$ and for each item $o \in M$, $E_{P,0}(o) = \emptyset$. Then, for each $r \in \{1, 2, \dots\}$, we define*
(i) *the set of items with positive supply after excluding the shares owned by agents in $\bigcup_{r'<r} E_{P,r'}(o)$, $M_{P,r} = \{o \in M : \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o} < 1\}$, and*
(ii) *for each item $o \in M_{P,r}$, $E_{P,r}(o) = \{j \in N : o = top(j, M_{P,r})\}$ to be the set of all agents eager for it.*

A random assignment P satisfies **ex-ante favoring-eagerness-for-remaining-items (ea-FERI)** *if for every $r \in \{1, 2, \dots\}$ and item $o \in M_{P,r}$, it holds for any agent $j \in N$ that if there exists an $r' < r$ such that $j \in E_{P,r'}(o)$, agent j is satisfied by items weakly preferred to o, i.e., $\sum_{o' \in U(\succ_j, o)} p_{j,o'} = 1$.*

Ea-FERI is a natural ex-ante extension of FERI for random assignments. Indeed, it is easy to see that for deterministic assignments, ea-FERI is equivalent to FERI. We also note that ea-FERI and ep-FERI do not imply each other. Please see Appendix A.1 for more details.

Remark 7. Ea-FERI implies ex-post FCM. Specifically, in any ea-FERI assignment P, for item $o \in M$ which is ranked top by some agents, i.e., $o \in M_{P,1}$ and $E_{P,1}(o) \neq \emptyset$, we have that $o \notin M_{P,r}$ with $r > 1$ by ea-FERI and the fact that $\sum_{o' \in U(\succ_j, o)} = 0$ for $j \in E_{P,1}(o)$, and it follows that every such o is allocated to one of the agents in $E_{P,1}(o)$ who rank o top among all the items in every deterministic assignment that constitutes the convex combination for P.

Proposition 3 below shows that ea-FERI implies sd-E.

Proposition 3. [ea-FERI⇒sd-E, sd-E⇏ea-FERI] *A random assignment satisfying ex-ante favoring-eagerness-for-remaining-items (ea-FERI) also satisfies sd-efficiency (sd-E), but not vice versa.*

Proof. **(ea-FERI ⇒ sd-E)** Assume for the sake of contradiction that P is ea-FERI, but not sd-E. By assumption and Lemma 2 (in Appendix A.3), we can find a set of agents $\{j_1, j_2, \cdots, j_h\}$ and items $M^* = \{o_1, o_2, \cdots, o_h\}$ such that $o_{i+1 \pmod h} \succ_{j_i} o_i$ with $p_{j_i,o_i} > 0$ with $i \leq h$. For each o_i, let r_i be the round where j_i consumes it. We note that r_i is unique for each o_i, because if item o_i is consumed by j_i at round r_i, then either item o_i is consumed until it is exhausted, or agent j_i is satisfied and therefore does not participate in consuming items at any subsequent round. Without loss of generality, let $o_{i'} = \arg\min_{o_i \in M^*} r_i$. Then we have that all the items in M^* are available at round $r_{i'}$ and $j_{i'} \in N_{o_{i'}}$, which means that $top(j_{i'}, M') = o_{i'}$ where M' is the set of all the available items at that round in Algorithm 3. Since $M^* \subseteq M'$, we have that $o_{i'} \succ_{i'} o_{i'+1}$, a contradiction to the assumption.

(sd-E⇏ ea-FERI) For the following preference profile R, the assignment P is the outcome of PS which satisfies sd-E:

$\succ_1$: $a \succ_1 b \succ_1 c$,
$\succ_2$: $a \succ_2 c \succ_2 b$,
$\succ_3$: $b \succ_3 a \succ_3 c$.

Assignment P

	a	b	c
1	1/2	1/4	1/4
2	1/2	0	1/2
3	0	3/4	1/4

We see that $E_{P,1}(b) = \{3\}$, $\sum_{o \in U(\succ_3, b)} = 3/4 < 1$, and $b \in M_{P,2}$, which violates ea-FERI. □

5.1 UPRE Satisfies ea-FERI, sd-WEF, and SETE

We propose the family of probabilistic respecting eagerness mechanisms (PRE) defined in Algorithm 3, and show that the uniform probabilistic respecting eagerness mechanism (UPRE), a member of PRE, satisfies the desirable fairness notions sd-WEF and SETE (Theorem 3). We also prove that not only does every PRE mechanism satisfy ea-FERI, but also every ea-FERI assignment must be the output of some member of PRE (Theorem 4). Therefore, UPRE satisfies ea-FERI, sd-WEF, and SETE.

Each member of the PRE family of mechanisms is specified by a parameter $\omega = (\omega_j)_{j \in N}$, denoted PRE$_\omega$. Each ω_j is an eating speed function which maps each time instance t to a rate of consumption for agent j such that $\int_0^1 \omega_j(t)\mathrm{d}t = 1$, $\omega_j(t) \geq 0$ for $t \in [0, 1]$, and $\omega_j(t) = 0$ for $t > 1$, i.e., agent j consumes exactly one unit of item during one unit of time. At the beginning of execution, we set $s(o) = 1$ to refer to the supply of item o, and set $t_j = 0$ to indicate the elapsed time each agent j has spent on consumption. At each round r, each agent j determines $top(j, M')$, her top item among the set M' consisting of every item o which remains available, i.e., all items with $s(o) > 0$. For each item $o \in M'$, N_o is the set of agents for whom o is the top item. All the agents in N_o consume o together for $\gamma_\omega(N_o,(t_j)_{j\in N},s(o))$ units of time. For any $N'\subseteq N$, elapsed consumption times $(t_j)_{j\in N}$, and supply s, we define:

$$\begin{aligned}\gamma_\omega(N',(t_j)_{j\in N},s) = \min(&\{\rho \mid \sum_{k\in N'}\int_{t_k}^{t_k+\rho}\omega_k(t)\mathrm{d}t = s\}\\ \cup &\{\rho\in[0,1] \mid \sum_{k\in N'}\int_{t_k}^{t_k+\rho}\omega_k(t)\mathrm{d}t = \sum_{k\in N'}\int_{t_k}^{1}\omega_k(t)\mathrm{d}t\}),\end{aligned} \tag{2}$$

Notice that for any agent j, $\int_{t_j}^1 \omega_j(t)\mathrm{d}t$ refers to her remaining demand. In words, Eq (2) requires that agents in N_o stop their consumption when either the supply of o is exhausted, or all of them are satisfied. Then the amount that agent j consumes at this round is the shares of o she gets in the final outcome, and we update the supply $s(o)$ and the elapsed time t_j.

Algorithm 3 Probabilistic respecting eagerness (PRE)

Input: An assignment problem (N,M), a strict linear preference profile R, a collection of eating functions $\omega=(\omega_j)_{j\in N}$.
$M'\leftarrow M$, $P\leftarrow 0^{n\times n}$, $s(o)\leftarrow 1$ for every o, and $t_j\leftarrow 0$ for every j.
while $M'\neq\emptyset$ **do**
  $N_o\leftarrow\{j\in N \mid top(j,M')=o\}$.
  for each item $o\in M'$ **do**
    Agents in N_o **consume** o.
      6.1: $\rho_o\leftarrow\gamma_\omega(N_o,(t_j)_{j\in N},s(o))$.
      6.2: For each $j\in N_o$, $p_{j,o}\leftarrow\int_{t_j}^{t_j+\rho_o}\omega_j(t)\mathrm{d}t$.
  $s(o)\leftarrow s(o)-\sum_{k\in N_o}\int_{t_k}^{t_k+\rho_o}\omega_k(t)\mathrm{d}t$.
  For each $j\in N_o$, $t_j\leftarrow t_j+\rho_o$.
  $M'\leftarrow M'\setminus\{o\in M' \mid s(o)=0\}$.
return P

We define the *uniform probabilistic respecting eagerness* mechanism (UPRE) to be the member of PRE (defined in Algorithm 3) in which every agent consumes items uniformly at the same speed during the time period $[0,1]$. We prove in Theorem 3 that UPRE satisfies sd-WEF and SETE. As a member of PRE, UPRE also satisfies ea-FERI by Theorem 4. We demonstrate the execution of UPRE in Example 4, and also show that UPRE runs in polynomial time in Appendix A.4.

Definition 4 (UPRE). *The* **uniform probabilistic respecting eagerness mechanism** *(UPRE) is a member of PRE, where every agent eats at a uniform eating speed of one unit of item per one unit of time, i.e., for each $j\in N$,*

$$\omega_j(t)=\begin{cases}1, & t\in[0,1],\\ 0, & t>1.\end{cases} \tag{3}$$

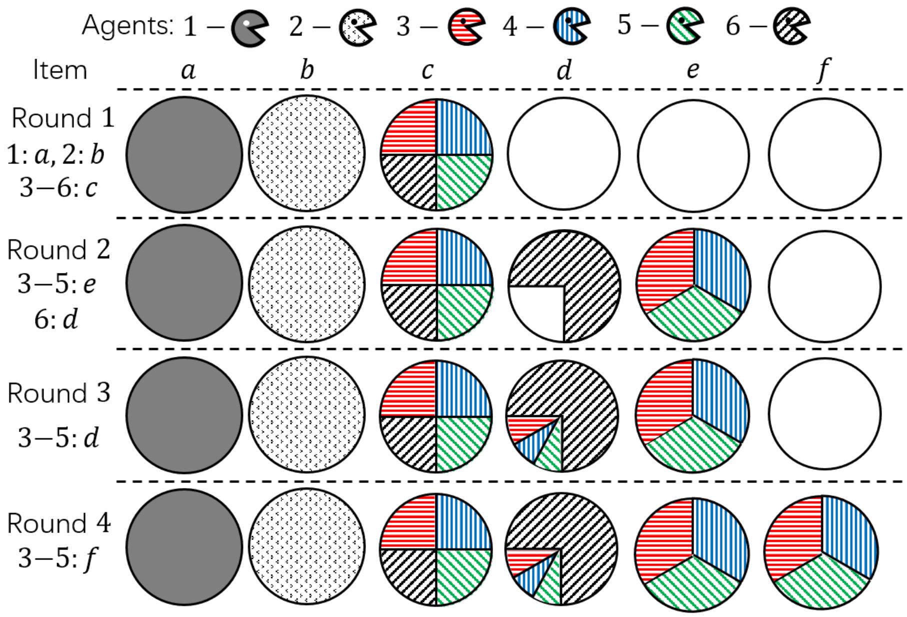


Figure 2: An example of the execution of a member of PRE (UPRE).

Example 4. In Figure 2 and the discussion below, we illustrate the execution of UPRE applied to the profile in Figure 1.

- At round 1, agents 1 and 2 consume a and b respectively, and other agents consume c. After consumption, agents 1 and 2 fully get a and b, respectively, and the other four agents each get 1/4 units of c. The supply of each consumed item is updated as $s(a) = s(b) = s(c) = 0$.

- At round 2, agents 3 - 5 consume e and each get 1/3 units such that $s(e) = 0$, and agent 6 consumes d till satisfied and gets 3/4 units, leaving $s(d) = 1/4$.

- At round 3, agents 3 - 5 consume d and each get 1/12 units each such that $s(d) = 0$.

- At round 4, agents 3 - 5 consume f and get 1/3 units each.

The following assignment is the final output of this procedure:

	a	b	c	d	e	f
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3-5	0	0	1/4	1/12	1/3	1/3
6	0	0	1/4	3/4	0	0

Theorem 3. *UPRE satisfies sd-weak-envy-freeness (sd-WEF) and strong equal treatment of equals (SETE).*

Proof sketch. (sd-WEF) Given a profile R, let $P = \text{UPRE}(R)$. For agents j and k with $P_k \succeq_j^{sd} P_j$, we show that at each round r, agent k applies for the same item o as j because otherwise,

(1) if o is consumed to exhaustion by j and other agents, then agent k can never obtain shares of o at later rounds, and

(2) if j is satisfied upon consuming o, i.e., satisfied by shares of items in $U(j, o)$, then agent k must get shares of an item ranked below o (according to $\succ_j$).

Both cases above imply that $P_j \succeq_j^{sd} P_k$ while $P_j \neq P_k$, a contradiction. Further, this means that at the end of each round r, both agents have the same elapsed consumption time, i.e., $t_k = t_j$, and therefore both agents consume the same shares of items since they have the same eating speeds. Together, this means $p_{j,o} = p_{k,o}$ for any o they have consumed, and it follows that $P_k = P_j$.

(SETE) We show that before consuming items not in $\succ_{j,k}$, agent j and k consume the same item o at each round r. If agent k consumes $o' \neq o$, then let $o \succ o'$ without loss of generality, which means that k does not consume the top item at round r, a contradiction to the execution of UPRE. Since the agents consume the same item at a round for the same length of time, we obtain that $p_{j,o} = p_{k,o}$ for each o appearing in $\succ_{j,k}$. □

Theorem 4. *Given a profile R, a random assignment P satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI) if and only if there exists an eating speed function ω such that $P = PRE_\omega(R)$.*

Proof sketch. (Satisfaction) Let $P = \text{PRE}_\omega(R)$ where ω is any collection of eating functions. We show that at every round r,

(1) $M_{P,r}$ and $E_{P,r}(o)$ are exactly the sets of items with remaining shares and agents who consume o, respectively, and

(2) if o is available for later rounds, then all agents in $E_{P,r}(o)$ are satisfied.

We see that (1) is trivially true for round 1. Agents in $E_{P,1}(o)$ are those in the set N_o on Line 4 of Algorithm 3. Also, they consume o, and stop as soon as either o is exhausted, or they are satisfied according to the consumption process, which means (2) is true for round 1. By Lines 7 and 9, we see that $s(o)$ is updated to ensure that M' does not contain item o' with $\sum_{k\in E_{P,1}(o')} p_{k,o'} = 0$. With the updated M', the condition (1) holds for $r = 2$ trivially, and we can prove condition (2) according to the selection of items to be consumed at round 2. The proof follows from an inductive argument along similar lines.

(Uniqueness) For any ea-FERI assignment Q, we find out a member of PRE with the following eating function such that $P = \text{PRE}_\omega(R)$ coincides with Q.

$$\omega_j(t) = \begin{cases} n \cdot q_{j,o}, & t \in [\frac{r-1}{n}, \frac{r}{n}], \text{ where } r = \min(\{\hat{r} \mid j \in E_{Q,\hat{r}}(o)\}), \\ 0, & \text{others.} \end{cases}$$

Constructing such a eating function is to ensure the following conditions successively for each round r in the execution of $\text{PRE}_\omega(R)$:

(1) items with remaining shares are the same, i.e., $M_{Q,r} = M_{P,r}$, and each agent is eager for the same item, i.e., $E_{Q,r}(o) = E_{P,r}(o)$;

(2) for any unsatisfied agent j which is going to consume item o at round r, she starts consumption at time $t_j = (r-1)/n$, and exactly consumes for $\rho_o = 1/n$ long; and

(3) after the consumption, for any $o \in M_{Q,r}$, each agent j eager for it obtains $q_{j,o}$ units of shares of item o, i.e., $p_{j,o} = q_{j,o}$.

It is easy to see that condition (1) above holds for $r = 1$. As for condition (2) when $r = 1$, we have $t_j = (r-1)/n = 0$, which is initially set on Line 2 of Algorithm 3, and $N_o = E_{P,1}(o) = E_{Q,1}(o)$ by Line 4. Then $\sum_{j \in N_o} p_{j,o} = s(o) = 1$ because otherwise, o has remaining shares for the later round while agents in $E_{Q,1}(o)$ are not satisfied, a violation to Q satisfying ea-FERI. By construction of ω, $\sum_{j \in E_{Q,1}(o)} \int_0^{1/n} \omega_j(t)\mathrm{d}t = \sum_{j \in E_{Q,1}(o)} q_{j,o} = 1$, and we can infer that the consumption time for o $\rho_o = 1/n$.

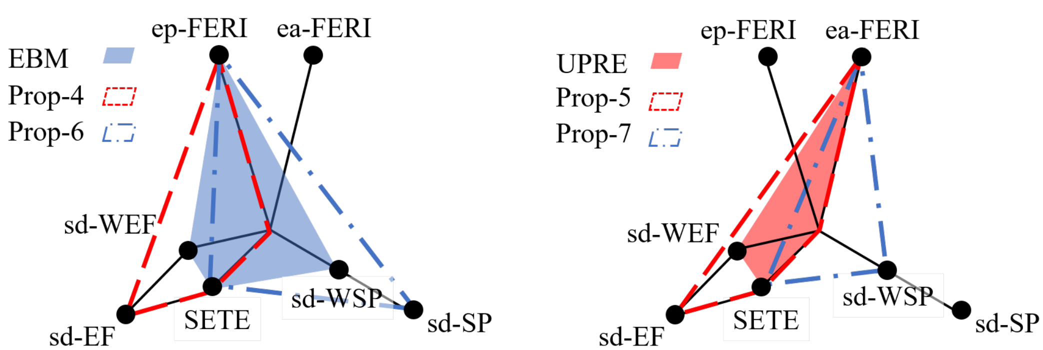
For $r = 1$, conditions (1) and (2) easily lead to condition (3). Moreover, with conditions (1) and (2) established for r, we can obtain them for $r+1$ following a similar analysis, and therefore condition (3) holds for $r+1$. In this way, we have $p_{j,o} = q_{j,o}$ for any $o \in M_{Q,r}$ with any $r \geq 1$. It implies that $Q = P$, which completes the proof. $\square$

With Proposition 3 and Theorem 4, we can conclude that every member of PRE also satisfies sd-E and ep-PE (Corollary 2).

Corollary 2. *For any collection of eating speed functions ω, PRE$_\omega$ satisfies ex-post Pareto-efficiency (ep-PE) and sd-efficiency (sd-E).*

6. Impossibility Results

A natural question that follows the positive results in Sections 4.1 and 5.1, is whether it is possible to design ep-FERI or ea-FERI mechanisms which provide a stronger guarantee of either efficiency, fairness, or strategyproofness. In this section, we show that it is impossible to design mechanisms that satisfy certain combinations of properties involving stronger variants of either fairness or strategyproofness. We summarize these results in the form of a spider graph in Figure 3. Each arm radiating away from the center of the graph represents a type of efficiency, fairness, or strategyproofness property. Each node represents a property, and the farther a node is away from the center of the graph, the stronger the property it represents. The nodes at the corners of an unshaded polygon, whose edges are broken lines, represent a combination of properties that are impossible to satisfy using any assignment mechanism. The properties satisfied by a mechanism are represented by the nodes at the corner of a shaded polygon.



(a) ep-FERI mechanisms (b) ea-FERI mechanisms

Figure 3: The impossibility results for FERI mechanisms.

Propositions 4 and 5 show that we cannot improve the fairness guarantee of ep-FERI or ea-FERI mechanisms from sd-WEF to sd-EF.

Proposition 4. *No mechanism simultaneously satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI) and sd-envy-freeness (sd-EF).*

Proof. We prove it with the following preference profile R. By FERI, one of agents in $\{1, 2, 3\}$ gets a, and agent 4 must get b. If assignment Q satisfies ep-FERI and SETE which is implied by sd-EF, then it is in the following form.

Preference Profile R

$\succ_1$: $a \succ_1 c \succ_1 b \succ_1 d$,
$\succ_2$: $a \succ_2 c \succ_2 b \succ_2 d$,
$\succ_3$: $a \succ_3 b \succ_3 c \succ_3 d$,
$\succ_4$: $b \succ_4 a \succ_4 d \succ_4 c$.

Assignment Q

	a	b	c	d
1	1/3	0	?	?
2	1/3	0	?	?
3	1/3	0	?	?
4	0	1	0	0

Then we do not have $Q_3 \succeq_3^{sd} Q_4$ since $\sum_{o' \in U(\succ_3, b)} q_{3,o'} < \sum_{o' \in U(\succ_3, b)} q_{4,o'}$, a contradiction to sd-EF. □

Proposition 5. *No mechanism simultaneously satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI) and sd-envy-freeness (sd-EF).*

As for strategyproofness, Proposition 6 shows with the weak fairness requirement of SETE, that sd-WSP cannot be improved to sd-SP for any ep-FERI mechanism, and Proposition 7 shows that even sd-WSP cannot be satisfied by ea-FERI mechanisms.

Proposition 6. *No mechanism simultaneously satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI), strong equal treatment of equals (SETE), and sd-strategy proofness (sd-SP).*

Proof. We prove it with the preference profile R in Proposition 4. If agent 3 misreports her preference as agent 4, i.e., $R' = (\succ_3', \succ_{-3})$ with $\succ_3' = \succ_4$, then one of agents 1 and 2 gets a, and one of agents 3 and 4 get b by FERI. For the remaining items $M' = \{c, d\}$, $top(1, M') = top(2, M') = c$ and $top(\succ_3', M') = top(4, M') = d$, which means that agent 1 (or 2) gets c when she does not get a, and agent 3 (or 4) gets d when she does not get b. Then the following assignment Q' is the only one satisfying ep-FERI and SETE for R'.

Preference Profile R'

$\succ_1$: $a \succ_1 c \succ_1 b \succ_1 d$,
$\succ_2$: $a \succ_2 c \succ_2 b \succ_2 d$,
$\succ_3'$: $b \succ_3 a \succ_3 d \succ_3 c$,
$\succ_4$: $b \succ_4 a \succ_4 d \succ_4 c$.

Assignment Q'

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

Comparing Q' with Q in Proposition 4 which is under the true preference R, we see that Q_3 does not dominate Q_3' on $U(\succ_3, b)$, which means that no ep-FERI and SETE mechanism can be sd-SP. □

Proposition 7. *No mechanism simultaneously satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI), strong equal treatment of equals (SETE), and sd-weak-strategy proofness (sd-WSP).*

Proposition 8 shows that the simultaneous satisfaction of ep-FERI and ea-FERI cannot be achieved given the fairness requirement SETE.

Proposition 8. *No mechanism simultaneously satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI), ex-ante favoring-eagerness-for-remaining-items (ea-FERI), and strong equal treatment of equals (SETE).*

7. Conclusion and Future Work

In this paper, we provide the first random mechanisms that are guaranteed to output ex-post FCM assignments simultaneously with other desirable properties of efficiency (ep-PE and sd-E), fairness (SETE and sd-WEF), and strategyproofness (sd-WSP) properties.

Our positive results expand the envelope for FCM and PE mechanisms along the dimensions of both fairness and strategyproofness as we illustrate in Figure 4, while our impossibility results, summarized in Figure 3, help define the limits of what may be possible. In Figure 4, similar to Figure 3, each arm radiating away from the center of the graph represents a type of property and each node represents a property. The nodes of each shaded polygon represent the properties satisfied by a mechanism. It also highlights an important open question for future work: Is there a mechanism that satisfies the combination of properties that are joined by the solid green line?

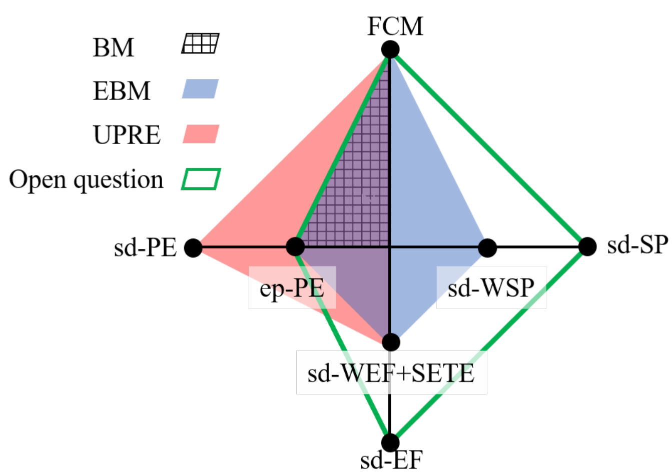


Figure 4: The properties of FCM mechanisms on PE, envy-freeness, and strategyproofness.

We hope that our results encourage the search for FCM and PE mechanisms which also satisfy other efficiency, fairness, and strategyproofness desiderata (like the "Open question" in Figure 4). For example, if FCM is deemed to be an indispensable property, an interesting open question is whether it is possible to design mechanisms that satisfy relaxations of one property, such as Pareto optimality, while satisfying stronger properties of fairness and strategyproofness, along the lines of similar work on constrained-optimal mechanisms in the school choice literature (Abdulkadiroğlu & Sönmez, 2003; Abdulkadiroğlu et al., 2017). Apart from notions of fairness based on the interpersonal comparison like envy-freeness, the question of how to design mechanisms which satisfy FCM together with alternate fairness desiderata based on the notions such as egalitarianism (e.g. minimizing the ranking of the item allocated to the worst-off agent) remains unexplored. Whether our impossibility results may be circumvented under natural domain restrictions, such as on the domain of preferences (e.g. approval or dichotomous preferences), is another interesting open question for future work.

Another natural direction for future work concerns generalizations of the assignment problem. An immediate question is whether our results may be extended to settings where ties or incomparability between items are allowed in agents' preferences (Katta & Sethuraman, 2006; Wang et al., 2020). When agents may demand multiple items (Heo, 2014; Kojima, 2009; Wang et al., 2020; Budish, 2011), the question of whether a mechanism can satisfy a natural extension of FCM together with other desiderata also remains open.

Acknowledgments

XG acknowledges the National Key R&D Program of China under Grant 2021YFF1201102 for support. LX acknowledges NSF #1453542, #1716333, #2106983 for support. YC acknowledges NSFC under Grants 62172016 and 61932001 for support. HW acknowledges NSFC under Grant 61972005 for support.

Appendix A. Additional Results

A.1 Relationships between Efficiency Properties

We summarize the efficiency properties we discuss in the main body with Figure 5.

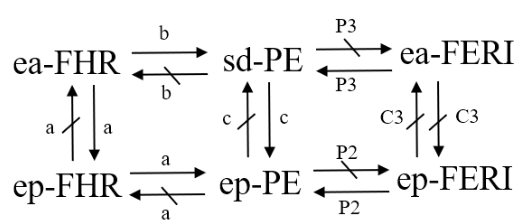


Figure 5: Relationship between ep-FERI, ea-FERI, ep-FHR, ea-FHR, ep-PE, and sd-E.

Note: The property X points to another property Y means that X implies Y. An arrow annotated with a, b, or c refers to a result due to Ramezanian and Feizi (2021), Chen et al. (2021), and Bogomolnaia and Moulin (2001), respectively, and an edge annotated with P refers to a Proposition in this paper.

Proposition 9. *A deterministic assignment satisfies favoring-eagerness-for-remaining-items (FERI) if and if only it satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI).*

Proof. In this proof, it will be useful to recall the definitions of the sets $T_{A,r}$, $M_{A,r}$, and $E_{A,r}(o)$ from Definitions 2 and 3: Given any deterministic assignment A, $T_{A,0} = M_{A,0} = E_{A,0}(o) = \emptyset$, and for each $r \in \{1, 2, \dots\}$,

$$T_{A,r} = \{o \in M : o = top(j, M \setminus \bigcup_{r'<r} T_{A,r'}) \text{ for some } j \in N \text{ with } A(j) \notin \bigcup_{r'<r} T_{A,r'}\}, \quad (4)$$

$$M_{A,r} = \{o \in M : \sum_{k \in \bigcup_{r'<r} E_{A,r'}(o)} A_{k,o} < 1\}, \quad (5)$$

$$E_{A,r}(o) = \{j \in N : o = top(j, M_{A,r})\} \text{ where } o \in M_{A,r}. \quad (6)$$

(FERI⇒ ea-FERI) For any A satisfying FERI, we show that for any $r \in \{1, 2, \dots\}$, the following two conditions hold:

Condition (1): $M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'}$, and

Condition (2): for any item $o \in M_{A,r^*}$ with $r^* > r$, $\sum_{\hat{o} \in U(\succ_j, o)} A_{j,\hat{o}} = 1$ for any $j \in E_{A,r}(o)$.

The proof proceeds by mathematical induction on the value of r.

Base Case. Condition (1) trivially holds for $r = 1$. We show that condition (2) holds by contradiction. Assume that there exist an item $o \in M_{A,r^*}$ with $r^* > 1$ and an agent $j \in E_{A,1}(o)$ such that $\sum_{\hat{o} \in U(\succ_j, o)} A_{j,\hat{o}} = 0$. Since $j \in E_{A,1}(o)$, we know that $o = top(j, M_{A,1})$ $= top(j, M)$ by Eq (6), which also means that $o \in T_{A,1}$ by Eq (4). By $o \in M_{A,r^*}$, we know that $\sum_{j \in \bigcup_{r' \leq r} E_{A,r'}(o)} A_{j,o} = 0 < 1$ by Eq (5), which also means that no agent $j' \in E_{A,1}(o) = \{k \in N \mid o = top(k, M)\}$ gets o, a contradiction to A satisfying FERI.

Induction Step. Supposing that conditions (1) and (2) hold for each of $r' = 1, \dots, r-1$, we prove that they also hold for r.

- Condition (1) holds for r. For any $r' < r$ and item $o' \in T_{A,r'}$, from the fact that A satisfies FERI, we know that $o' = top(A^{-1}(o'), M_{A,r'})$. It means that $A^{-1}(o') \in E_{A,r'}(o')$ and $\sum_{j \in \bigcup_{r'<r} E_{A,r'}(o')} A_{j,o'} = 1$, and therefore $o' \notin M_{A,r}$ by Eq (5). It is easy to prove the opposite direction that $o \in T_{A,r}$ for any $o \notin M_{A,r}$ with a similar argument. Together they lead to condition (1) for r.

- Condition (2) holds for r. For any item $o \in M_{A,r^*}$ with $r^* > r$, we have that $\sum_{j \in \bigcup_{r'<r^*} E_{A,r'}(o)} A_{j,o} < 1$ by Eq (5), and it means that $A_{j',o} = 0$ for any $j' \in E_{A,r'}(o)$ with $r' \leq r$ since A is deterministic. Moreover, we have that $o \notin T_{A,r}$; otherwise, there exists an agent j with $o = top(j, M_{A,r})$ by the fact that A satisfies FERI, and therefore $j \in E_{A,r}(o)$ by Eq (6), which means that $\sum_{k \in \bigcup_{r'<r^*} E_{A,r'}(o)} A_{k,o} = 1$, a contradiction to $r < r^*$ and $o \in M_{A,r^*}$ which implies that $\sum_{k \in \bigcup_{r'<r^*} E_{A,r'}(o)} A_{k,o} = 0$ by Eq (5). With $o \notin T_{A,r}$, for any $j \in E_{A,r}(o)$, i.e., $o = top(j, M_{A,r})$ by Eq (6), we have that $A(j) \in T_{A,r'}$ with some $r' < r$ by Eq (4). Again by condition (1) for r', we know that $A(j) = top(j, M_{A,r'}) \succ_j o$ since $o \in M_{A,r^*} \subseteq M_{A,r'}$, and therefore $\sum_{o' \in U(\succ_j, o)} A_{j,o'} = 1$, i.e. condition (2) for r.

With the mathematical induction above, we obtain that condition (2) holds for any r, and therefore the FERI assignment A satisfies ea-FERI.

(ea-FERI$\Rightarrow$ FERI) For any A satisfying ea-FERI, we show that for any $r \in \{1, 2, \dots\}$, the following two conditions hold:

(1) $M \setminus \bigcup_{r'<r} T_{A,r'} = M_{A,r}$, and

(2) $o = top(A^{-1}(o), M_{A,r})$ for every item $o \in T_{A,r}$.

The proof proceeds by mathematical induction on the value of r.

Base Case. Condition (1) trivially holds for $r = 1$. We show condition (2) holds by contradiction. If $top(A^{-1}(o), M_{A,1}) = top(A^{-1}(o), M) \neq o$, then $A^{-1}(o) \notin E_{A,1}(o)$ by Eq (6). Accordingly, we know that $\sum_{k \in E_{A,1}(o)} A_{k,o} = 0 < 1$, and therefore $o \in M_{A,2}$ by Eq (5), However, for any $j \in E_{P,1}(o) \neq \emptyset$, $\sum_{o' \in (\succ_j, o) A_{j,o'}} = A_{j,o} = 0 < 1$, a contradiction to A satisfying ea-FERI.

Induction Step. Supposing that condition (1) and (2) hold for each of $r' = 1, \dots, r-1$, we prove that they also hold for r.

- Condition (1) holds for r. For every $o \in M_{A,r}$, we have that $\sum_{j \in \bigcup_{r'<r} E_{A,r}(o)} A_{j,o} < 1$ by Eq (6). Since A is deterministic, it also implies that

$$\text{for any } j \in \bigcup_{r'<r} E_{A,r'}(o),\ A_{j,o} = 0. \tag{7}$$

It follows that $o \notin \bigcup_{r'<r} T_{A,r'}$; otherwise, we have that $o = top(j, M_{A,r'})$ where $j = A^{-1}(o)$ by condition (2) for some $r' < r$, which means that $\sum_{k \in \bigcup_{r'<r} E_{A,r'}(o)} A_{k,o} = 1$, a contradiction to Eq (7). It is easy to prove the opposite direction that $o \in M_{A,r}$ for any $o \notin \bigcup_{r'<r} T_{A,r'}$ with a similar argument. Together they mean that $M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'}$, i.e., condition (1) holds for r.

- Condition (2) holds for r. For any item $o \in T_{A,r}$, assume for the sake of contradiction that $o \neq top(A^{-1}(o), M_{A,r})$, which means that $A(j) \neq o$ for any j with $o = top(j, M_{A,r})$, i.e., $j \in E_{A,r}(o)$ by Eq (6). Since $o \in T_{A,r}$, then by condition (1) for r which we have just proved, we have that $o \in M_{A,r}$. Since A is deterministic, we also have the following equation by Eq (5):

$$\sum_{j \in \bigcup_{r'<r} E_{A,r'}(o)} A_{j,o} = 0 < 1. \tag{8}$$

With the assumption and Eq (8), we have that $\sum_{j \in \bigcup_{r'<r+1} E_{A,r'}(o)} A_{j,o} = 0 < 1$, which means that $o \in M_{A,r+1}$ by Eq (5). Again by $o \in T_{A,r}$, we know that there exists $j' \in E_{A,r}(o)$, i.e., $o = top(j', M_{A,r})$, with $A(j') \notin \bigcup_{r'<r} T_{A,r'}$ by Eq (4). It also means that $A(j') \in M_{A,r}$ by condition (1) for r. By the assumption, we know that $A(j') \neq o = top(j', M_{A,r})$ and therefore $o \succ_{j'} A(j')$. Then we obtain that $\sum_{\hat{o} \in U(\succ_{j'}, o)} p_{j',\hat{o}} = 0$ with $o \in M_{A,r+1}$ and $j' \in E_{A,r}(o)$, a contradiction to the assumption that A satisfies ea-FERI. By the contradiction, we have that $o = top(A^{-1}(o), M_{A,r})$ for every item $o \in T_{A,r}$, i.e., condition (2) holds for r.

With the mathematical induction above, we establish that condition (2) holds for any r, which means that the ea-FERI assignment A satisfies FERI. □

Corollary 3. [ep-FERI$\not\Rightarrow$ ea-FERI, ea-FERI$\not\Rightarrow$ ep-FERI] *A random assignment satisfying ex-ante favoring-eagerness-for-remaining-items (ea-FERI) do not need to satisfy ex-post favoring-eagerness-for-remaining-items (ep-FERI), and vice versa.*

Proof. **(ep-FERI$\not\Rightarrow$ ea-FERI)** It follows from the fact that EBM satisfies ep-FERI (Theorem 1), but EBM does not satisfy ea-FERI (Proposition 15).

(ea-FERI$\not\Rightarrow$ ep-FERI) It follows from the fact that UPRE satisfies ea-FERI (Theorem 4), but UPRE does not satisfy ep-FERI (Proposition 18). □

Definition 5. (Abraham et al., 2007) *Given an instance, a deterministic assignment A satisfies popularity (POP) if there does not exist another A' such that $|\{j \in N : A'(j) \succ_j A(j)\}| > |\{j \in N : A(j) \succ_j A'(j)\}|$.*

Lemma 1. (Abraham et al., 2007) *An assignment A is popular if and only if every agent is assigned* (1) *her most preferred item, or* (2) *the most preferred item which is not ranked first by any agent.*

Proposition 10. *A popular (POP) deterministic assignment satisfies favoring-eagerness-for-remaining-items (FERI), but not vice versa.*

Proof. **(POP $\Rightarrow$ FERI)** By Lemma 1, an assignment A is popular if and only if for every agent j, $A(j) = top(j, M)$ or $A(j) = top(j, M')$ where $M' = \{o \in M | o = top(k, M)$ for any $k \in N\}$.

By Definition 2, we observe that:

(i) For $r = 1$, $T_{A,1} = \{top(j, M) | j \in N\}$ for any $j \in N$. By the characterization above, any $o \in T_{A,1}$ is assigned to one of the agents who rank it as their top item in M, i.e. $o = top(A^{-1}(o), M,)$, which satisfies FERI for $r = 1$.

(ii) For $r = 2$, $T_{A,2} = \{o \in M : o = top(j, M \setminus T_{A,1}), A(j) \notin T_{A,1}\}$. We note that $M' = M \setminus T_{A,1}$ because by (i), any o satisfies $o \in T_{A,1}$ if and only if $o = top(k, M)$ for some agent k. According to the characterization of POP, for any j with $A(j) \notin T_{A,1}$, we trivially have $A(j) \neq= top(j, M)$ and therefore $A(j) = top(j, M')$. It also means that for any $o \in T_{A,2}$, $o = top(A^{-1}(o), M')$, which satisfies FERI for $r = 2$.

(iii) for $r > 2$, $T_{A,r} = \emptyset$ because $A(j) = top(j, M) \in T_{A,1}$ or $A(j) = top(j, M') \in T_{A,2}$, and we have that A satisfies FERI trivially.

(FERI $\not\Rightarrow$ POP) Consider the profile in Figure 1 and the assignment A^* in Example 2 which satisfies FERI. Now, consider the assignment A' below:

$$A' : 1 \leftarrow a,\ 2 \leftarrow b,\ 3 \leftarrow f,\ 4 \leftarrow c,\ 5 \leftarrow e,\ 6 \leftarrow d.$$

Notice that A' is more popular than A^*: $A'(4) \succ_4 A^*(4)$, $A'(5) \succ_4 A^*(5)$, while $A^*(3) \succ_4 A'(3)$, and $A'(j) = A^*(j)$ for $j \in \{1, 2, 6\}$. Therefore, FERI does not imply POP. □

A.2 Rank-maximality is Not Compatible with Fairness and Strategyproofness

We also consider the compatibility of rank-maximality (RM) with fairness and strategyproofness.

Definition 6. (Irving et al., 2006) *Given an instance, a deterministic assignment A satisfies rank-maximality (RM) if there is no assignment A' such that its* signature *dominates A, where the signature of A is an n-vector $\mathbf{x} = (x_r)_{r \leq n}$ such that for each $r \in [n]$, the r-th component is the number of agents who are allocated their r-th ranked item, and a signature $\mathbf{x}$ dominates signature $\mathbf{y}$ if there exists r' such that $x_{r'} > y_{r'}$ and for every $r'' < r'$, $x_{r''} \geq y_{r''}$.*

Since RM implies FHR (Belahcene et al., 2021), it automatically means that ex-post RM (ep-RM) suffers the same incompatibility with fairness and strategyproofness as FHR. In fact, ep-RM is incompatible even with sd-WEF alone as we show in Proposition 11.

Proposition 11. *No mechanism satisfies ex-post rank-maximality (ep-RM) and sd-weak-envy-freeness (sd-WEF) simultaneously.*

Proof. Let R be the profile in Figure 1. By RM implying FCM, a and b must be assigned to agents 1 and 2, respectively. Although agents 3-6 all rank c on top, in any RM assignment, item c can only be allocated to agent 6 and $\{d, e, f\}$ to agents 3-5, which leads to a signature $\mathbf{y} = (3, 1, 1, 1, 0, 0)$. Otherwise, if c is not assigned to agent 6 in an RM assignment, then by RM implying FHR, agent 6 can only get f since $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$ for $j \in \{3, 4, 5\}$, and agents 3-5 get $\{c, d, e\}$, which results in the signature $\mathbf{x} = (3, 1, 1, 0, 0, 1)$ dominated by $\mathbf{y}$, a contradiction. Then, since any random assignment P satisfying ep-RM is a convex combination over the set of possible RM assignments, we have that for any item o, $\sum_{o' \in U(\succ_3, o)} p_{3,o'} \leq 1 = p_{6,c} = \sum_{o' \in U(\succ_6, o)} p_{6,o'}$, and it is strict for items other than f, a violation of sd-WEF. □

A.3 Properties that RP, PS, BM, ABM, EBM, PR, and UPRE Fail to Satisfy

Propositions 12 and 13 show that RP and PS are not first-choice maximal, and therefore they do not satisfy ep-FERI since FERI implies first-choice maximality.

Proposition 12. *RP does not satisfy ex-post first-choice maximality.*

Proof. We show it with the instance with the following profile:

$$\succ_1: \; a \succ_1 b \succ_1 c,$$
$$\succ_2: \; a \succ_2 c \succ_2 b,$$
$$\succ_3: \; b \succ_3 a \succ_3 c.$$

The following A is one possible output of RP when the priority order is $2 \rhd 1 \rhd 3$. In A, we see that only agent 2 gets her first choice. However, there exists another assignment A' as shown below where both agents 1 and 3 get their first choices, which means that RP does not satisfy FCM.

Assignment A

	a	b	c
1	0	1	0
2	1	0	0
3	0	0	1

Assignment A'

	a	b	c
1	1	0	0
2	0	0	1
3	0	1	0

□

Proposition 13. *PS does not satisfy ex-post first-choice maximality.*

Proof. **(neither ep-FERI nor FCM)** we continue to use the instance with the profile in Proposition 12. The following P is the outcome of PS. We see that A in Proposition 12 must be among the deterministic assignments which constitute the convex combination for P, which means that PS does not satisfy FCM, and therefore it is not ep-FERI.

Assignment P

	a	b	c
1	1/2	1/4	1/4
2	1/2	0	1/2
3	0	3/4	1/4

□

Proposition 14. *BM with a uniform probability distribution over all the priority orders of agents does not satisfy ex-post favoring-eagerness-for-remaining-items (ep-FERI), ex-ante favoring-eagerness-for-remaining-items (ea-FERI) or sd-weak-envy-freeness (sd-WEF), but satisfies strong equal treatment of equals (SETE).*

Proof. We refer to BM with a uniform probability distribution as BM^u.

(not ep-FERI) For the instance with profile in Figure 1, the assignment indicated by circled item is one possible outcome of BM given the priority $1 \rhd 2 \rhd 3 \rhd 4 \rhd 5 \rhd 6$, which does not satisfy FERI as we discuss in Example 2, and therefore it is not ep-FERI.

(not ea-FERI) This follows from the fact that it is not sd-E (Chen et al., 2021).

(not sd-WEF) This follows from Proposition 1 and the fact that it satisfies SETE (shown below) and ep-FHR (Ramezanian & Feizi, 2021).

(SETE) Let $P = \mathbb{E}(\text{BM}^u(R))$ for any given profile R. For any agents j, k and their common prefix $\succ_{j,k}$, given a priority order $\rhd$ with $j \rhd k$, if j gets an item o appearing in $\succ_{j,k}$, then it is easy to see that k gets o given $\rhd'$ which just swaps the positions of j and k in $\rhd$. Due to the assumption, we know that any such pair of priorities $\rhd$ and $\rhd'$ have the equal probability to be drawn, and therefore we have that $p_{j,o} = p_{k,o}$, which means SETE. □

We recall Lemma 2 from Bogomolnaia and Moulin (2001) used in Proposition 15 and the proof of Proposition 3.

Lemma 2. *(Bogomolnaia & Moulin, 2001) Given a preference profile R and a random assignment P, let $\tau(P,R)$ be the relation over all the items such that: if there exists an agent j such that $o_a \succ_j o_b$ and $p_{j,o_b} > 0$, then $o_a \tau(P,R) o_b$. The random assignment P is sd-E if and only if $\tau(P,R)$ is acyclic.*

Proposition 15. *EBM with a uniform probability distribution over all the priority orders of agents does not satisfy ex-post favoring-higher-ranks (ep-FHR), sd-efficiency (sd-E), ex-ante favoring-eagerness-for-remaining-items (ea-FERI), ex-ante favoring-higher-ranks(ea-FHR), sd-envy-freeness (sd-EF) or sd-strategyproofness (sd-SP).*

Proof. **(not ep-FHR)** For the profile in Figure 1, one of its possible outcome is the assignment A^* in Example 2 which does not satisfy FHR, and therefore EBM is not ep-FHR.

(not sd-E) We show it by the instance with following R:

$$\begin{aligned}
\succ_1&: a \succ_1 b \succ_1 c \succ_1 \text{others},\\
\succ_2&: a \succ_2 b \succ_2 d \succ_2 \text{others},\\
\succ_3&: a \succ_3 b \succ_3 e \succ_3 \text{others},\\
\succ_4&: a \succ_4 b \succ_4 f \succ_4 \text{others},\\
\succ_{5\text{-}7}&: a \succ b \succ g \succ c \succ d \succ x \succ y \succ \text{others},\\
\succ_{8\text{-}10}&: a \succ b \succ h \succ e \succ f \succ y \succ x \succ \text{others}.
\end{aligned}$$

The following are two possible outcomes of EBM (where $j \leftarrow o$ means agent j gets item o):

$$\begin{aligned}
A :&1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow e, 4 \leftarrow f, 5 \leftarrow g,\\
&6 \leftarrow c, 7 \leftarrow d, 8 \leftarrow h, 9 \leftarrow y, 10 \leftarrow x.\\
A' :&1 \leftarrow c, 2 \leftarrow d, 3 \leftarrow a, 4 \leftarrow b, 5 \leftarrow g,\\
&6 \leftarrow x, 7 \leftarrow y, 8 \leftarrow h, 9 \leftarrow e, 10 \leftarrow f
\end{aligned}$$

Let $P = \mathbb{E}(\text{AM}(R))$. Then $p_{7,y} > 0$ and $p_{10,x} > 0$. With $x \succ_7 y$ and $y \succ_{10} x$ and Lemma 2, we have that $x\tau(P,R)y$ and $y\tau(P,R)x$, which means that P is not sd-E.

(neither ea-FERI nor ea-FHR) It follows from the fact that EBM is not sd-E.

(neither sd-EF nor sd-SP) This follows from Propositions 4 and 6 and the fact that it satisfies ep-FERI and SETE. □

Proposition 16. *ABM satisfies SETE, but does not satisfy ex-post favoring-higher-ranks (ep-FHR), sd-efficiency (sd-E), ex-ante favoring-eagerness-for-remaining-items (ea-FERI), ex-ante favoring-higher-ranks(ea-FHR), or sd-envy-freeness (sd-EF).*

Proof. We refer to ABM with a uniform probability distribution as ABM^u.

(not ep-FHR) It follows from the fact that no mechanism satisfies ep-FHR with SETE and sd-WSP (Ramezanian & Feizi, 2021).

(not sd-E, ea-FERI, or ea-FHR) Consider the preference profile R , and the two deterministic assignments A, A' in Proposition 15 which are FERI since EBM satisfies ep-FERI. According to the proof of Theorem 2, we know that any FERI assignment is a possible output of ABM with certain priority. Then A and A' are among the deterministic assignments that constitute convex combinations for $P = \mathbb{E}(\text{EBM}^u(R))$ as the priority is chosen uniformly. From the proof of Proposition 15, we know that P is not sd-E, therefore not ea-FERI and ea-FHR.

(not sd-EF) It follows from Proposition 4 and the fact that ABM satisfies ep-FERI.

(SETE) Let R be any preference profile. For any two agents j and k and a given priority $\rhd$ with $j \rhd k$, let $\rhd'$ be another priority which only swaps j and k in $\rhd$, $A = \text{ABM}^{\rhd}(R)$, and $A' = \text{ABM}^{\rhd'}(R)$. If $A(j)$ appears in $\succ_{j,k}$, it is easy to see that $A'(k) = A(j)$. Furthermore, if $A(k)$ also appears in $\succ_{j,k}$, then $A'(j) = A(k)$. Let $P = \mathbb{E}(\text{EBM}^u(R))$. Since the pair of priorities like $\rhd$ and $\rhd'$ have the same probability to be chosen, we see that for any item o appearing in $\succ_{j,k}$, $p_{j,o} = p_{k,o}$, which means that P satisfies SETE. □

Proposition 17. *PR does not satisfy ex-post favoring-eagerness-for-remaining-items (ep-FERI) or ex-ante favoring-eagerness-for-remaining-items (ea-FERI), but satisfies SETE.*

Proof. **(not ep-FERI)** We show it by the instance with R in Figure 1. Let $P = \text{PR}(R)$ is shown in the following.

Assignment P

	a	b	c	d	e	f
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3-5	0	0	1/4	1/3	1/3	1/12
6	0	0	1/4	0	0	3/4

Let A be the deterministic assignment indicated by circled items in Figure 1, and we see that A must be among the deterministic assignments which constitute the convex combination for P. We know that A does not satisfy FERI as shown in Example 2, which means that P does not satisfy ep-FERI.

(not ea-FERI) We continue to use the instance with R in Figure 1. In the assignment P above, it is easy to obtain that $M_{P,2} = \{d, e, f\}$ since agents in $E_{P,1}(o)$ for $o \in \{a, b, c\}$ owns all the shares of o. ($M_{P,r}$ and $E_{P,r}(o)$ are defined in Definition 3). Then we have that $E_{P,2}(d) = \{6\}$ and $\sum_{j \in \bigcup_{r<3} E_{P,d}} p_{j,d} = p_{6,d} = 0 < 1$, which means that $d \in M_{P,3}$ while $\sum_{o \in U(6,d)} p_{6,o} = 1/4 < 1$, which violates ea-FERI.

(SETE) PR satisfies equal-rank envy-freeness by (Chen et al., 2021) which requires that in $P = \text{PR}(R)$ for the given preference profile R, for any agents j, k and item o with $rk(j, o) = rk(k, o)$, $\sum_{o' \succ_j, o} p_{j,o'} + p_{k,o} \leq \sum_{o' \in U(\succ_j, o)} p_{j,o'}$. Then for any item o appearing in $\succ_{j,k}$, $rk(j, o) = rk(k, o)$, and therefore $\sum_{o' \succ_j, o} p_{j,o'} + p_{k,o} = \sum_{o' \in U(\succ_j, o)} p_{j,o'}$, i.e., $p_{j,o} = p_{k,o}$, which means that PR satisfies SETE. □

Proposition 18. *UPRE does not satisfy ex-post favoring-eagerness-for-remaining-items (ep-FERI), ex-post favoring-higher-ranks (ep-FHR), ex-ante favoring-higher-ranks (ea-FHR), sd-envy-freeness (sd-EF), or sd-strategyproofness (sd-WSP).*

Proof. **(not ep-FERI)** This follows from Proposition 8 and the fact that it satisfies ea-FERI by Theorem 4 and SETE by Theorem 3.

(not ep-FHR) For the profile R in Figure 1, the following assignment P is the outcome of UPRE.

Assignment P

	a	b	c	d	e	f
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3-5	0	0	1/4	1/12	1/3	1/3
6	0	0	1/4	3/4	0	0

The deterministic assignment A^* in the following, where $j \leftarrow o$ means agent j is allocated item o, is the one in Example 2 which is not FERI. It is easy to see that A^* must be among those which constitute the convex combination for P, which means that P does not satisfy ep-FHR.

$$A^* : 1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow c, 4 \leftarrow e, 5 \leftarrow f, 6 \leftarrow d$$

(not ea-FHR) We continue to use the instance with R in Figure 1. In the assignment P above , where $p_{6,d} > 0$ but $rk(3,d) < rk(2,d)$ and $\sum_{o \in U(\succ_3, d)} p_{3,o} < 1$, which violates ea-FHR.

(not sd-EF, not sd-WSP) It follows from Propositions 5 and 7 and the fact that it satisfies ea-FERI and SETE. □

A.4 Running Time Analysis for EBM and UPRE

Proposition 19. *Given a profile R, the deterministic assignment EBM(R) can be computed in polynomial time in the number of agents if G is a polynomial time algorithm*[3].

Proof. In Algorithm 1, Line 2 is the initial setting which takes $O(n^2)$ time, and the **While** loop is executed at most n times because there is at least one item is allocated in each round. Below, we analyze the time for each line in the main body of the **While** loop.

For Line 5, identifying the top item for each agent among $M' \subseteq M$, takes $O(n \cdot n)$ time.

For Line 6, it issues a lottery for each o over N_o and the implementation runs in polynomial time with respect to n by the condition. Here the range is $|N_o| < n$, which means the implementation of lottery is in polynomial time with respect to n.

Line 7 take $O(n')$ time where n' is the number of items being allocated at that round, and it takes $O(n)$ time in total since at most n items are allocated in one run. Together we have that Algorithm 1 runs in polynomial time. □

3. Such algorithm exists, like Xorshift RNG (Thomson, 1958) and linear congruential generator (Marsaglia, 2003)

Remark 8. Although it is in polynomial time for EBM to output a deterministic assignment as shown in Proposition 19, there is no guarantee for the time complexity of computing the expected results of EBM. We conjecture that it is #P-complete to compute the expected output of EBM, just as computing the expected result of RP (where the priority order is generated randomly and uniformly) is #P-complete (Saban & Sethuraman, 2015).

Proposition 20. *Given a profile R, the random assignment UPRE(R) can be computed in polynomial time in the number of agents.*

Proof. Recall that UPRE is Algorithm 3 using Eq (3) as eating functions, and then $\int_{t_j}^{t_j+\rho} \omega_j \mathrm{d}t = \min(\rho, 1-t_j)$ when $t_j < 1$, which can be computed in $O(1)$ time. In Algorithm 3, Line 2 is the initial setting which takes $O(n^2)$ time. The **While** loop is executed at most n times because an agent consumes different items in each round. Below, we analyze the time for each line in the main body of the **While** loop.

On **Line 4**, identifying the top item for each agent among $M' \subseteq M$, takes $O(n \cdot n)$ time.

For **Line 6.1**, for each o, we can compute ρ_o with the following steps:

Step 1. Sort agents in N_o by $1-t_j$ in increasing order and obtain the sequence $j_1, j_2, \ldots, j_{n_o}$ where $n_o = |N_o|$, which takes $O(n^2)$ time since $n_o \leq n$;

Step 2. For each $i \in \{1, \ldots, n_o\}$, test if $\sum_{k \in N_o} \int_{t_k}^{t_k+\rho} \omega_k(t)\mathrm{d}t \leq s(o)$ with $\rho = 1 - t_{j_i}$ and stop when it is not. This takes $O(n)$ time;

Step 3. If $i' < n_o$ is the maximum value for which the test in step (2) passes, then $\rho_o \geq 1 - t_{j_{i'}}$ and computing $\rho_o = \max(\{\rho \mid \rho \cdot (n_o - i') + \sum_1^{i'}(1-t_i) \leq s(o)\}) = \frac{s(o) - \sum_{i=1}^{i'}(1-t_i)}{n_o}$ takes $O(1)$ time; if $i = n_o$, then $\rho_o = 1 - t_{j_{n_o}}$.

In this way, we have that Line 6.1 runs in $O(n^2)$ time.

Line 6.2 needs us to perform one integration for each agent and can also be computed in polynomial time since each integration can be done in $O(1)$ time.

Line 7 updates the supply of each $o \in M'$, Line 8 updates t_j for each agent j, and Line 9 checks if $s(o) = 0$, each of which needs addition/subtraction for no more than n times.

Together we have that UPRE runs in polynomial time. □

A.5 EBM is Not A Member of ABM

We show that EBM is not a member of ABM by proving that there does not exist a distribution π over all the priority orders such that $\mathrm{ABM}^\pi(R) = \mathrm{EBM}(R)$ for any preference profile R.

For an instance of assignment problems with $N = \{1, 2, \ldots, 5\}$ and $M = \{a, b, \ldots, e\}$, there are ${}^5P_5 = 120$ priority orders in total. For the preference R^* where all the agents have an identical preference, we trivially have that there are ${}^5P_5 = 120$ possible outcomes of EBM, each with the same probability. By Theorem 2, we also have that each possible EBM(R) corresponds to a unique priority order. Then $\mathbb{E}(\mathrm{EBM}(R^*))$ coincides with $\mathbb{E}(\mathrm{ABM}^u(R^*))$ where u is the uniform distribution over all the priority orders.

Now we show that EBM is different from ABM^u by comparing the probability of agent 1 obtaining c in the outcomes of EBM and ABM applied to the profile R for 5 agents below:

$$\text{1-2}: a \succ c \succ \{others\}, \quad \text{3-5}: b \succ c \succ \{others\}.$$

First we consider EBM. Let $A = \text{EBM}(R)$ be a possible outcome of EBM and $P = \mathbb{E}(\text{EBM}(R))$. In the first round, EBM issues a lottery for a among agents 1-3 and one for b among $4, 5$. Then $Pr(A(1) \neq a) = 1/2$. In the second round, EBM issues a lottery for c, and the number of participants is always 3, which means that $Pr(A(1) = c \mid A(1) \neq a) = 1/3$. It follows that $p_{1,c} = Pr(A(1) = c) = 1/6$.

Then we consider ABM^u. Let $Q = \mathbb{E}(\text{ABM}^u(R))$. Agent 1 gets c when in the priority order,

(1) agent 2 is ranked above agent 1,

(2) at least two agents $j, k \in \{3, 4, 5\}$, i.e., the agents who do not get b, are ranked below agent 1.

The priority orders satisfying this where $j = 4$ and $k = 5$ are the ones corresponding to the topological orderings inFigure 6, of which there are 6 as listed below:

$$\begin{aligned} &3 \rhd 2 \rhd 1 \rhd 4 \rhd 5, \quad 3 \rhd 2 \rhd 1 \rhd 5 \rhd 4, \\ &2 \rhd 3 \rhd 1 \rhd 4 \rhd 5, \quad 2 \rhd 3 \rhd 1 \rhd 5 \rhd 4, \\ &2 \rhd 1 \rhd 3 \rhd 4 \rhd 5, \quad 3 \rhd 1 \rhd 3 \rhd 5 \rhd 4. \end{aligned}$$

With fact that j, k are chosen in $\{3, 4, 5\}$, there are $6 \cdot {}^2C_3 = 18$ priority orders where agent 1 gets c, i.e., $q_{1,c} = 18/120 = 3/20$.

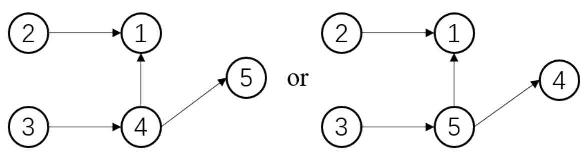


Figure 6: Possible topological orders of priority orders where agent 1 gets c.

Together we see that $q_{1,c} \neq p_{1,c}$, which means that $Q \neq P$. Therefore, while ABM^π is uniquely equal to EBM for the preference profile R^* where agents have an identical preference, ABM^π is not equal to EBM for R above. It follows that EBM is not a member of ABM.

Appendix B. Omitted Proofs

Theorem 1. *EBM satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI), sd-weak-envy-freeness (sd-WEF), strong equal treatment of equals (SETE), and sd-weak-strategyproofness (sd-WSP).*

Proof. Consider an arbitrary instance (N, M) and a strict linear preference profile R, and let $P = \mathbb{E}(\text{EBM}(R))$.

Part 1: $\mathbb{E}(\textbf{EBM}(R))$ is ep-FERI.

Let $A = \text{EBM}(R)$ be any one of the possible outcomes of EBM applied to R. We prove by mathematical induction that the items to be assigned at each round r of Algorithm 1 are exactly those in $T_{A,r}$ (defined in Definition 2), and that A is FERI by showing that every item $o \in T_{A,r}$ is assigned to an agent most eager for it in round r, i.e.,

$$o = top(A^{-1}(o), M \setminus \bigcup_{r^*<r} T_{A,r^*}). \tag{9}$$

Base case. When $r = 1$, any $o \in T_{A,1}$ satisfies that $o = top(j, M)$ for some j. In Algorithm 1, all such agents are in N_o on Line 5, and o is assigned to one of them by Line 6 at that round. It means that $o = top(A^{-1}(o), M)$, which is equivalent to Eq (9) when $r = 1$.

Inductive step. Consider the case that $r > 1$. Suppose that for each $r' < r$, EBM(R) assigns the items in $T_{A,r'}$ during round r' and it holds for every item $o' \in T_{A,r'}$, that $o' = top(A^{-1}(o'), M \setminus \bigcup_{r^*<r'} T_{A,r^*})$. We will show that at the end of round r, for every $o \in T_{A,r}$, $o = top(A^{-1}(o), M \setminus \bigcup_{r^*<r} T_{A,r^*})$. By the assumption and Line 3, we have that at the beginning of round r:

- the set of remaining items is $M' = M \setminus \bigcup_{r^*<r} T_{A,r^*}$, since items in $\bigcup_{r^*<r} T_{A,r^*}$ are allocated before round r, and
- for any agent $k \in N'$ who has not received any item yet, it holds that $A(k) \notin T_{A,r'}$ for any $r' < r$.

Therefore, if there exists an agent $j \in N_o \subseteq N'$ (i.e., $o = top(j, M')$) on Line 5, then an agent in N_o gets o by Line 6, i.e., $A^{-1}(o) \in N_o$, which implies Eq (9).

By the induction hypothesis we have that Eq (9) holds for any $o \in T_{A,r}$ with $r \geq 1$, which means that A satisfies FERI. It follows that $\mathbb{E}$(EBM(R)) is ep-FERI.

Part 2: $\mathbb{E}$(EBM(R)) is sd-WEF.

Before proceeding with the proof we introduce some notation for convenience:

- For ease of exposition, we will refer to each possible execution of EBM as a "world", denoted w, and EBM$^w(R)$ to be the corresponding deterministic assignment output by EBM. It is easy to see that if $w \neq w'$, then EBM$^w(R) \neq$ EBM$^{w'}(R)$. Let $W(R)$ be the set of all possible worlds for the given instance with R, and W for short when R is clearly given in the context. The probability of w, denoted $Pr(w)$, can be computed according to the lotteries in each round.

- We use l to refer to a lottery and $N(l)$ be the set of agents who participate in l. Let $L(w, r)$ denote the set of lotteries in round r of world w (w can be omitted when clear), and $r(w)$ be the total rounds of w. Specially, l_o^w refers to the lottery for item o in w. Then we have that

$$Pr(w) = \Pi_{l \in L(r), r \leq r(w)} \frac{1}{|N(l)|} = \Pi_{o \in M} \frac{1}{|N(l_o^w)|}$$

since every item can only be allocated once through a lottery. Let $Pr(W') = \sum_{w \in W'} Pr(w)$ for $W' \subseteq W$. If W' is the set of all the worlds with the same lotteries and winners for the first r rounds, then $Pr(W') = \Pi_{l \in L(r'), r' \leq r} \frac{1}{|N(l)|}$. For $P = \mathbb{E}$(EBM(R)), we have that $p_{j,o} = Pr(\{w \in W \mid \text{EBM}^w(R)(j) = o\})$, i.e., the probability of all the worlds where j gets items o.

- For ease of exposition, we use $M_r^w, N_r^w, N_{o,r}^w$ to refer to the values of variables M', N', N_o at the beginning of each round r during the execution of Algorithm 1 in the world w, and omit w when it is clear from the context.

Start of proof of Part 2. Consider an arbitrary pair of agents j and k such that $P_k \succeq_j^{sd} P_j$ and $P_j \neq P_k$. Without loss of generality, let $\succ_j$ be $o_1 \succ_j o_2 \succ_j \cdots \succ_j o_n$. We show by mathematical induction on the rank $i = 1, 2, \ldots, n$ with respect to $\succ_j$, that the following conditions hold:

Condition (1): if there exits a round r such that $j \in N_{o_i,r}$ and $k \in N_r$ in a certain world $w \in W$, then $k \in N_{o_i,r}$,

Condition (2): if in a certain world $w \in W$, j gets some $o \succ_j o_i$ at round r', then for $r > r'$ with $k \in N_r$ and $top(j, M_r) = o_i$, we have that $k \in N_{o_i,r}$, and

Condition (3): $p_{k,o_i} = p_{j,o_i}$.

Base case. First we prove conditions (1)-(3) for $i = 1$. Since no $o \succ_j o_1$, we have condition (2) trivially true. For every possible world $w \in W$, we have that $j \in N_{o_1,1}$ and $k \in N_1 = N$. If $k \notin N_{o_1,1}$, then k does not participate in the lottery for o_1 , and she does not get o_1 in any w, which means that $p_{k,o_1} = 0 < p_{j,o_1}$, a contradiction to $P_k \succeq_j^{sd} P_j$. Therefore we have condition (1) for $i = 1$. It follows that $p_{j,o_1} = Pr(\{w \in W \mid \text{EBM}^w(R)(j) = o_1\}) = Pr(\{w \in W \mid \text{EBM}^w(R)(k) = o_1\}) = p_{k,o_1}$, i.e., condition (3) holds for $i = 1$.

Inductive step. Assume that conditions (1)-(3) hold for $i' < i$, we show that they also hold for i. We show that $p_{k,o_i} \leq p_{j,o_i}$ by comparing the probabilities of worlds where j gets o_i with those where k gets o_i in the following cases (i)-(iii).

Case i: Consider any world $w' \in W$ where agents j do not get items better than o_i according to $\succ_j$, and agents k do not get items better than o_i according to $\succ_k$.

We first show that we do not need to consider the case that there does not exist a round r such that $j \in N_{o_i,r}$ in world w'. If such r does not exist, $\text{EBM}^{w'}(R)(j) \neq o_i$, and there does not exists r^* with $k \in N_{o_i,r^*}$ either. Otherwise, if such r^* exists, $o_i \in M_{r^*}$. Let $o_{i^*} = top(j, M_{r^*}) \succ_j o_i$, which means that $i > i^*$. By condition (1) for $i' < i$, when j applies for $o_{i'}$, k does too. It follows that $i^* \geq i$ since $o_{i^*} \neq o_i$, a contradiction. Together we see that both agents do not get o_i in w', and therefore w' is out of discussion.

Then we consider the case that there exists a round r such that $j \in N_{o_i,r}$. Let W_1 be the set of worlds where lotteries and winners are the same as w' for any round $r' < r$, and therefore all the worlds in W_1 have the same $M_r, N_r, N_{o,r}$ as w' for r. By selection of w', we have that $j, k \in N_r$. In the following, we compare the probabilities that j and k get o_i in W_1 by cases.

Case i (a): If $k \in N_{o_i,r}$, then she participates in the lottery for o_i at round r and her chance to win is equal to j's, which means that

$$\begin{aligned}
&Pr(\{w \in W_1 \mid \text{EBM}^w(R)(j) = o_i\})\\
=&Pr(W_1) \cdot \frac{1}{|N(l_{o_i})|}\\
=&Pr(\{w \in W_1 \mid \text{EBM}^w(R)(k) = o_i\}).
\end{aligned}$$

Case i (b): If $k \notin N_{o_i,r}$, then by $j \in N_{o_i,r} \neq \emptyset$, o_i is allocated to some agent in $N_{o_i,r}$ and never appears in later rounds, which means that

$$Pr(\{w \in W_1 \mid \text{EBM}^w(R)(j) = o_i\}) > Pr(\{w \in W_1 \mid \text{EBM}^w(R)(k) = o_i\}) = 0. \tag{10}$$

Case ii: In this case, we consider the world where one of agents j and k gets an item better than o_i with respect to $\succ_j$. For any world $w_k \in W$ in which agent j gets item $o_h \succ_j o_i$ at round r' and k does not get any item better than o_i with respect to $\succ_k$, let r satisfy $o_i = top(j, M_r)$.

First we show that we do not need to consider the case where such an r does not exist in w_k. In that case, we have that $k \notin N_{o_i,r^*}$ for any r^*. Otherwise, if $k \in N_{o_i,r^*}$ for some r^*, then $k \in N_{r^*}$. Let $o_{i^*} = top(j, M_{r^*})$. We have that $o_{i^*} \succ_j o_i$, i.e, $i^* < i$, and the fact that $o_{i^*} \neq o_i$ contradicts condition (1) for i^* if $j \in N_{r^*}$, and condition (2) for i^* if $j \notin N_{r^*}$. Therefore $k \notin N_{o_i,r^*}$ for any r^* if such r does not exist, which means that $\text{EBM}^{w_k}(R)(k) \neq o_i$ and we do not need to consider w_k.

Then we consider the case where such r exists. Recall that agent j gets item $o_h \succ_j o_i$ at round r', and r satisfy $o_i = top(j, M_r)$. By condition (1) for $i' \leq h$, neither of agents applies for any o with $o_h \succ_j o$ before round r', and it follows that $r > r'$. Let W_k be the set of worlds where lotteries and winners are the same as w_k for any round $r^* < r$. Correspondingly, we find a set of worlds W_j such that

- for any round in $\{1, \ldots, r'-1, r'+1, \ldots, r-1\}$, lotteries and winners are the same as w_k,
- for round r', lotteries are the same as w_k, and so do winners except the one for item o_h, and
- agent k wins the lottery of o_h at round r'.

We have that $W_j \neq \emptyset$, because k participates in the lottery for o_h at round r' since $k \in N_{o_h,r'}$ by condition (1) for h, which means that k is possible to win o_h instead of j, and then j participates in the same lotteries instead of k does in w_k till round r by condition (2) for $h < i' < i$. By construction of W_j and W_k, $Pr(W_j) = Pr(W_k)$. For any $w \in W_j$ and $w' \in W_k$, we have that $M_r^w = M_r^{w'}$, $j \in N_r^w$, $k \in N_r^{w'}$, and $N_r^w \setminus \{j\} = N_r^{w'} \setminus \{k\}$. By selection of r such that $o_i = top(j, M_r)$, we obtain that $j \in N_{o,r}^w$. In the following, we compare the probabilities that j and k get o_i in W_j and W_k respectively by cases.

Case ii (a): If $o_i = top(k, M_r)$, i.e., $k \in N_{o_i,r}^{w'}$, then by construction of W_j and W_k, $N(l_{o_i}^w) \setminus \{j\} = N(l_{o_i}^{w'}) \setminus \{k\}$. It follows that $|N(l_{o_i}^w)| = |N(l_{o_i}^{w'})|$ and

$$\begin{aligned}&Pr(\{w \in W_j \mid \text{EBM}^w(R)(j) = o_i\}) = Pr(W_j) \cdot \frac{1}{|N(l_{o_i}^w)|} = Pr(W_k) \cdot \frac{1}{|N(l_{o_i}^{w'})|}\\ =&Pr(\{w \in W_k \mid \text{EBM}^w(R)(k) = o_i\}).\end{aligned}$$

Case ii (b): If $o_i \neq top(k, M_r)$, i.e., $k \notin N_{o_i,r}^{w'}$, we discuss in case of $N_{o_i,r}^{w'}$. When $N_{o_i,r}^{w'} \neq \emptyset$, then o_i is allocated to some agent in $N_{o_i,r}^{w'}$, which means that $Pr(\{w \in W_j \mid \text{EBM}^w(R)(j) = o_i\}) > Pr(\{w \in W_k \mid \text{EBM}^w(R)(k) = o_i\}) = 0$. When $N_{o_i,r}^{w'} = \emptyset$, we have that $N_{o_i,r}^w = \{j\}$. It means that j is the only applicant for o_i, i.e., $|N(l_{o_i}^w)| = 1$, and therefore gets it in any $w \in W_j$. As for agent k, she applies for $o' \neq o_i$ at round r in $w' \in W_k$, and $o_i \succ_j o'$ by the selection of o_i. It follows that

$$\begin{aligned}&Pr(\{w \in W_j \mid \mathrm{EBM}^w(R)(j) = o_i\}) = Pr(W_j) \cdot \frac{1}{|N(l^w_{o_i})|} = Pr(W_j)\\ >&Pr(W_k) - Pr(\{w \in W_k \mid \mathrm{EBM}^w(R)(k) = o'\})\\ \geq&Pr(\{w \in W_k \mid \mathrm{EBM}^w(R)(k) = o_i\}).\end{aligned} \tag{11}$$

Concluding the inductive step. From cases i and ii, we have covered all the worlds that agent k can possibly get item o_i, and we obtain that $p_{k,o_i} \leq p_{j,o_i}$. With the assumption that $P_k \succeq^{sd}_j P_j$ and condition (3) holds for $i' < i$, it follows that $p_{k,o_i} \geq p_{j,o_i}$. Therefore we have $p_{k,o_i} = p_{j,o_i}$, i.e., condition (3) holds for i. The equality also requires that

- If in the world w, $j \in N_{o_i,r}$ and $k \in N_r$, then w belongs to case i where agent j do not get items better than o_i according to $\succ_j$ and agents k do not get items better than o_i according to $\succ_k$. By case i, we have that $k \in N_{o_i,r}$, i.e. condition (1) for rank i. Otherwise if $k \notin N_{o_i,r}$, then it fits into case i (b), and we have Eq (10) which leads to $p_{k,o_i} < p_{j,o_i}$, a contradiction to assumption that $P_k \succeq^{sd}_j P_j$.

- If in the world w, agent j gets some $o \succ_j o_i$ at round r', and there exists round r such that $k \in N_r$ with $o_i = top(j, M_r)$, then w belongs to case ii where agent j gets an item better than o_i according to $\succ_j$ while agents k do not get items better than o_i according to $\succ_k$. By case ii (b), we have that $o_i = top(k, M_r)$, i.e. $k \in N_{o_i,r}$, and it follows that condition (2) holds for rank i. Otherwise if $o_i \neq top(k, M_r)$, then it fits into case ii (b), and we have Eq (11) which leads to $p_{k,o_i} < p_{j,o_i}$, a contradiction to assumption that $P_k \succeq^{sd}_j P_j$.

With the induction above, we prove that $p_{k,o_i} = p_{j,o_i}$ for any i. It follows that if $P_k \succ_j P_j$, $P_k = P_j$.

Part 3: $\mathbb{E}$(EBM(R)) is sd-WSP. We continue to use the new notations introduced at the beginning of Part 2. Without loss of generality, let $\succ_j$ be $o_1 \succ_j o_2 \succ_j \cdots$. Let profile $R' = (\succ'_j, \succ_{-j})$ where $\succ'_j$ is any preference that agent j misreports, and $Q = \mathrm{EBM}(R')$. Assume that $Q_j \succeq^{sd} P_j$. We show by mathematical induction on the rank $i = 1, 2, \ldots$ with respect to agent j, that the following conditions hold:

Condition (1): when $j \in N^w_{o_i,r}$ in a world $w \in W(R)$, for any $w' \in W(R')$ where lotteries and winners are the same as w before round r, we have that $j \in N^{w'}_{o_i,r}$, and

Condition (2): $p_{j,o_i} = q_{j,o_i}$.

Base case. First, we show condition (1) for $i = 1$. It is easy to see that in any $w \in W(R)$, j applies for o_1 at round 1, i.e., $j \in N^w_{o_1,1}$. We claim that $j \in N^{w'}_{o_1,1}$ for any $w' \in W(R')$. Otherwise, if $j \notin N^{w'}_{o_1,1}$ in some w', we show that both of the possible cases below lead to a contradiction to our assumption that $Q_j \succeq^{sd} P_j$.

- When $N^{w'}_{o_1,1} \neq \emptyset$, o_1 is assigned to some agent in $N^{w'}_{o_1,1}$ in w'. It follows that $p_{j,o_1} > q_{j,o_1} = 0$, a contradiction to the assumption.

• When $N^{w'}_{o_1,1} = \emptyset$, i.e., $N^{w}_{o_1,1} = \{j\}$, o_1 is assigned to the only applicant j in w, while she applies for item $o' \neq o_1$ in w' and $o_1 \succ_j o'$ trivially. It follows that

$$\begin{aligned} p_{j,o_1} =& Pr(\{w \in W(R) \mid \text{EBM}^w(R)(j) = o_1\}) = 1 \\ >& 1 - Pr(\{w \in W(R') \mid \text{EBM}^w(R')(j) = o'\}) \\ \geq& Pr(\{w \in W(R') \mid \text{EBM}^w(R')(j) = o_1\}) = q_{j,o_1}, \end{aligned}$$

a contradiction to the assumption.

In this way, we have $j \in N^{w'}_{o_1,1}$ for any $w' \in W(R')$, i.e., condition (1) for $i = 1$, which means that $|N(l^w_{o_1})| = |N(l^{w'}_{o_1})|$ and

$$\begin{aligned} p_{j,o_1} =& Pr(\{w \in W(R) \mid \text{EBM}^w(R)(j) = o_1\} = \frac{1}{|N(l^w_{o_1})|} \\ =& \frac{1}{|N(l^{w'}_{o_1})|} = Pr(\{w \in W(R') \mid \text{EBM}^w(R')(j) = o_1\}) = q_{j,o_1}, \end{aligned}$$

i.e., condition (2) for $i = 1$.

Inductive step. Supposing conditions (1) and (2) hold for $i' < i$, we show that they also hold for i. First we show condition (1) for i. For an arbitrary world $w^* \in W(R)$ with $j \in N^{w^*}_{o_i,r}$, let $W_1 \subseteq W(R)$ and $W_2 \subseteq W(R')$ be the sets of worlds where lotteries and winners are the same as w^* before round r with respect to R and R', respectively. By construction of W_1 and W_2, $Pr(W_1) = Pr(W_2)$. For any $w \in W_1$ and $w' \in W_2$, $M^w_r = M^{w'}_r$, $N^w_r = N^{w'}_r$, and $N^w_{o,r} = N^{w'}_{o,r}$ for any $o \in M^w_r \setminus \{o_i\}$. We have that $j \in N^w_{o_i,r}$ by the selection of w^*, and we claim that $j \in N^{w'}_{o_i,r}$ for any $w' \in W(R')$. Otherwise, if $j \notin N^{w'}_{o_i,r}$ in some w', we show that both of the possible cases below lead to a contradiction to our assumption that $Q_j \succeq^{sd} P_j$.

• When $N^{w'}_{o_i,r} \neq \emptyset$, item o_i is assigned to some agent in $N^{w'}_{o_i,r}$ in w'. It follows that

$$\begin{aligned} & Pr(\{w \in W_1 \mid \text{EBM}^w(R)(j) = o_i\}) = Pr(W_1) \\ >& Pr(\{w \in W_2 \mid \text{EBM}^w(R')(j) = o_i\}) = 0. \end{aligned} \tag{12}$$

With condition (1) for $i' < i$, in world w', agent j can only apply for o_i at round $r' \geq r$ not earlier than she does in w, which means that $p_{j,o_i} > q_{j,o_i} = 0$ with Eq (12). Together with condition (2) for $i' < i$, we have a contradiction to the assumption that $Q_j \succeq^{sd} P_j$.

• When $N^{w'}_{o_i,r} = \emptyset$, i.e., $N^w_{o_i,r} = \{j\}$, item o_i is assigned to the only applicant j in w while she applies for item $o' \neq o_i$ in w' and $o_i \succ_j o'$ by the selection. It follows that

$$\begin{aligned} & Pr(\{w \in W_1 \mid \text{EBM}^w(R)(j) = o_i\}) = Pr(W_1) \\ >& Pr(W_2) - Pr(\{w \in W_2 \mid \text{EBM}^w(R')(j) = o'\}) \\ \geq& Pr(\{w \in W_2 \mid \text{EBM}^w(R')(j) = o_i\}). \end{aligned}$$

This means that $p_{j,o_i} > q_{j,o_i}$, a contradiction to the assumption that condition (2) holds for $i' < i$.

In this way, we have condition (1) for i, which means that $|N(l^w_{o_i})| = |N(l^{w'}_{o_i})|$ and

$$Pr(\{w \in W_1 \mid \text{EBM}^w(R)(j) = o_i\} = Pr(W_1) \cdot \frac{1}{|N(l^w_{o_i})|}$$
$$= Pr(W_2) \cdot \frac{1}{|N(l^{w'}_{o_i})|} = Pr(\{w \in W_2 \mid \text{EBM}^w(R')(j) = o_1\}),$$

which implies $p_{j,o_i} = q_{j,o_i}$, i.e., condition (2) for i.

By mathematical induction, we have that $p_{j,o} = q_{j,o}$ for any o, and therefore if $Q_j \succeq^{sd} P_j$, that $Q_j = P_j$.

Part 4: $\mathbb{E}(\mathbf{EBM}(R))$ is SETE.

For agents j and k, we prove that $p_{j,o} = p_{k,o}$ for any item o appearing in $\succ_{j,k}$. We compare probability of possible worlds where agent j gets $o \in U(\succ_{j,k}, o_m)$ with those where agent k gets o.

First we consider the world w where j gets o at round r and k gets $o' \in U(\succ_{j,k}, o_m)$ at round r'. Let w' satisfy j gets o' at round r, k gets o at round r', and the results of other lotteries keep the same as w. In w', we see that k wins the lottery for o instead of j, and j participates in lotteries at rounds $r+1$ to r' instead of k. We also see that for every lottery l_o, $|N(l_o)|$ keeps the same in worlds w and w'. Therefore we have that $Pr(w) = Pr(w')$.

Then we consider the world w where j gets o at round r and k does not get items appearing in $\succ_{j,k}$. Let o' be the last item k applies for in $\succ_{j,k}$ at round r', and W_j be the set of worlds which are the same as w from rounds 1 to r'. Here the probability of W_j can also be computed as $Pr(W_j) = \Pi_{l \in L(r^*), r^* \leq r'} \frac{1}{|N(l)|}$. We construct another set W_k such that for any $w \in W_j$, (i) the winners of lotteries are the same as w at round 1 to $r-1$, (ii) the winner of $l(o)$ is k at round r, and any other $l \in L(r)$ is the same as w, (iii) j participates in lotteries at rounds $r+1$ to r' instead of k. Then we see that for every lottery $l \in L(r^*)$ with $r^* \leq r'$, $|N(l)|$ are the same in any world $w \in W_j$ and $w' \in W_k$. Therefore we have that $Pr(W_j) = Pr(W_k)$.

Together we have that $p_{j,o} = p_{k,o}$ for any o appearing in $\succ_{j,k}$. □

Theorem 2. *Given a profile R, a random assignment P satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI) if and only if there exists a probability distribution π over all the priorities such that $P = \mathbb{E}(ABM^\pi(R))$.*

Proof. We provide the satisfaction part for the sake of completeness.
(Satisfaction) To show ABM satisfies ep-FERI, we first prove by mathematical induction that given a profile R, $A = \text{ABM}^{\rhd}(R)$ satisfies FERI for any $\rhd$.

Base case. At round 1 of $\text{ABM}^{\rhd}$, every agent applies for their top ranked items. For any item o, let $N^1_o = \{j : o = top(j, M)\}$, i.e., the set of agents who rank o highest. We also note that o with $N^1_o \neq \emptyset$ satisfies $o \in T_{A,1} \subseteq M$ trivially. According to the priority order $\rhd$, item o is assigned to the agent with the highest priority among N^1_o. It follows that $o = top(A^{-1}(o), M)$, which meets the requirement of FERI when $r = 1$.

Induction Step. Assume that A meets the requirement of FERI for any $r' < r$, For round r, let M^r and N^r be the set of available items and unsatisfied agents at the beginning of that round, respectively. By construction, it follows that $M^r = M \setminus \bigcup_{r'<r} T_{A,r'}$, and that for any $j \in N^r$, $A(j) \notin \bigcup_{r'<r} T_{A,r'}$. In Algorithm 2, every agent in N^r applies for their top ranked items among M^r. For any item o, let $N_o^r = \{j : o = top(j, M^r)\}$, i.e., the set of agents who rank o highest among M^r. We note that o with $N_o^r \neq \emptyset$ satisfies $o \in T_{A,r}$ by the construction. Then o is assigned to the agent ranked highest in $\rhd$ among N_o^r. It follows that $o = top(A^{-1}(o), M^r)$, and we see that it meets the requirement of FERI for r.

By induction, we have that the outcome of $\mathrm{ABM}^{\rhd}(R)$ satisfies FERI for any profile R, and therefore $\mathrm{ABM}^{\pi}(R) = \sum \pi(\rhd) * \mathrm{ABM}^{\rhd}(R)$ satisfies ep-FERI by definition. □

To prove Theorems 3 and 4, we show the following claim (where $M_{P,r}$ and $E_{P,r}(o)$ are defined in Definition 3).

Lemma 3. *Given any preference profile R and any member f of PRE, let $P = f(R)$. Then, for any round r, it holds that:*
(i) $M_{P,r} = M'$ and $E_{P,r}(o) = N_o$ for each $o \in M_{P,r}$, where M' is the set of items with remaining supply, and N_o is the set of agents who are eager for item o at the beginning of round r during the execution of Algorithm 3.

(ii) *for any agent j and item o^* with $top(j, M_{P,r-1}) \succ_j o^* \succ_j top(j, M_{P,r})$, it holds that $p_{j,o^*} = 0$.*

(iii) *for any round $r^* > r$, and any item $o \in M_{P,r^*}$, it holds that for any $j \in E_{P,r}(o)$,*
$$\sum_{o' \in U(\succ_j, o)} p_{j,o'} = 1. \tag{13}$$

Proof. The proof proceeds by mathematical induction on the value of r.

Base case. When $r = 1$, we see that $s(o)$ is initially set to 1 with respect to the supply of item and $M_{P,1} = M$ which is also the initial value of M' on Line 3 of Algorithm 3. Therefore $E_{P,1}(o) = \{j \mid top(j, M')\} = N_o$ by Line 4 for round 1. Together we have (i) for $r = 1$.

Besides, since no such $o^* \succ_j o = top(j, M_{P,1})$ exists for any agent j, we have that (ii) holds for $r = 1$ trivially.

Since t_j is set to 0 for any $j \in N$ and $\sum_{k \in N_o} \int_0^1 \omega_k(t)\mathrm{d}t \geq s(o) = 1$, o is consumed to exhaustion by agents in N_o at round 1. It also means that $\sum_{k \in E_{P,1}(o)} p_{k,o} = 1$, $o \notin M_{P,r^*}$ with $r^* > 1$, and therefore (iii) holds trivially.

Inductive step. Supposing that (i)-(iii) holds for any $r' < r$, we show that it also holds for r. In Algorithm 3, at the beginning of round r, M' only contains item o with positive supply, i.e., $s(o) > 0$, after consumption of previous rounds by Line 8. Because (i) holds for $r' < r$, we have that only agents in $\bigcup_{r'<r} E_{P,r'}(o)$ are able to consume o before round r, which means that $s(o) = 1 - \sum_{j \in \bigcup_{r'<r} E_{P,r'}(o)} p_{j,o}$, and therefore $M' = M_{P,r}$. Then we have that $N_o = \{j \mid o = top(j, M')\} = E_{P,r}(o)$. Together we have that (i) holds for r.

Then we show that (ii) holds for r. For any agent j and item o^* such that $top(j, M_{P,r-1}) \succ_j o^* \succ_j top(j, M_{P,r})$, it means that $o^* \notin M_{P,r}$ and agent j cannot get shares of o^* at round $r-1, r$ or later rounds.

- If $top(j, M_{P,r-1}) = top(j, M_{P,r})$, we have that (ii) is trivially true.
- If $top(j, M_{P,r-1}) \neq top(j, M_{P,r})$ and $o^* = top(j, M_{P,r'})$ for some $r' < r-1$, we know that j consumes $top(j, M_{P,r-1})$ at round $r-1$, and o^* at round r' because (i) holds for

r', which means that $top(j, M_{P,r-1})$ and o^* are available at round r'. It follows that both items $o^*, top(j, M_{P,r-1}) \in M_{P,r'}$, a contradiction to $top(j, M_{P,r-1}) \succ_j o^*$. Therefore $o^* \neq top(j, M_{P,r'})$ for any $r' < r-1$, i.e., agent j does not get shares of o^* before round $r-1$.
Together we have that $p_{j,o^*} = 0$, i.e., (ii) holds for r.

Finally, we show (iii) holds for r. For any $o \in M'$, if $o \in M_{P,r^*}$ for some $r^* > r$, then

$$\sum_{k \in \bigcup_{r'' \leq r} E_{P,r''}(o)} p_{k,o} < \sum_{k \in \bigcup_{r'' \leq r^*} E_{P,r''}(o)} p_{k,o} \leq 1. \tag{14}$$

Eq (14) implies that o is still available after the consumption at round r. With $\omega_j(t) = 0$ when $t > 1$ for any $j \in N$, it follows that

$$p_{j,o} = \begin{cases} \int_{t_j}^{1} \omega_j(t)\mathrm{d}t, & t_j < 1 \\ 0, & t_j \geq 1. \end{cases} \tag{15}$$

With (i) for r', we know that for any $o' = top(j, M_{P,r'})$ with $j \in N_o$ and $r' < r$, agent $j \in E_{P,r'}(o')$, which means that o' is consumed by j if agent j is not satisfied at round r'. Moreover, $o' \in U(\succ_j, o)$ due to $M_{P,r} \subseteq M_{P,r'}$, and we have that items consumed by j in time period $[0, t_j]$ are not worse than o according to $\succ_j$. Then with Eq (15) and (ii) for $r' \leq r$, $\sum_{\hat{o} \in U(\succ_j, o)} p_{j,\hat{o}} = \int_0^1 \omega_j(t)\mathrm{d}t$=1, i.e., Eq (13), which complete the proof of (iii) for current r. □

Theorem 3. *UPRE satisfies sd-weak-envy-freeness (sd-WEF) and strong equal treatment of equals (SETE).*

Proof. Given an instance with R, let $P = \text{UPRE}(R)$.

Part 1: UPRE(R) is sd-WEF.

Assume that there exist agents j and j' such that $P_{j'} \succeq_j^{sd} P_j$. Without loss of generality, let o_r be the item such that $j \in N_{o_r}$ at round r, and we have the following claim:

Claim 1. *For $r' < r$, if $o_r \neq o_{r'}$, then $o_r \succ_j o_{r'}$.*

The claim holds because by Lemma 3 (i), $o_r = top(j, M_{P,r})$ and $o_{r'} \in M_{P,r'} \subseteq M_{P,r}$.

We prove by mathematical induction that the following conditions hold for any round r with $t_j < 1$:

Condition (1): $t_{j'} = t_j$,

Condition (2): $p_{j,o'} = p_{j',o'} = 0$ for any o' with $o_{r-1} \succ_j o' \succ_j o_r$,

Condition (3): $j' \in N_{o_r}$, and

Condition (4): $p_{j,o_r} = p_{j',o_r}$.

Base case. With $j \in N_{o_1}$ at round 1, we know that $t_k = 0$ for every k, i.e., condition (1) holds for $r = 1$, and $o_1 = top(j, M') = top(j, M)$ by Lemma 3 (i).

Condition (2) is trivially true since no item $o' \succ_j o_1$ exists.

Then, we show that condition (3) holds for $r = 1$. Item o_1 is consumed to exhaustion at this round by Line 6.1 because $\sum_{k \in N_{o_1}} \int_{t_k}^{t_k+1} \omega_k(t)\mathrm{d}t \geq s(o_1) = 1$, which means that no agent can get o_1 at any round $r^* > 1$. Therefore $p_{j,o_1} > 0$ due to $j \in N_{o_1}$. If $j' \notin N_{o_1}$, then j' does no consume o_1, which means that $p_{j',o_1} = 0 < p_{j,o_1}$, a contradiction to the assumption that $P_{j'} \succeq_j^{sd} P_j$. Then we have that $j' \in N_{o_1}$, i.e., condition (3) holds for $r = 1$.

Because $t_j = t_{j'} = 0$, $p_{j,o_1} = \int_0^{\rho_{o_1}} \omega_j(r)\mathrm{d}t = \int_0^{\rho_{o_1}} \omega_{j'}(r)\mathrm{d}t = p_{j',o_1}$ by Eq (3), i.e., condition (4) for $r = 1$.

Inductive step. Supposing that conditions (1)-(4) hold for any $r' < r$, we show that they also hold for r with $t_j < 1$.

We see that condition (1) trivially holds for r, i.e., $t_{j'} = t_j$ due to the fact that by condition (3) for any $r' < r$, both $t_{j'}$ and t_j increase by the same value $\rho_{o_{r'}}$ on Line 6.1 in each round r'.

Then, we prove condition (2) for r. Here $o_r \neq o_{r-1}$, because otherwise we know that after the consumption at round $r-1$, o_{r-1} is not exhausted, which means that agent $j \in N_{o_{r-1}}$ is satisfied. It follows that $t_j \geq 1$ at the beginning of round r, which we do not need to consider.

For any o' with $o_{r-1} \succ_j o' \succ_j o_r$, $p_{j,o'} = 0$ by Lemma 3 (i) and (ii), and we show that $p_{j',o'} = 0$. We have that $o' \notin M'$ at round r because $j \in N_{o_r}$, i.e., $top(j, M') = o_r$, which means that o' is unavailable for round $r^* \geq r$. With Claim 1, o' is not consumed by j at any round $r' < r$, and therefore it is also not consumed by j' according to condition (3) for $r' < r$. Then o' is never consumed by j or j', which means that $p_{j,o'} = p_{j',o'} = 0$, i.e., condition (2) for r.

Next, we prove condition (3) for r. We consider the following cases.

• If $o_r \in M_{P,r+1}$, then by Claim 1 and condition (2) for $r' \leq r$, it must hold that $\sum_{o^* \succ_j o_r} p_{j,o^*} = \int_0^{t_j} \omega_j(r)\mathrm{d}t = t_j < 1$. By Lemma 3 (i), $j \in N_{o_r} = E_{P,r}(o)$. By Lemma 3 (iii) and $o_r \in M_{P,r+1}$, $\sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*} = 1$. By condition (1) for r that $t_{j'} = t_j < 1$, j' is not satisfied at the beginning of round r. If $j' \in N_{o'}$ with $o' \neq o_r$, it means that j' consumes o' at round r and $p_{j',o'} > 0$, and $o_r \succ_j o'$ since $o_r = top(j, M')$. By conditions (2) and (4) for $r' < r$, and condition (2) for r which we just prove, we have that

$$\sum_{o^* \succ_j o_r} p_{j,o^*} = \sum_{o^* \succ_j o_r} p_{j',o^*}. \tag{16}$$

Therefore,

$$\sum_{o^* \in U(\succ_j, o_r)} p_{j',o^*} < 1 - p_{j',o'} < 1 = \sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*},$$

a contradiction to the assumption that $P_{j'} \succeq_j^{sd} P_j$.

• If $o_r \notin M_{P,r+1}$, then o_r is consumed to exhaustion by agents in N_{o_r} at round r, and no agent consumes o_r after round r. Since $j \in N_{o_r}$ and $t_j < 1$, $p_{j,o_r} > 0$. If $j' \notin N_{o_r}$, then j' does not consume o_r at round $r^* \geq r$. Moreover, j' also does not consume o_r before round r by condition (3) for $r' < r$, which means that $p_{j',o_r} = 0$. With Eq (16), we have that $\sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*} > \sum_{o^* \in U(\succ_j, o_r)} p_{j',o^*}$, a contradiction to the assumption that $P_{j'} \succeq_j^{sd} P_j$. Together we show that $j \in N_{o_r}$.

Finally, condition (4) holds for r trivially because by Eq (3), and conditions (1) and (3) for r, we have that

$$p_{j,o_r} = \int_{t_j}^{t_j+\rho_{o_r}} \omega_j(r)\mathrm{d}t = \int_{t_{j'}}^{t_{j'}+\rho_{o_r}} \omega_{j'}(r)\mathrm{d}t = p_{j',o_r}.$$

By the induction, we have conditions (2) and (4) for any r, i.e., $p_{j,o} = p_{k,o}$ for any item o, which means that $P_{j'} = P_j$ if $P_{j'} \succeq_j^{sd} P_j$.

Part 2: UPRE(R) is SETE.

We show that before consuming items not in $\succ_{j,k}$, $t_j = t_k$, and j and k consume the same item in each round by mathematical induction on rounds.

Base case. At round 1, we know that both j and k consume the most preferred item o in $\succ_{j,k}$. We have $p_{j,o} = p_{k,o}$ by Eq (3) and $t_j = t_k = 0$ which is set initially at the beginning of Algorithm 3.

Inductive step. Supposing that j and k consume the same item and get the same shares for each round $r' < r$, we prove that this is also true for round r. By the inductive assumption, we trivially have that $t_j = t_k$ at the beginning of r. Let j consume o, and k consume o'. Here we do not need to consider the case that both o, o' are not in $\succ_{j,k}$. If $o \neq o'$, we assume that $o \succ o'$ without loss of generality, and therefore o must be in $\succ_{j,k}$. It means that o is available at round r, but k consumes o', a contradiction to the selection of top items. Therefore $o = o'$, and $p_{j,o} = p_{k,o}$ by $t_j = t_k$ and Eq (3).

By induction, we have that $p_{j,o} = p_{k,o}$ for every o in $\succ_{j,k}$. □

Lemma 4 below illustrates how shares of items must be allocated in order to satisfy ea-FERI, which is used in proving the uniqueness part of Theorem 4 and impossibility results for ea-FERI.

Lemma 4. *Given P satisfying ea-FERI, for any r and $o \in M_{P,r}$, we define the remaining shares of item o excluding those owned by agents in $E_{P,r'}(o)$ with any $r' < r$,*

$$s_{P,r}(o) = 1 - \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o},$$

and the remaining demand of agent j for items ranked below the item $top(j, M_{P,r-1})$,

$$d_{P,r}(j) = 1 - \sum_{o' \in U(\succ_j, top(j, M_{P,r-1}))} p_{j,o'}.$$

For any $j \in E_{P,r}(o) \neq \emptyset$,
(i) *for any o^* with $top(j, M_{P,r-1}) \succ_j o^* \succ_j o$, it holds that $p_{j,o^*} = 0$.*

(ii) *if the total remaining demand of agents eager for item o surpasses the remaining shares of o, i.e. $\sum_{k \in E_{P,r}(o)} d_{P,r}(k) \geq s_{P,r}(o)$, then $o \notin M_{P,r^*}$ for any $r^* > r$ and remaining shares of o are allocated to these agents, i.e., $\sum_{k \in E_{P,r}(o)} p_{k,o} = s_{P,r}(o)$.*

(iii) *if the total remaining demand of agents eager for item o does not surpass the remaining shares of o, i.e., $\sum_{k \in E_{P,r}(o)} d_{P,r}(k) \leq s_{P,r}(o)$, then these agents' demands are satisfied by shares of o, i.e., $p_{j,o} = d_{P,r}(j)$.*

Proof. (i) By the condition, we have that $o^* \notin M_{P,r}$, i.e., $\sum_{k\in\bigcup_{r'<r} E_{P,r'}(o)} p_{k,o} = 1$, which means that $j \notin E_{P,*}(o^*)$ with $^* \geq r$. We also have that $j \notin E_{P,r'}(o^*)$ for any $r' < r$, since $top(j, M_{P,r'}) \succ_j o^*$. Together we have that j does not eager for o^*, which leads to $p_{j,o^*} = 0$.

(ii) Assume that $o \in M_{P,r^*}$ with $r^* > r$. By (i) for r and $\sum_{k\in E_{P,r}(o)} d_{P,r}(k) \geq s_{P,r}(o)$, there exists agent $j' \in E_{P,r}(o)$ such that $\sum_{o'\in U(\succ_{j'},o)} p_{j',o'} = d_{P,r}(j') + p_{j',o} < 1$, a contradiction to P satisfying ea-FERI.

(iii) Assume that there exists $j' \in E_{P,r}(o)$ with $p_{j',o} < d_{P,r}(j')$. Then by $s_{P,r+1}(o) > s_{P,r}(o) - \sum_{k\in E_{P,r}(o)} d_{P,r}(k) \geq 0$, we have that $\sum_{o'\in U(\succ_{j'},o)} p_{j',o'} < 1$ and $o \in M_{P,r+1}$, a contradiction. □

Theorem 4. *Given a profile R, a random assignment P satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI) if and only if there exists an eating speed function ω such that $P = PRE_\omega(R)$.*

Proof. **(Satisfaction)** Let $P = \mathrm{PRE}_\omega(R)$ where ω is any collection of eating functions. By Lemma 3 (iii), we have that for any item $o \in M_{P,r}$, $\sum_{o'\in U(\succ_j,o)} p_{j,o'} = 1$ for any $j \in E_{P,r'}(o)$ with $r' < r$, which means that P satisfies ea-FERI.

(Uniqueness) Given Q satisfying ea-FERI, we prove that it coincides with the outcome $P = \mathrm{PRE}_\omega(R)$ where the eating functions in ω are as defined in Eq (17) for each agent j:

$$\omega_j(t) = \begin{cases} n \ \cdot \ q_{j,o}, & t \in [\frac{r-1}{n}, \frac{r}{n}], \text{ where } r = \min(\{\hat{r} \mid j \in E_{Q,\hat{r}}(o)\}), \\ 0, & \text{others.} \end{cases} \tag{17}$$

We prove by mathematical induction that the following conditions hold for any round r:
Condition (1): $M_{P,r} = M_{Q,r}$, and $E_{P,r}(o) = E_{Q,r}(o)$ for each $o \in M_{Q,r}$.

Condition (2): for any $j \in E_{Q,r}(o)$, if agent j is not satisfied by items in $U(\succ_j, top(j, M_{Q,r-1})$, i.e., $\sum_{o'\in U(\succ_j,top(j,M_{Q,r-1}))} p_{j,o'} < 1$, the start time $t_j = (r-1)/n$ and the consumption time $\rho_o = 1/n$, and

Condition (3): for any $j \in E_{Q,r}(o)$ and $o \in M_{Q,r}$, $p_{j,o} = q_{j,o}$.

Base case. When $r = 1$, we trivially have that $M_{Q,1} = M' = M$ and $E_{Q,1}(o) = N_o$ for any $o \in M$ at round 1 in Algorithm 3, and each $j \in N_o$ consumes o. With Lemma 3 (i), we have that condition (1) holds for $r = 1$.

Then we show condition (2) holds for $r = 1$. By Line 2, $s(o)$, the supply of o, is set to 1 for any $o \in M'$, and for any $j \in E_{Q,r}(o)$, t_j is set to $0 = (r-1)/n$. Since $\sum_{k\in N_o}\int_{t_k}^{1} \omega_k(t)\mathrm{d}t \geq s(o) = 1$, $\rho_o = \min\{\rho \mid \sum_{k\in N_o}\int_0^\rho \omega_k(t)\mathrm{d}t = s(o)\}$ by Line 6.1, and o is consumed to exhaustion. We also have that $\sum_{k\in N_o} p_{k,o} = 1$ for P. Otherwise, $o \in M_{P,2}$, and there exists $j' \in E_{P,1}(o)$ with $\sum_{o'\in U(\succ_j,o)} p_{j',o'} = p_{j',o} < 1$, a contradiction to P satisfying ea-FERI by the satisfaction part above. Similarly, with Q satisfying ea-FERI, $\sum_{k\in N_o} q_{j,o} = 1 = \sum_{k\in N_o} p_{k,o}$. Therefore by Eq (17), $\sum_{k\in N_o}\int_0^{1/n} \omega_k(t)\mathrm{d}t = 1$, which means that ρ_o, the time for consuming o, is exactly $1/n$. Together we have that condition (2) holds for $r = 1$.

With $\rho_o = 1/n$, $p_{j,o} = \int_0^{1/n} \omega_j(t)\mathrm{d}t = q_{j,o}$, i.e., condition (3) holds for $r = 1$.

Inductive step. Supposing that conditions (1)-(3) hold for $r' < r$, we show they also hold for r. First, since conditions (1) and (3) hold for $r' < r$, we trivially have that condition (1) holds for r.

Next we show condition (2) holds for r. By Lemma 3 (i) and condition (1) for r which we just prove, it holds that $M_{Q,r} = M'$ and $E_{Q,r}(o) = N_o$ for each $o \in M_{Q,r}$. By Line 8 and condition (2) for $r-1$, we have that for any $j \in E_{Q,r}(o)$ with $\sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} p_{j,o'} < 1$, $t_j = (r-1)/n$.

Then we show that the consumption time $\rho_o = 1/n$. Let $N'_o = \{k \in N_o \mid \sum_{o' \in U(\succ_k, top(j, M_{Q,r-1}))} p_{k,o'} < 1\}$, and we see that agent $j' \in N_o \setminus N'_o$ does not consume o at round r since they have been satisfied with $\int_0^{t_j} \omega_j(t)\mathrm{d}t = \sum_{o' \in U(\succ_{j'}, top(j, M_{Q,r-1}))} p_{j',o'} \geq 1$. Besides, we define $M^* = \{o^* \mid k \in E_{Q,r'}(o^*) \text{ with } r' < r\} = \{o^* \mid k \in E_{P,r'}(o^*) \text{ with } r' < r\}$ by conditions (1) and (3) for $r' < r$. We show that $\rho_o = 1/n$ in both cases about $s(o)$, now the remaining shares of o at round r in Algorithm 3. We also note that with Lemma 3 (i), $s(o) = \sum_{k \notin \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o} = \sum_{k \notin \bigcup_{r'<r} E_{Q,r'}(o)} q_{k,o}$, which means that $s(o)$ is also the total shares of o owned by agents not in $E_{Q,r'}(o)$ with $r' < r$ in assignment Q.

- If $\sum_{k \in N_o} \int_{t_k}^1 \omega_k(t)\mathrm{d}t \geq s(o)$, then o is consumed to exhaustion, i.e. $\sum_{k \in N'_o} p_{k,o} = s(o)$. Otherwise, assume for the sake of contradiction that $\sum_{k \in N'_o} p_{k,o} < s(o)$. Then by the assumption, $o \in M_{P,r+1}$, and there exists $j' \in E_{P,r}(o)$ with $p_{j',o} < \int_{t_j}^1 \omega_j(t)\mathrm{d}t$, which means that $\sum_{o' \in U(\succ_{j'}, o)} p_{j',o'} = \sum_{o' \in M^*} p_{j',o} + p_{j',o} < \int_0^1 \omega_j(t)\mathrm{d}t = 1$ by Lemma 4 (i), a contradiction to P satisfying ea-FERI. Since Q also satisfies ea-FERI, we have that $o \notin M_{Q,r+1}$, and with conditions (1) and (3) for $r' < r$, we have that $\sum_{k \in N'_o} q_{k,o} = s(o) = \sum_{k \in N'_o} p_{k,o}$. We also note that $r = \min(\{\hat{r} \mid j \in E_{Q,\hat{r}}(o)\})$ for any $j \in N'_o$, because otherwise $j \in E_{Q,r'}(o) = E_{P,r'}(o)$ with $r' < r$ and $o \in M_{P,r}$, while $\sum_{o' \in U(\succ_j, o)} p_{j,o'} \leq \sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} p_{j,o'} < 1$, a contradiction to P satisfying ea-FERI. Then by Eq (17) and Line 6.1 of Algorithm 3,

$$\rho_o = \min\{\rho \mid \sum_{k \in N_o} \int_{t_k}^{t_k+\rho} \omega_k(t)\mathrm{d}t = s(o)\} = 1/n.$$

- If $\sum_{k \in N_o} \int_{t_k}^1 \omega_k(t)\mathrm{d}t < s(o)$, then $o \in M_{P,r+1}$, and we have that all the agents in N'_o are satisfied, i.e., $\sum_{k \in N'_o} p_{k,o} = \sum_{k \in N_o} \int_{t_k}^{t_k+\rho_o} \omega_k(t)\mathrm{d}t = \sum_{k \in N_o} \int_{t_k}^1 \omega_k(t)\mathrm{d}t$. Otherwise, there exists $j' \in E_{P,r}(o)$ who is not satisfied with $\sum_{o' \in U(\succ_{j'}, o)} p_{j',o'} = \sum_{o' \in M^*} p_{j',o} + p_{j',o} < 1$ by Lemma 4 (i), a contradiction to P satisfying ea-FERI. With conditions (1) and (3) for $r' < r$, we also have that

$$\sum_{k \in N'_o} q_{k,o} \leq \sum_{k \in N'_o} (1 - \sum_{o' \in M^*} q_{k,o'}) = \sum_{k \in N'_o} (1 - \sum_{o' \in M^*} p_{k,o'}) = \sum_{k \in N'_o} \int_{t_k}^1 \omega_k(t)\mathrm{d}t = \sum_{k \in N'_o} p_{k,o}. \quad (18)$$

We also claim that $\sum_{k \in N'_o} q_{j,o} = \sum_{k \in N'_o} (1 - \sum_{o' \in M^*} q_{k,o'})$ in Eq (18). Otherwise, $o \in M_{Q,r+1}$ since $\sum_{k \in N'_o} q_{k,o} < \sum_{k \in N'_o} p_{k,o} < s(o)$, and there exists $j' \in N'_o$ with $\sum_{o' \in U(\succ_{j'}, o)} q_{j',o} = \sum_{o' \in M^*} q_{j',o} + q_{j',o} < 1$ by Lemma 4 (i), a contradiction to Q satisfying ea-FERI. With $r = \min(\{\hat{r} \mid j \in E_{Q,\hat{r}}(o)\})$ for any $j \in N'_o$, and Eq (17) and (18), we obtain that $\rho_o = 1/n$.

Together we have that condition (2) holds for r.

Finally, we show that condition (3) holds for r. For any $j \in E_{Q,r}(o)$, if $j \in E_{Q,r'}(o)$ with some $r' < r$, then we have the proof trivially by the fact that condition (3) holds for r'. Next we consider the case that $j \notin E_{Q,r'}(o)$, i.e., $o \neq top(j, M_{Q,r'})$ with any $r' < r$. We show $q_{j,o} = p_{j,o}$ in both of the possible cases below:

- If $\sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} p_{j,o'} < 1$, then by the fact that condition (2) for r which we just proved above, $p_{j,o} = \int_{(r-1)/n}^{r/n} \omega_k(t) \mathrm{d}t = q_{j,o}$.

- If $\sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} p_{j,o'} = 1$, then $\sum_{o' \in U(\succ_j, top(j, M_{P,r-1}))} p_{j,o'} = 1$ by condition (1) for $r-1$, which means that j is satisfied before round r by Lemma 4 (i) and does not consume o since $o \neq top(j, M_{Q,r'}) = top(j, M_{P,r'})$ for any $r' < r$. Therefore $p_{j,o} = 0$. As for Q, by condition (3) for $r' < r$, $\sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} q_{j,o'} \geq \sum_{o' \in U(\succ_j, top(j, M_{Q,r-1}))} p_{j,o'} = 1$. Because $o \neq top(j, M_{Q,r-1})$ and $o \in M_{P,r} = M_{Q,r} \subseteq M_{Q,r-1}$ by condition (1) for r, we have that $top(j, M_{Q,r-1}) \succ_j o$, and therefore $q_{j,o} = 0 = p_{j,o}$.

Together we have that condition (3) holds for r.

From the induction above, we have that conditions (1) and (3) hold for any r, i.e., for any r, we have that $M_{P,r} = M_{Q,r}$, $E_{P,r}(o) = E_{Q,r}(o)$ for any $o \in M_{Q,r}$, and $p_{j,o} = q_{j,o}$ for any $j \in E_{Q,r}(o)$. In Algorithm 3, shares of o are only allocated to agents in $E_{P,r}(o)$ in each round, and o is exhausted at the end, which means that $p_{j',o} = 0$ if $j' \notin E_{P,r}(o) = E_{Q,r}(o)$ for any r. With the fact that the supply of all the items is fully allocated to agents, it follows that $q_{j',o} = 0$ if $j' \notin E_{Q,r}(o)$ for any r by condition (3). Together we have that $P = Q$. □

Proposition 5. *No mechanism simultaneously satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI) and sd-envy-freeness (sd-EF).*

Proof. For ease of reading, we recall the preference profile R and assignment Q used in Proposition 4, which are used in the following proof.

Preference Profile R

$\succ_1$: $a \succ_1 c \succ_1 b \succ_1 d$,
$\succ_2$: $a \succ_2 c \succ_2 b \succ_2 d$,
$\succ_3$: $a \succ_3 b \succ_3 c \succ_3 d$,
$\succ_4$: $b \succ_4 a \succ_4 d \succ_4 c$.

Assignment Q

	a	b	c	d
1	1/3	0	?	?
2	1/3	0	?	?
3	1/3	0	?	?
4	0	1	0	0

For any assignment P satisfying ea-FERI, we have that $E_{P,1}(a) = \{1,2,3\}$ and $E_{P,1}(b) = \{4\}$. It follows that $\sum_{k \in E_{P,1}(a)} d_{P,1}(k) > s_{P,1}(a)$, and therefore $a \notin M_{P,r}$ with $r > 1$ by Lemma 4 (ii), which means that only agents $1,2$ and 3 get shares of a. It also follows that agent 4 fully gets b for the same token. Then any assignment satisfying ea-FERI and SETE (implied by sd-EF) is in the form of Q, but Q does not satisfy sd-EF as we have shown in Proposition 4. □

Proposition 7. *No mechanism simultaneously satisfies ex-ante favoring-eagerness-for-remaining-items (ea-FERI), strong equal treatment of equals (SETE), and sd-weak-strategy proofness (sd-WSP).*

Proof. Assume such mechanism f exists. Let R be:

$$\begin{aligned}
\succ_1&: a \succ_1 b \succ_1 \cdots \succ_1 h \succ_1 c,\\
\succ_2&: a \succ_2 h \succ_2 \cdots \succ_2 b \succ_2 c,\\
\succ_{3\text{-}7}&: c \succ d \succ e \succ f \succ g \succ b \succ h \succ a,\\
\succ_8&: c \succ_8 d \succ_8 b \succ_8 e \succ_8 f \succ_8 g \succ_8 h \succ_8 a.
\end{aligned}$$

Let $P = f(R)$. Recall the notations in Lemma 4 that $s_{P,r}(o) = 1 - \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o}$, and $d_{P,r}(j) = 1 - \sum_{o' \succ_j o} p_{j,o'}$ for any $j \in E_{P,r}(o)$.

- For $r = 1$, by Lemma 4 (ii), since $E_{P,1}(a) = \{1, 2\}$ and $\sum_{k \in E_{P,1}(a)} d_{P,1}(k) > s_{P,1}(a)$, only agents 1 and 2 gets shares of a. It follows that only agents in $E_{P,1}(c) = \{3, \ldots, 8\}$ gets c for the same token. Then we have $p_{1,a} = p_{2,a} = 1/2$, $p_{j,c} = 1/6$ for $j \in \{3, \ldots, 8\}$ by SETE.

- For $r = 2$, $M_{P,2} = \{b, d, \ldots, h\}$, $E_{P,2}(b) = \{1\}$, $E_{P,2}(h) = \{2\}$, and $E_{P,2}(d) = \{3, \ldots, 8\}$. With $\sum_{k \in E_{P,2}(d)} d_{P,2}(k) > s_{P,2}(d)$, we have that $p_{j,d} = 1/6$ for $j \in \{3, \ldots, 8\}$ by Lemma 4 (ii) and SETE. With $\sum_{k \in E_{P,2}(b)} d_{P,2}(k) \leq s_{P,2}(b)$, $p_{1,b} = d_{P,2}(1) = 1/2$ by Lemma 4 (iii), and it follows that $p_{2,h} = d_{P,2}(2) = 1/2$ for the same token.

- For $r = 3$, $M_{P,3} = \{b, e, \ldots, h\}$, $E_{P,3}(b) = \{8\}$, and $E_{P,3}(e) = \{3, \ldots, 7\}$. With Lemma 4 (ii) and $\sum_{k \in E_{P,3}(d)} d_{P,3}(k) > s_{P,3}(d)$, we have that $p_{8,b} = 1/2$. With $\sum_{k \in E_{P,3}(e)} d_{P,3}(k) > s_{P,3}(e)$, $p_{j,e} = 1/5$ for $j \in \{3, \ldots, 7\}$ by SETE.

With the analysis above, we have assignment P in the following form.

Assignment P

	a	b	c	d	e	f	g	h
1	1/2	1/2	0	0	0	0	0	0
2	1/2	0	0	0	0	0	0	1/2
3-7	0	0	1/6	1/6	1/5	?	?	?
8	0	1/2	1/6	1/6	0	?	?	?

If agent 8 misreports her preference as

$$\succ_8': c \succ_8' d \succ_8' e \succ_8' b \succ_8' f \succ_8' g \succ_8' h \succ_8' a,$$

then let $P' = f(R')$ for $R' = (\succ_8', \succ_{-8})$.

- The analysis for P' with $r = 1$ and 2 is the same as P.

- For $r = 3$, $M_{P',3} = \{b, e, \ldots, h\}$ and $E_{P',3}(e) = \{3, \ldots, 8\}$. With $\sum_{k \in E_{P',3}(e)} d_{P',3}(k) > s_{P',3}(e)$, $p'_{j,e} = 1/6$ for $j \in \{3, \ldots, 8\}$ by SETE.

- For $r = 4$, $M_{P',4} = \{b, f, g, h\}$, $E_{P',4}(b) = \{8\}$, and $E_{P',4}(f) = \{3, \ldots, 7\}$. With Lemma 4 (iii) and $\sum_{k \in E_{P',4}(b)} d_{P',4}(k) = s_{P',4}(b)$, $p'_{8,b} = d_{P',4}(8) = 1/2$.

Then we obtain the assignment P' in the following form.

Assignment P'

	a	b	c	d	e	f	g	h
1	1/2	1/2	0	0	0	0	0	0
2	1/2	0	0	0	0	0	0	1/2
3-7	0	0	1/6	1/6	1/6	?	?	?
8	0	1/2	1/6	1/6	1/6	0	0	0

We see that P'_8 strictly dominates P_8 for $\sum_{o\in U(8,e)} p_{8,o} = 5/6 < 1 = \sum_{o\in U(8,e)} p'_{8,o}$ and $\sum_{o\in U(8,o')} p_{8,o} \leq \sum_{o\in U(8,o')} p'_{8,o}$ for other $o' \in M$, a contradiction to the fact that f is sd-WSP. $\square$

Proposition 8. *No mechanism simultaneously satisfies ex-post favoring-eagerness-for-remaining-items (ep-FERI), ex-ante favoring-eagerness-for-remaining-items (ea-FERI), and strong equal treatment of equals (SETE).*

Proof. Assume that there exists a mechanism f satisfying ea-FERI and SETE. Let $P = f(R)$ for the following preference profile R, and then we show that P is not ep-FERI,

$$
\begin{aligned}
\succ_{1,2}&: a_1 \succ_1 a_2 \succ_1 a_3 \succ_1 \text{others}\\
\succ_{3}&: a_1 \succ_3 a_2 \succ_3 a_4 \succ_3 \text{others}\\
\succ_{4,5}&: b_1 \succ_4 b_2 \succ_4 b_3 \succ_4 \text{others}\\
\succ_{6}&: b_1 \succ_6 b_2 \succ_6 b_4 \succ_6 \text{others}\\
\succ_{7\text{-}17}&: c_1 \succ_7 c_2 \succ_7 c_3 \succ_7 c_4 \succ_7 c_5 \succ_7 c_6 \succ_7 \text{others}\\
\succ_{x}&: c_1 \succ_x c_2 \succ_x c_3 \succ_x a_3 \succ_x b_3 \succ_x c_5 \succ_x c_4\\
&\quad \succ_x c_6 \succ_x \text{others}
\end{aligned}
$$

- For $r = 1$, $M_{P,1} = M, E_{P,1}(a_1) = \{1,2,3\}, E_{P,1}(b_1) = \{4,5,6\}$ and $E_{P,1}(c_1) = \{7,\dots,17,x\}$. By Lemma 4 (ii), item $a_1, b_1, c_1 \notin M_{P,2}$ since $d_{P,1}(j) = 1$ for any $j \in N$. Then we have $p_{j,a_1} = 1/3$ for $j \in E_{P,1}(a_1)$, $p_{i',b_1} = 1/3$ for $j' \in E_{P,1}(b_1)$ and $p_{j^*,c_1} = 1/12$ for $j^* \in E_{P,1}(c_1)$ by SETE.

- For $r = 2$, $M_{P,2} = \{a_2, a_3, a_4, b_2, b_3, b_4, c_2, \dots, c_6, \dots\}, E_{P,2}(a_2) = \{1,2,3\}, E_{P,2}(b_2) = \{4,5,6\}$ and $E_{P,2}(c_2) = \{7,\dots,17,x\}$. Similar to $r = 1$, by Lemma 4 (ii), $a_2, b_2, c_2 \notin M_{P,3}$. Then we have $p_{j,a_2} = 1/3$ for $j \in E_{P,2}(a_2)$, $p_{i',b_2} = 1/3$ for $j' \in E_{P,2}(b_2)$ and $p_{j^*,c_2} = 1/12$ for $j^* \in E_{P,2}(c_2)$ by SETE.

- For $r = 3$, $M_{P,3} = \{a_3, a_4, b_3, b_4, c_3, \dots, c_6, \dots\}$, $E_{P,3}(a_3) = \{1,2\}$, $E_{P,3}(a_4) = \{3\}$, $E_{P,3}(b_3) = \{4,5\}, E_{P,3}(b_4) = \{6\}$, and $E_{P,3}(c_3) = \{7,\dots,17,x\}$. By Lemma 4 (iii), $p_{j,a_3} = 1/3$ for $j \in E_{P,3}(a_3)$, $p_{3,a_4} = 1/3$, $p_{j',b_3} = 1/3$ for $j' \in Ej_{P,3}(b_3)$, and $p_{6,b_4} = 1/3$. Since $\sum_{\hat{o}\in U(\succ_j, top(j,M_{P,3}))} = 1$ for any agent $j \in N' = \{1,\dots,6\}$, their allocations have been determined and we do not need to consider them for $r > 3$.

- For $r = 4$, $M_{P,4} = \{a_3, a_4, b_3, b_4, c_4, c_5, c_6, \dots\}$. Then $E_{P,4}(c_4) \setminus N' = \{7,\dots,17\}$ and $E_{P,4}(a_3) \setminus N' = \{x\}$. By Lemma 4 (ii), $a_3, c_4 \notin M_{P,5}$, $p_{x,a_3} = 1/3$ and $p_{j',c_4} = 1/11$ for $j' \in E_{P,4}(c_4)$ by SETE.

- For $r = 5$, $M_{P,5} = \{a_4, b_3, b_4, c_5, c_6, \dots\}$, $E_{P,5}(c_5) \setminus N' = \{7, \dots, 17\}$ and $E_{P,5}(b_3) \setminus N' = \{x\}$. By Lemma 4 (ii), $b_3, c_5 \notin M_{P,6}$, $p_{x,b_3} = 1/3$ and $p_{j',c_5} = 1/11$ for $j' \in E_{P,5}(c_5)$ by SETE.

- For $r = 6$, $M_{P,5} = \{a_4, b_4, c_6, \dots\}$, $E_{P,6}(c_6) \setminus N' = \{7, \dots, 17, x\}$. By Lemma 4 (ii), $c_6 \notin M_{P,7}$ and $p_{j',c_6} = 1/12$ for $j' \in E_{P,6}(c_6)$ by SETE.

We show the part of P which has been determined by $r \leq 6$ in the following P(i) for agents $\{1, 2, 3, x\}$ over items $\{a_1, \dots, a_4\}$, P(ii) for agents $\{4, 5, 6, x\}$ over items $\{b_1, \dots, b_4\}$, and P(iii) for agents $\{7, \dots, 17, x\}$ over items $\{c_1, \dots, c_6\}$.

Assignment P(i)

	a_1	a_2	a_3	a_4
1	1/3	1/3	1/3	0
2	1/3	1/3	1/3	0
3	1/3	1/3	0	1/3
x	0	0	1/3	0

Assignment P(ii)

	b_1	b_2	b_3	b_4
4	1/3	1/3	1/3	0
5	1/3	1/3	1/3	0
6	1/3	1/3	0	1/3
x	0	0	1/3	0

Assignment P(iii)

	c_1	c_2	c_3	c_4	c_5	c_6
7-17	1/12	1/12	1/12	1/11	1/11	1/12
x	1/12	1/12	1/12	0	0	1/12

There exists an assignment A with $A(x) = c_6$ among the deterministic assignments which constitute the convex combination for P. In the following, we prove that none of such A is FERI. According to P, a_3 is assigned to one of $\{1, 2\}$ in A since agent x does not get it. Due to the fact that $\succ_1 = \succ_2$, let $A(1) = a_3$ without loss of generality. With the fact that only agents in $\{1, 2, 3\}$ can get $\{a_1, a_2\}$, we have that agents $\{2, 3\}$ get $\{a_1, a_2\}$. It follows that b_3 is assigned to one of $\{4, 5\}$ and $\{4, 5, 6\}$ get $\{b_1, b_2, b_3\}$ for the same token. Due to the fact that $\succ_4 = \succ_5$, let $A(4) = b_3$ without loss of generality, and therefore $\{5, 6\}$ get $\{b_1, b_2\}$. Agents in $\{7, \dots, 17\}$ get the rest items, and for ease of exposition, let agent j_i with $i \in \{1, \dots, 6\}$ satisfy $j_i \in \{7, \dots, 17\}$ and $j_i = A^{-1}(c_i)$. We further have the following analysis about checking if A satisfies FERI:

- For $r = 1$, $T_{A,1} = \{a_1, b_1, c_1\}$ because $M_1 = M$, $top(j, M_1) = a_1$ for $j \in \{1, 2, 3\}$, $top(j', M_1) = b_1$ for $j' \in \{4, 5, 6\}$, and $top(j^*, M_1) = c_1$ for $j^* \in \{7, \dots, 11, x\}$. Then one of $\{2, 3\}$ gets a_1, one of $\{4, 5\}$ gets b_1, and agent j_1 gets c_1 by FERI.

- For $r = 2$, no matter which $j \in \{2, 3\}$ gets a_1 and which $j' \in \{4, 5\}$ gets b_1, $T_{A,2} = \{a_2, b_2, c_2\}$ because for $M_2 = M \setminus T_{A,1}$, $top(j, M_2) = a_2$ for $j \in \{1, 2, 3\}$, $top(j', M_2) = b_2$ for $j' \in \{4, 5, 6\}$, and $top(j^*, M_2) = c_2$ for $j^* \in \{7, \dots, 11, x\}$. Then the rest one of $\{2, 3\}$ gets a_2, the rest one of of $\{4, 5\}$ gets b_2, and agent j_2 gets c_2 by FERI.

- For $r = 3$, we do not consider $j' \in \{2, 3, 5, 6, j_1, j_2\}$ because $A(j') \in \bigcup_{r'<3} T_{A,r'}$. We obtain that $T_{A,3} = \{a_3, b_3, c_3\}$ because for $M_3 = M \setminus \bigcup_{r'<3} T_{A,r'}$, $top(1, M_3) = a_3$, $top(4, M_3) = b_3$, and $top(j, M_3) = c_3$ for $j \in \{7, \dots, 11, x\}$. Then agent 1 gets a_3, agent 4 gets b_3, and agent j_3 gets c_3 by FERI.

- For $r = 4$, we do not consider $j' \in \{1, \ldots, 6, j_1, j_2, j_3\}$. We obtain that $T_{A,4} = \{c_4, c_5\}$ because for $M_4 = M \setminus \bigcup_{r'<4} T_{A,r'}$, $top(j, M_4) = c_4$ for $j \in \{7, \ldots, 11\}$ and $top(x, M_4) = c_5$. However, we have that $A^{-1}(c_5) = j_5$ and $top(j_5, M_4) = c_4$, which violates FERI.

With the analysis above, we have that A does not satisfy FERI, and therefore P does not satisfy ep-FERI, which means that f does not satisfy ep-FERI, ea-FERI, and SETE simultaneously. $\square$

References

Abdulkadiroğlu, A., & Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, *93*(3), 729–747.

Abdulkadiroğlu, A., Che, Y.-K., Pathak, P. A., Roth, A. E., & Tercieux, O. (2017). Minimizing justified envy in school choice: The design of new orleans' oneapp. Working paper 23265, National Bureau of Economic Research.

Abdulkadiroğlu, A., Pathak, P. A., Roth, A. E., & Sönmez, T. (2006). Changing the boston school choice mechanism: Strategy-proofness as equal access. Working paper. Accessed: 2021-09-08.

Abdulkadiroğlu, A., & Sönmez, T. (1998). Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, *66*(3), 689–702.

Abraham, D., Chen, N., Kumar, V., & Mirrokni, V. S. (2006). Assignment problems in rental markets. In *Internet and Network Economics*, pp. 198–213, Berlin, Heidelberg. Springer.

Abraham, D., Irving, R., Kavitha, T., & Mehlhorn, K. (2007). Popular matchings. *SIAM J. Comput.*, *37*(4), 1030–1045.

Abraham, D. J. (2009). *Matching Markets: Design and Analysis*. Ph.D. thesis, School of Computer Science, Carnegie Mellon University.

Alcalde, J. (1996). Implementation of stable solutions to marriage problems. *Journal of Economic Theory*, *69*(1), 240–254.

Bartholdi, J. J., & Orlin, J. B. (1991). Single transferable vote resists strategic voting. *Social Choice and Welfare*, *8*(4), 341–354.

Belahcene, K., Mousseau, V., & Wilczynski, A. (2021). Combining fairness and optimality when selecting and allocating projects. In *30th International Joint Conference on Artificial Intelligence (IJCAI-21)*, pp. 38–44.

Bogomolnaia, A., & Moulin, H. (2001). A new solution to the random assignment problem. *Journal of Economic Theory*, *100*(2), 295–328.

Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, *119*(6), 1061–1103.

Chen, Y., Harless, P., & Jiao, Z. (2021). The probabilistic rank random assignment rule and its axiomatic characterization. Working paper. Accessed: 2021-09-06.

Chen, Y., & Sönmez, T. (2002). Improving efficiency of on-campus housing: An experimental study. *American Economic Review*, *92*(5), 1669–1686.

Dur, U., Mennle, T., & Seuken, S. (2018). First-choice maximal and first-choice stable school choice mechanisms. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pp. 251–268, New York, USA. ACM.

Dur, U. M. (2019). The modified Boston mechanism. *Mathematical Social Sciences*, *101*, 31–40.

Foley, D. K. (1966). *Resource Allocation and the Public Sector*. Ph.D. thesis, Yale University.

Garg, N., Kavitha, T., Kumar, A., Mehlhorn, K., & Mestre, J. (2010). Assigning papers to referees. *Algorithmica*, *58*(1), 119–136.

Ghodsi, A., Sekar, V., Zaharia, M., & Stoica, I. (2012). Multi-resource fair queueing for packet processing. In *Proceedings of the ACM SIGCOMM 2012 Conference on Applications, Technologies, Architectures, and Protocols for Computer Communication*, pp. 1–12, New York, USA. ACM.

Ghodsi, A., Zaharia, M., Hindman, B., Konwinski, A., Shenker, S., & Stoica, I. (2011). Dominant resource fairness: Fair allocation of multiple resource types. In *Proceedings of the 8th USENIX Conference on Networked Systems Design and Implementation*, pp. 323–336, Boston, USA.

Grandl, R., Ananthanarayanan, G., Kandula, S., Rao, S., & Akella, A. (2014). Multi-resource packing for cluster schedulers. In *Proceedings of the 2014 ACM Conference on SIGCOMM*, pp. 455–466, New York, USA. ACM.

Hare, T. (1861). *A Treatise on The Election of Representatives, Parliamentary and Municipal*. Longman, Green, Longman, & Roberts.

Harless, P. (2018). Immediate acceptance without priorities: preserving efficiency while respecting rank. Working paper. Accessed: 2021-09-06.

Heo, E. J. (2014). Probabilistic assignment problem with multi-unit demands: A generalization of the serial rule and its characterization. *Journal of Mathematical Economics*, *54*, 40–47.

Hylland, A., & Zeckhauser, R. (1979). The efficient allocation of individuals to positions. *Journal of Political Economy*, *87*(2), 293–314.

Irving, R. W., Kavitha, T., Mehlhorn, K., Michail, D., & Paluch, K. E. (2006). Rank-maximal matchings. *ACM Transactions on Algorithms*, *2*(4), 602–610.

Katta, A.-K., & Sethuraman, J. (2006). A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, *131*(1), 231–250.

Kojima, F. (2009). Random assignment of multiple indivisible objects. *Mathematical Social Sciences*, *57*(1), 134–142.

Kojima, F., & Ünver, M. U. (2014). The "Boston" school-choice mechanism: an axiomatic approach. *Economic Theory*, *55*(3), 515–544.

Marsaglia, G. (2003). Xorshift RNGs. *Journal of Statistical Software*, *8*(14), 1–6.

Mennle, T., & Seuken, S. (2021). Partial strategyproofness: Relaxing strategyproofness for the random assignment problem. *Journal of Economic Theory*, *191*, 105–144.

Miralles, A. (2009). School choice: the case for the boston mechanism. In *Auctions, Market Mechanisms and Their Applications*, pp. 58–60, Berlin, Heidelberg. Springer.

Nesterov, A. S. (2017). Fairness and efficiency in strategy-proof object allocation mechanisms. *Journal of Economic Theory*, *170*, 145–168.

Osborne, M. J. (2004). *An Introduction to Game Theory*, chap. 12, pp. 377–392. Oxford University Press.

Paluch, K. (2013). Capacitated rank-maximal matchings. In *International Conference on Algorithms and Complexity*, pp. 324–335, Berlin, Heidelberg. Springer.

Pathak, P. A. (2017). *What Really Matters in Designing School Choice Mechanisms*, Vol. 1 of *Econometric Society Monographs*, p. 176–214. Cambridge University Press.

Pathak, P. A., & Sönmez, T. (2008). Leveling the playing field: Sincere and sophisticated players in the boston mechanism. *American Economic Review*, *98*(4), 1636–1652.

Ramezanian, R., & Feizi, M. (2021). Ex-post favoring ranks: A fairness notion for the random assignment problem. *Review of Economic Design*, *25*, 157–176.

Roth, A. E., Sönmez, T., & Ünver, M. U. (2005). Pairwise kidney exchange. *Journal of Economic Theory*, *125*(2), 151–188.

Saban, D., & Sethuraman, J. (2015). The complexity of computing the random priority allocation matrix. *Mathematics of Operations Research*, *40*(4), 1005–1014.

Segal-Halevi, E., Hassidim, A., & Aziz, H. (2020). Fair allocation with diminishing differences. *Journal of Artificial Intelligence Research*, *67*, 471–507.

Shapley, L., & Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, *1*(1), 23–37.

Thomson, W. (1958). A modified congruence method of generating pseudo-random numbers. *The Computer Journal*, *1*(2), 83–83.

Varian, H. R. (1973). Equity, envy, and efficiency. *Journal of Economic Theory*, *9*(1), 63–91.

Wang, H., Sikdar, S., Guo, X., Xia, L., Cao, Y., & Wang, H. (2020). Multi-type resource allocation with partial preferences. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pp. 2260–2267, New York. AAAI Press.

Zhou, L. (1990). On a conjecture by gale about one-sided matching problems. *Journal of Economic Theory*, *52*(1), 123–135.