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A Logic of East and West

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Abstract

We propose a logic of east and west (LEW) for points in 1D Euclidean space. It formalises primitive direction relations: east (E), west (W) and indeterminate east/west (I_{ew}). It has a parameter $\tau \in \mathbb{N}_{>1}$, which is referred to as the level of indeterminacy in directions. For every $\tau \in \mathbb{N}_{>1}$, we provide a sound and complete axiomatisation of LEW, and prove that its satisfiability problem is NP-complete. In addition, we show that the finite axiomatisability of LEW depends on τ: if $\tau = 2$ or $\tau = 3$, then there exists a finite sound and complete axiomatisation; if $\tau > 3$, then the logic is not finitely axiomatisable. LEW can be easily extended to higher-dimensional Euclidean spaces. Extending LEW to 2D Euclidean space makes it suitable for reasoning about not perfectly aligned representations of the same spatial objects in different datasets, for example, in crowd-sourced digital maps.

1. Introduction

This work is motivated by the problem of matching spatial objects in different geospatial datasets and verifying logical consistency of *SameAs* matching relations. Geospatial datasets contain spatial information (e.g., geometries and coordinates) and semantic information (e.g., classifications, names, functions) of spatial objects. A matching relation $SameAs(a, b)$ states that two spatial objects a, b in different datasets refer to the same object in the real world. It is challenging to verify logical consistency of *SameAs* matching relations with respect to spatial information. One main reason is that the same real-world object is often represented using different geometries or coordinates in different geospatial datasets. To tolerate slight differences in geometric representations, a number of qualitative distance logics have been proposed for reasoning about qualitative distances between spatial objects from different datasets (Du et al., 2013; Du & Alechina, 2016). However, these spatial logics do not cover the direction aspect, which is an important dimension of spatial information. In this work, we propose new qualitative direction logics for validating matching relations with respect to qualitative directions between spatial objects.

Several qualitative spatial or temporal calculi have been developed for formalizing and reasoning about direction or ordering relations (Aiello et al., 2007; Ligozat, 2012). These include the point calculus (Vilain & Kautz, 1986) which defines three ordering relations $<$ (less than), $>$ (greater than) and eq (equal) for points in 1D Euclidean space, Allen's calculus (Allen, 1983), the cardinal direction calculus which extends the point calculus to 2D Euclidean space (Ligozat, 1998), the rectangle algebra (Balbiani et al., 1998), the $2n$-star calculi which generalize the cardinal direction calculus by introducing a variable n referring to the granularity for defining direction relations (Renz & Mitra, 2004), and the cardinal direction relations between regions (Skiadopoulos & Koubarakis, 2004, 2005). Beside these formalisms where directions are defined using binary relations, there exist several spatial formalisms which define directions using ternary relations. These spatial formalisms include the $\mathcal{LR}$ calculus (Scivos & Nebel, 2004), the flip-flop calculus (Ligozat, 1993), the double-cross calculus (Freksa, 1992) and the 5-intersection calculus (Billen & Clementini, 2004), where relations like left, right, after, between, before, are defined.

In this paper, we propose a logic of east and west (LEW) for points in 1D Euclidean space. LEW has three primitive direction relations: E (east), W (west) and I_{ew} (indeterminate east/west). Based on the primitive relations, direction relations dE (definitely east), sE (somewhat east), nEW (neither east nor west), sW (somewhat west) and dW (definitely west) are defined. Every individual name a is interpreted as a point x_a in 1D Euclidean space (i.e., x_a is a real number, as the x coordinate of a). The truth condition of each LEW direction relation over individual names a and b is expressed using a linear inequality over x_a and x_b.

Differing from the point calculus (Vilain & Kautz, 1986), the direction relations in LEW are defined with respect to a margin of error $\sigma \in \mathbb{R}_{>0}$ for tolerating slight differences in geometric representations in different geospatial datasets, and a level of indeterminacy in directions $\tau \in \mathbb{N}_{>1}$.

Differing from the work on disjunctive linear relations (Jonsson & Bäckström, 1998), linear constraints (Koubarakis & Skiadopoulos, 2000; Ostuni et al., 2021) and the INDU calculus (Pujari et al., 1999), we take an axiomatisation-based approach and explore the existence of finite sound and complete axiomatisations of LEW, with the aim of developing rule-based reasoners based on a complete set of axioms as was done by Du et al. (2015).

Over Euclidean spaces, there exist some sound and complete axiomatisations for spatial formalisms (Tarski, 1959; Szczerba & Tarski, 1979; Tarski & Givant, 1999; Balbiani et al., 2007; Trybus, 2010); however, none of them considers direction relations. Here, for every level of indeterminacy $\tau \in \mathbb{N}_{>1}$, we provide a sound and complete axiomatisation for LEW. Some spatial logics, which can encode directions, are undecidable, e.g., the compass logic (Marx & Reynolds, 1999) and SpPNL (Morales et al., 2007). The satisfiability problem of some spatial logics (e.g., Cone by Montanari et al., 2009 and SOSL by Walega and Zawidzki, 2019) are PSPACE-complete. Here, for every level of indeterminacy $\tau \in \mathbb{N}_{>1}$, we show that the satisfiability problem of LEW is NP-complete. These results were presented by Du, Alechina, and Cohn (2020) for a 2D extension of LEW, i.e., a logic of directions. In this paper, we provide additional finite axiomatisability results. The finite axiomatisability of LEW depends on τ: if $\tau = 2$ or $\tau = 3$, then there exists a finite sound and complete axiomatisation; if $\tau > 3$, then it is not finitely axiomatisable.

The rest of this paper is structured as follows. Section 2 introduces the logic of east and west (LEW) and its higher-dimensional extensions. Section 3 presents the axiomatisations of LEW. Sections 4-6 present the soundness and completeness, non-finite axiomatisability, decidability and complexity results, respectively. Section 7 discusses this work in the wider context of qualitative spatial and temporal reasoning. Section 8 summarises the main results of this paper and directions for future work.

2. A Logic of East and West

We first present a logic of east and west (LEW) for points in 1D Euclidean space, then extend it to higher-dimensional Euclidean spaces.

2.1 Syntax and Semantics

LEW defines three primitive direction relations: east (E), west (W) and indeterminate east/west (I_{ew}).

Definition 1 (The language of LEW) *Let Ind be a set of individual names. The language $L(LEW, Ind)$ (we omit Ind for brevity below) is defined inductively as follows:*

$$\phi := E(a,b) \mid W(a,b) \mid I_{ew}(a,b) \mid \neg\phi \mid \phi \wedge \psi$$

where $a, b \in Ind$. We assume $\phi \vee \psi =_{def} \neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi =_{def} \neg(\phi \wedge \neg\psi)$, $\phi \leftrightarrow \psi =_{def} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, $\perp =_{def} \phi \wedge \neg\phi$ to be rewrite rules.

The lower case letters a, b, c, d, e and o, possibly with subscripts or superscripts, are usually used to denote individual names in Ind. The language $L(LEW)$ is a subset of the language of first-order logic (Brachman & Levesque, 2004). It does *not* include universal quantifiers, existential quantifiers or function symbols. Its predicate symbols are restricted to those for qualitative directions.

The language $L(LEW)$ is interpreted over 1D Euclidean models based on 1D Euclidean space $\mathbb{R}$. Figure 1 illustrates the primitive relations with respect to the point 0.

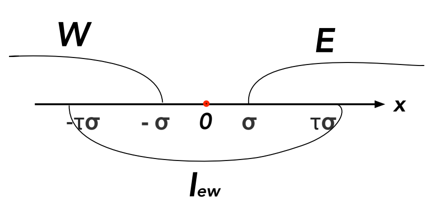

Figure 1: primitive direction relations in LEW

Definition 2 (1D Euclidean τ-model) *A* 1D Euclidean τ-model M *is a structure $(\mathcal{I}, \sigma, \tau)$, where $\mathcal{I}$ is an interpretation function which maps each individual name in Ind to a real number (e.g., a point in the x axis), $\sigma \in \mathbb{R}_{>0}$ is a margin of error, and $\tau \in \mathbb{N}_{>1}$ refers to the level of indeterminacy in directions.*

The parameter τ is defined as a natural number rather than a real. In practice, an integer τ is always likely to be sufficiently expressive.

Definition 3 (Truth definition) *A formula ϕ in $L(LEW)$ is true in the 1D Euclidean τ-model $M = (\mathcal{I}, \sigma, \tau)$ written $M \models_{LEW} \phi$ in virtue of these inductive clauses, following their syntax:*

$M \models_{LEW} W(a, b)$ *iff* $x_a - x_b < -\sigma$*;*

$M \models_{LEW} E(a, b)$ *iff* $x_a - x_b > \sigma$*;*

$M \models_{LEW} I_{ew}(a, b)$ *iff* $-\tau\sigma \leq x_a - x_b \leq \tau\sigma$*;*

$M \models_{LEW} \neg\phi$ *iff* $M \not\models_{LEW} \phi$*;*

$M \models_{LEW} \phi \wedge \psi$ *iff* $M \models_{LEW} \phi$ *and* $M \models_{LEW} \psi$*,*

where $a, b \in Ind$*,* $\mathcal{I}(a)= x_a$*,* $\mathcal{I}(b)= x_b$*,* ϕ, ψ *are formulas in* $L(LEW)$*.*

A formula in $L(LEW)$ is τ-satisfiable if it is true in some 1D Euclidean τ-model. A formula ϕ in $L(LEW)$ is τ-valid if it is true in all 1D Euclidean τ-models (hence if its negation is not τ-satisfiable). For every $\tau \in \mathbb{N}_{>1}$, LEW is the set of all τ-valid formulas in $L(LEW)$.

On a more general note, a logic in a given propositional language is the set of all formulas in the language which are valid from a certain point of view (Chagrov & Zakharyaschev, 1997). A (first-order) theory is any set of first-order sentences (Balbiani et al., 2007). In this sense, for every $\tau \in \mathbb{N}_{>1}$, LEW is a theory.

As shown by Lemma 1 below, σ is a scaling factor.

Lemma 1 *For every $\tau \in \mathbb{N}_{>1}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{>0}$, if a formula ϕ in $L(LEW)$ is true in a 1D Euclidean τ-model $M = (\mathcal{I}, \sigma_1, \tau)$, then it is true in a 1D Euclidean τ-model $M' = (\mathcal{I}', \sigma_2, \tau)$ provided that for every individual name a in Ind, $\mathcal{I}'(a) = \frac{\mathcal{I}(a)\sigma_2}{\sigma_1}$.*

The proof is by straightforward verification of truth conditions in Definition 3.

We introduce the following definitions as 'syntactic sugar'.

Definition 4 (Definitely, Somewhat, Neither-Nor)

definitely west $dW(a,b) =_{def} W(a,b) \wedge \neg I_{ew}(a,b)$

somewhat west $sW(a,b) =_{def} W(a,b) \wedge I_{ew}(a,b)$

neither east nor west $nEW(a, b) =_{def} \neg E(a, b) \wedge \neg W(a, b)$

somewhat east $sE(a, b) =_{def} E(a, b) \wedge I_{ew}(a, b)$

definitely east $dE(a, b) =_{def} E(a, b) \wedge \neg I_{ew}(a, b)$

As shown in Figure 2, these five relations are jointly exhaustive and pairwise disjoint. By Definitions 3 and 4, $M \models_{LEW} dE(a, b)$ iff $(x_a - x_b) \in (\tau\sigma, \infty)$, where ∞ denotes infinity; $M \models_{LEW} sE(a, b)$ iff $(x_a - x_b) \in (\sigma, \tau\sigma]$. We call $(\tau\sigma, \infty)$ the range of $dE(a, b)$, $(\sigma, \tau\sigma]$ the range of $sE(a, b)$. As τ decreases, the range of $dE(a, b)$ becomes wider, the range of $sE(a, b)$ becomes narrower. If τ is allowed to be 1, then $dE(a, b)$ becomes $E(a, b)$ and $sE(a, b)$ becomes $\perp$.

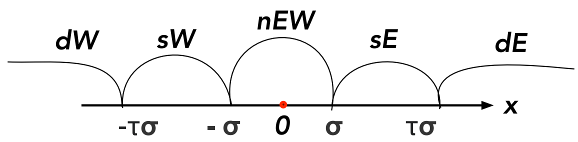

Figure 2: five jointly exhaustive and pairwise disjoint relations in LEW

2.2 Extensions of LEW

A logic of north and south (LNS) is defined similarly over points in the vertical axis (i.e., y axis). LNS is a variant of LEW, but over 'vertical' rather than 'horizontal' 1D Euclidean space. Distinguishing 'vertical' and 'horizontal' 1D Euclidean space here enables more intuitive definitions of direction relations. The primitive direction relations of LNS are north (N), south (S) and indeterminate north/south (I_{ns}). Similar to Definition 4, we define 'definitely north' (dN), 'somewhat north' (sN), 'neither north nor south' (nNS), 'somewhat south' (sS) and 'definitely south' (dS).

A logic over a higher-dimensional Euclidean space can be defined similarly. A logic of directions (LD) is a 2D extension of LEW. It contains all the primitive direction relations in LEW and LNS. As shown in Table 1, in LD, there exist $5 \times 5 = 25$ jointly exhaustive and pairwise disjoint relations, each of which is a combination of one of the relations dN, sN, nNS, sS, dS and one of the relations dW, sW, nEW, sE, dE. For example, for any pair of individual names a, b, the formula $dN\,dW(a, b)$ holds iff a is definitely to the north *and* definitely to the west of b.

	dW	sW	nEW	sE	dE
dN	$dN\,dW$	$dN\,sW$	$dN\,nEW$	$dN\,sE$	$dN\,dE$
sN	$sN\,dW$	$sN\,sW$	$sN\,nEW$	$sN\,sE$	$sN\,dE$
nNS	$nNS\,dW$	$nNS\,sW$	$nNS\,nEW$	$nNS\,sE$	$nNS\,dE$
sS	$sS\,dW$	$sS\,sW$	$sS\,nEW$	$sS\,sE$	$sS\,dE$
dS	$dS\,dW$	$dS\,sW$	$dS\,nEW$	$dS\,sE$	$dS\,dE$

Table 1: 25 jointly exhaustive and pairwise disjoint direction relations in LD

In the following sections, we present the soundness and completeness, finite axiomatisability, decidability and complexity results for LEW. The results for LNS and the higher-dimensional extensions of LEW can be obtained similarly. The point calculus (Vilain & Kautz, 1986) and the cardinal direction calculus (Ligozat, 1998) can be seen as a special case of LEW and LD, respectively, if σ is allowed to be 0. There exist different (from LEW) extensions of the point calculus and Allen's calculus (Allen, 1983), for examples, INDU for Allen's intervals with a comparison of their lengths (Pujari et al., 1999) and an algebra of granular temporal relations for both points and intervals (Cohen-Solal et al., 2015).

3. Axiomatisations

This section presents sound and complete axiomatisations of LEW: LEW^τ for every level of indeterminacy $\tau \in \mathbb{N}_{>1}$ (Section 3.1), LEW^2_{fin} for $\tau = 2$ (Section 3.2), and LEW^3_{fin} for $\tau = 3$ (Section 3.3). Each of them contains a finite sound and complete axiomatisation of the classical propositional logic (Giero, 2016). An axiomatisation is also referred to as a calculus, an axiomatic system (Chagrov & Zakharyaschev, 1997) or a proof system (van Benthem, 2010). It is sound, if it derives *only* the valid formulas; it is complete, if it derives *all* the valid formulas. LEW^2_{fin} and LEW^3_{fin} both contain a *finite* number of axioms. The development of finite sound and complete axiomatisations is useful for developing rule-based reasoners and generating explanations for any detected logical contradiction.

3.1 LEW^τ

For every level of indeterminacy $\tau \in \mathbb{N}_{>1}$, the following calculus LEW^τ is sound and complete for LEW. Here a and b, sometimes with subscripts, are meta variables which may be instantiated by any individual name in *Ind*. An instance of an axiom is a formula in $L(LEW)$ obtained by instantiating every meta variable in the axiom by an individual name in *Ind*. For example, by Axiom 1, for every individual name a in *Ind*, the formula $\neg W(a, a)$ is an instance of Axiom 1 and it is τ-valid. AS 5 is an axiom schema, where n is the number of conjuncts in the antecedent of an axiom, $number(\alpha)$ denotes the number of occurrences of α in the sequence $R_1, \ldots, R_n$. Note that $number(\alpha)$ is not in $L(LEW)$ but a meta-language notation introduced to compactly define axioms.

PL A finite sound and complete axiomatisation of classical propositional logic

Axiom 1 $\neg W(a, a)$

Axiom 2 $E(a, b) \leftrightarrow W(b, a)$

Axiom 3 $I_{ew}(a, b) \rightarrow I_{ew}(b, a)$

Axiom 4 $W(a, b) \vee E(a, b) \vee I_{ew}(a, b)$

AS 5 For any $n \in \mathbb{N}_{>1}$, if for every integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$, then $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0) \rightarrow \perp$ is an axiom.

MP Modus ponens: $\phi,\ \phi \rightarrow \psi\ \vdash\ \psi$

By AS 5, if $number(W)$, $number(dW)$, $number(\neg E)$ and $number(\neg dE)$ are all equal to one, then $W(a_0, a_1) \wedge \neg dE(a_1, a_2) \wedge \neg E(a_2, a_3) \wedge dW(a_3, a_0) \rightarrow \perp$ is an axiom, as shown in Figure 3.

The notion of τ-derivability $\Gamma \vdash_{LEW^\tau} \phi$ in the LEW^τ calculus is standard. A formula ϕ in $L(LEW)$ is τ-derivable if $\vdash_{LEW^\tau} \phi$. Γ is τ-inconsistent if for some formula ϕ it τ-derives both ϕ and $\neg\phi$ (otherwise it is τ-consistent).

$$\begin{tikzcd}
a_0 \arrow[r, "W"] & a_1 \arrow[d, "\neg dE"] \arrow[dl, phantom, "\perp"] \\
a_3 \arrow[u, "dW"] & a_2 \arrow[l, "\neg E"]
\end{tikzcd}$$

Figure 3: An example axiom in AS 5

Though for every $\tau \in \mathbb{N}_{>1}$, LEW^{τ} is sound and complete, it contains infinitely many axioms. The finite axiomatisability of LEW depends on τ: if $\tau = 2$ or $\tau = 3$, then it is finitely axiomatisable; otherwise, it is not. Below we present two finite sound and complete axiomatisations: LEW^2_{fin} for $\tau = 2$ and LEW^3_{fin} for $\tau = 3$.

3.2 LEW^2_{fin}

For $\tau = 2$, the following finite calculus LEW^2_{fin} is sound and complete for LEW. It replaces AS 5 in LEW^2 with Axioms 6-12.

PL, MP, Axioms 1-4

Axiom 6 $W(a,b) \wedge \neg dE(b,c) \wedge W(c,a) \to \perp$

Axiom 7 $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,a) \to \perp$

Axiom 8 $W(a,b) \wedge \neg E(b,c) \wedge W(c,d) \wedge \neg E(d,a) \to \perp$

Axiom 9 $W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \to \perp$

Axiom 10 $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \to \perp$

Axiom 11 $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \to \perp$

Axiom 12 $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \to \perp$

Axiom 6 above states that for every three individual names a, b, c in Ind, the formula $W(a,b) \wedge \neg dE(b,c) \wedge W(c,a) \to \perp$ is 2-valid, as shown in Figure 4. By Definitions 3 and 4, it is τ-valid for any $\tau \leq 2$. Axiom 7 states that for every three individual names a, b, c in Ind, the formula $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,a) \to \perp$ is 2-valid. By Definitions 3 and 4, it is τ-valid for any $\tau \geq 2$. Any instance of any other axiom in LEW^2_{fin} is τ-valid for every $\tau \in \mathbb{N}_{>1}$.

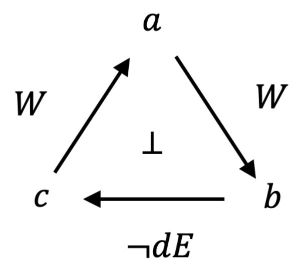

Figure 4: Axiom 6 in LEW^2_{fin}

3.3 LEW^3_{fin}

For $\tau = 3$, the following finite calculus LEW^3_{fin} is sound and complete for LEW. It replaces AS 5 in LEW^3 with Axioms 8-19.

PL, MP, Axioms 1-4, 8-12

Axiom 13 $W(a, b) \wedge W(b, c) \wedge W(c, d) \wedge \neg dE(d, a) \rightarrow \bot$

Axiom 14 $\neg E(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \wedge dW(d, a) \rightarrow \bot$

Axiom 15 $W(a, b) \wedge W(b, c) \wedge \neg E(c, d) \wedge \neg E(d, a) \rightarrow \bot$

Axiom 16 $\neg dE(a, b) \wedge W(b, c) \wedge \neg E(c, d) \wedge dW(d, a) \rightarrow \bot$

Axiom 17 $dW(a, b) \wedge \neg E(b, c) \wedge W(c, d) \wedge \neg dE(d, a) \rightarrow \bot$

Axiom 18 $W(a, b) \wedge dW(b, c) \wedge \neg E(c, d) \wedge \neg dE(d, a) \rightarrow \bot$

Axiom 19 $W(a, b) \wedge \neg dE(b, c) \wedge \neg E(c, d) \wedge dW(d, a) \rightarrow \bot$

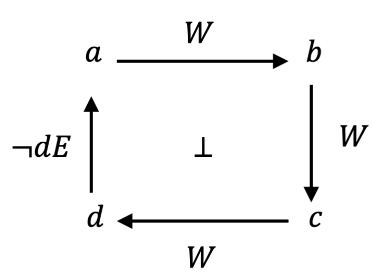

Figure 5: Axiom 13 in LEW^3_{fin}

Axiom 13 states that for every four individual names a, b, c, d, the formula $W(a, b) \wedge W(b, c) \wedge W(c, d) \wedge \neg dE(d, a) \rightarrow \bot$ is 3-valid, as shown in Figure 5. By Definitions 3 and 4, it is τ-valid for any $\tau \leq 3$. Axiom 14 states that for every four individual names a, b, c, d, the formula $\neg E(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \wedge dW(d,a) \rightarrow \perp$ is 3-valid. By Definitions 3 and 4, it is τ-valid for any $\tau \geq 3$. Any instance of any other axiom in LEW^3_{fin} is τ-valid for every $\tau \in \mathbb{N}_{>1}$.

LEW^2_{fin} and LEW^3_{fin} contain common axioms, e.g., 1-4, 8-12. Axiom 7 is not in LEW^3_{fin}, because any instance of Axiom 7 can be derived using Axioms 14, 1 and 2: by Axiom 14, the formula $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,a) \wedge \neg E(a,a) \rightarrow \perp$ is 3-valid; by Axioms 1 and 2, $\neg E(a,a)$ is 3-valid; hence $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,a) \rightarrow \perp$ is 3-valid.

For every $\tau \in \mathbb{N}_{>1}$, Axioms 8-12, 15-19 specify all the τ-valid formulas of the form $R_1(a,b) \wedge R_2(b,c) \wedge R_3(c,d) \wedge R_4(d,a) \rightarrow \perp$, where for every integer i such that $1 \leq i \leq 4$, R_i is in $\{W, dW, \neg E, \neg dE\}$, $number(W) = number(\neg E)$ and $number(dW) = number(\neg dE)$. Axioms 15-19 are not in LEW^2_{fin}, because when $\tau = 2$, any instance of them can be derived using Axiom 6, then Definition 4, Axioms 2 and 3 together or Axiom 2 alone, then Axiom 7.

4. Soundness and Completeness

This section will show that LEW^τ, LEW^2_{fin} and LEW^3_{fin} are sound and complete (i.e., every derivable formula is valid and every valid formula is derivable):

Theorem 1 *For every $\tau \in \mathbb{N}_{>1}$, the LEW^τ calculus is sound and complete for 1D Euclidean τ-models.*

Theorem 2 *The LEW^2_{fin} calculus is sound and complete for 1D Euclidean 2-models.*

Theorem 3 *The LEW^3_{fin} calculus is sound and complete for 1D Euclidean 3-models.*

4.1 Deciding Linear Inequalities by Computing Loop Residues

In our proofs, we use results on solving systems of linear inequalities over reals. To make the presentation self-contained, we first recap the definitions from Shostak (1981). The convention by Shostak (1981) is: the lower case letters x, y and v, possibly with subscripts or superscripts, denote real variables; a, b and c, possibly with subscripts or superscripts, denote real numbers. Let S be a set of linear inequalities of the form $ax+by \leq c$, where x, y are real variables, a, b, c are real numbers. If S has a solution which assigns each variable in S a real number, then S is *satisfiable*. Without loss of generality, we assume one of the variables in S, denoted as v_0, is special, appearing only with coefficient zero. It is called the 'zero variable'. All other variables in S have nonzero coefficients.

Recall that in graph theory, a graph is a pair (V, E), where V is a set of vertices and E is a set of edges. The graph G for S contains a vertex for each variable in S and an edge for each inequality, where each vertex is labelled with its associated variable and each edge is labelled with its associated inequality. For example, the edge labelled with $ax + by \leq c$ connects the vertex labelled with x and the vertex labelled with y.

Let P be a path through G, given by a sequence $v_1, \ldots, v_{n+1}$ of vertices and a sequence $e_1, \ldots, e_n$ of edges, where $n \geq 1$. The triple sequence for P is

$$(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_n, b_n, c_n)$$

where for each integer i such that $1 \leq i \leq n$, the inequality labelling e_i is $a_i v_i + b_i v_{i+1} \leq c_i$. A path is a *loop* if its first and last vertices are the same. A loop is *simple* if its intermediate vertices are distinct. A path P is said to be *admissible* if for every integer i such that $1 \leq i \leq n-1$, b_i and a_{i+1} have opposite signs (one is strictly positive and the other is strictly negative). The definitions and results that follow apply to admissible paths.

The *residue inequality* of an admissible path P is defined as the inequality obtained from P by applying transitivity to the inequalities labelling its edges. The *residue* r_p of P is defined as the triple (a_p, b_p, c_p),

$$(a_p, b_p, c_p) = (a_1, b_1, c_1) * (a_2, b_2, c_2) * \cdots * (a_n, b_n, c_n)$$

where $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ is the triple sequence for P and $*$ is the binary operation on triples defined by

$$(a, b, c) * (a', b', c') = (kaa', -kbb', k(ca' - c'b))$$

where $k = a'/|a'|$. The operation $*$ is associative. The *residue inequality* of P is the inequality $a_p x + b_p y \leq c_p$, where x, y are the first and last vertices of P. For example, if P is a path over three vertices v_1, v_2, v_3 and two edges labelled with $v_1 - v_2 \leq 1$ and $v_2 - v_3 \leq 1$, respectively, then the residue inequality of P is $v_1 - v_3 \leq 2$.

Lemma 2 *(Shostak, 1981) Any assignment of real numbers to variables that satisfies the inequalities labelling an admissible path P also satisfies the residue inequality of P.*

Let P be an admissible *loop* with an initial vertex x. By Lemma 2, any assignment satisfying the inequalities along P also satisfies $a_p x + b_p x \leq c_p$. If $a_p + b_p = 0$ and $c_p < 0$, then the residue inequality of P is false, and P is called an *infeasible loop.*

Let G be the graph for S. The simple admissible loops of G are enumerated modulo cyclic permutation and reversal[1]. A *closure* G' of G is obtained by adding a new edge labelled with the residue inequality for each simple admissible loop P of G. A graph is *closed* if it is a closure of itself.

Theorem 4 *(Shostak, 1981) Let S be a set of linear inequalities of the form $ax + by \leq c$, where x, y are real variables, a, b, c are real numbers, a, b are not equal to zero at the same time; G be a closed graph for S. Then S is satisfiable iff G has no infeasible simple loop.*

Theorem 4 is for inequalities of the form $ax + by \leq c$ only. It was extended to include both strict and non-strict inequalities (Shostak, 1981). We say an admissible path is *strict* if at least one of its edges is labelled with a strict inequality, i.e., an inequality of the form $ax + by < c$. A strict admissible loop P with residue (a_p, b_p, c_p) is infeasible, if $a_p + b_p = 0$ and $c_p \leq 0$. Lemma 3 and Corollary 1 are stated for any set of inequalities of the form $(x - y) \sim c$, where x, y are real variables, $\sim$ is $\leq$ or $<$, and c is a real number.

Lemma 3 *(Shostak, 1981) Let S be a set of linear inequalities of the form $(x - y) \sim c$, where x, y are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. Then the graph for S is closed.*

Corollary 1 *(Litvintchouk & Pratt, 1977; Pratt, 1977; Shostak, 1981) Let S be a set of linear inequalities of the form $(x - y) \sim c$, where x, y are real variables, $\sim$ is $\leq$ or $<$, and c is a real number; G be a graph for S. Then S is satisfiable iff G has no infeasible simple loop.*

1. If a loop P' is a permutation of a loop P, then there are paths Q and R such that $P = QR$ and $P' = RQ$. The reversal of a path $v_1, v_2, \ldots, v_{n+1}$ is a path $v_{n+1}, v_n, \ldots, v_1$.

4.2 Soundness and Completeness of LEW^τ

For every $\tau \in \mathbb{N}_{>1}$, to prove the soundness of the LEW^τ calculus, we show that every τ-derivable $L(LEW)$ formula ϕ is τ-valid. The proof of soundness is by an easy induction on the length of the derivation of ϕ. By Definitions 3 and 4, every instance of every axiom in LEW^τ is τ-valid and modus ponens preserves validity.

To prove completeness, we will show that for every $\tau \in \mathbb{N}_{>1}$, *if a finite set of $L(LEW)$ formulas Σ is τ-consistent, then there is a 1D Euclidean τ-model satisfying it.* Any finite set of formulas Σ can be rewritten as a formula ψ that is the conjunction of all the formulas in Σ. The set Σ is τ-consistent iff ψ is τ-consistent iff its negation is not τ-derivable. If there is a 1D Euclidean τ-model M satisfying Σ, then M satisfies ψ, hence its negation is not τ-valid. Therefore, by showing that 'if Σ is τ-consistent, then there exists a 1D Euclidean τ-model satisfying it', we show that 'if $\neg\psi$ is not τ-derivable, then $\neg\psi$ is not τ-valid'. By contraposition we get completeness.

Following Definition 5, the truth conditions of any set of formulas in $L(LEW)$ can be expressed as sets of inequalities of the form $(x_1 - x_2) \sim c$, where x_1, x_2 are real variables, $\sim$ is $\leq$ or $<$, and c is a real number.

Definition 5 (τ-σ-translation) *The 'τ-σ-translation' function $tr(\tau, \sigma)$ is defined as follows:*

$tr(\tau,\sigma)(W(a,b)) = (x_a - x_b < -\sigma)$;

$tr(\tau,\sigma)(E(a,b)) = (x_b - x_a < -\sigma)$;

$tr(\tau,\sigma)(dW(a,b)) = (x_a - x_b < -\tau\sigma)$;

$tr(\tau,\sigma)(dE(a,b)) = (x_b - x_a < -\tau\sigma)$;

$tr(\tau,\sigma)(\neg\phi) = \neg(tr(\tau,\sigma)(\phi))$, *where ϕ is a formula of one of the forms $W(a,b)$, $E(a,b)$, $dW(a,b)$ and $dE(a,b)$; $\neg(z_1 - z_2 < c) = (z_2 - z_1 \leq -c)$.*

Now we can state the proof of Theorem 1.

Proof. Take an arbitrary integer $\tau > 1$. To prove completeness, we show that if a finite set of formulas Σ in $L(LEW)$ is τ-consistent, then there is a 1D Euclidean τ-model satisfying it.

The proof idea is as follows. We take a finite τ-consistent set of formulas Σ. We rewrite it as a single formula in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m > 0$. This formula is τ-satisfiable, iff at least one of its disjuncts is τ-satisfiable. We proceed by contradiction. Suppose all disjuncts ϕ_i are *not* τ-satisfiable. Take an arbitrary disjunct ϕ_i. Then ϕ_i is not τ-satisfiable, iff the graph G_i of a set of linear inequalities S_i has an infeasible simple loop P. From P, we obtain $L(LEW)$ formulas as conjuncts in ϕ_i. Applying the axioms and axiom schemas in the LEW^τ calculus, we show $\perp$ is τ-derivable from ϕ_i. Since $\perp$ is τ-derivable from every ϕ_i, then $\perp$ is τ-derivable from Σ, which contradicts the assumption that Σ is τ-consistent.

Now we work this idea out in detail. Suppose a finite set of formulas Σ in $L(LEW)$ is τ-consistent. We obtain Σ' by rewriting every $I_{ew}(a,b)$ in Σ as $\neg dW(a,b) \wedge \neg dE(a,b)$. By

Axiom 4 and Definition 4, Σ and Σ' are logically equivalent. The set Σ' can be rewritten as a formula ϕ that is the conjunction of all the formulas in Σ'. We rewrite ϕ in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m > 0$ and every literal is of one of the forms: $W(a,b)$, $E(a,b)$, $dW(a,b)$, $dE(a,b)$, and their negations. Then ϕ is τ-satisfiable, iff at least one of its disjuncts is τ-satisfiable.

We proceed by contradiction. Suppose every disjunct ϕ_i of ϕ is *not* τ-satisfiable, where $1 \leq i \leq m$. Take an arbitrary disjunct ϕ_i. A set of linear inequalities S_i is obtained by translating every literal in ϕ_i according to Definition 5. The inequalities in S_i are of the form $(x_a - x_b) \sim c$, where x_a, x_b are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. By Corollary 1, the disjunct ϕ_i is *not* τ-satisfiable iff the graph G_i of S_i has an infeasible simple loop P. The loop P is either strict or non-strict. Let s denote the sum of the constants around P. Based on the definition of infeasible loops, if P is strict, then $s \leq 0$; otherwise, $s < 0$. By Definition 5, if a strict inequality $x_a - x_b < c$ is in S_i, then c is $-\sigma$ or $-\tau\sigma$; if a non-strict inequality $x_a - x_b \leq c$ is in S_i, then c is σ or $\tau\sigma$. Recall that τ and σ are both positive numbers. If P is non-strict, then all the inequalities labelling it are of the form $x_a - x_b \leq c$, where $c > 0$, and the sum of all such c is positive. This contradicts the fact that $s < 0$ for non-strict infeasible loops. Therefore P is strict and $s \leq 0$. By the number of vertices in P, there are two cases.

1. The loop P contains at least two vertices. Without loss of generality, let us assume that P consists of vertices $x_{a_0}, x_{a_1}, \ldots, x_{a_{n-1}}$, where $n > 1$. Since P is admissible, the linear inequalities labelling P are of the form $(x_{a_0} - x_{a_1}) \sim c_1, \ldots, (x_{a_{n-1}} - x_{a_0}) \sim c_n$, where $\sim$ is $\leq$ or $<$, and for every integer i such that $1 \leq i \leq n$, c_i is $-\sigma$, σ, $-\tau\sigma$ or $\tau\sigma$. We translate the linear inequalities labelling P to formulas as follows. We translate every linear inequality of the form $x_a - x_b < -\sigma$ to $W(a,b)$; every $x_a - x_b < -\tau\sigma$ to $dW(a,b)$; every $x_a - x_b \leq \sigma$ to $\neg E(a,b)$; every $x_a - x_b \leq \tau\sigma$ to $\neg dE(a,b)$. In this way, from P we obtain a sequence of formulas of the form $R_1(a_0,a_1), \ldots, R_n(a_{n-1},a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$. Since $s \leq 0$, we have $\mathit{number}(W) + \tau * \mathit{number}(dW) \geq \mathit{number}(\neg E) + \tau * \mathit{number}(\neg dE)$. By AS 5, we have $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0) \rightarrow \perp$. By Definition 5, for every occurrence of $W(a,b)$ in $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, the formula $W(a,b)$ or $E(b,a)$ is a conjunct in ϕ_i; for every occurrence of $dW(a,b)$, the formula $dW(a,b)$ or $dE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg E(a,b)$, the formula $\neg E(a,b)$ or $\neg W(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg dE(a,b)$, the formula $\neg dE(a,b)$ or $\neg dW(b,a)$ is a conjunct in ϕ_i. By Axiom 2, we have $W(a,b) \leftrightarrow E(b,a)$. By Definition 4, Axioms 2 and 3, we have $dW(a,b) \leftrightarrow dE(b,a)$. Therefore, $\perp$ is τ-derivable from ϕ_i.

2. Otherwise, P contains a single vertex. Since $s \leq 0$, the linear inequality labelling P is of the form $x_a - x_a < c$, where c is $-\sigma$ or $-\tau\sigma$. We translate any linear inequality of the form $x_a - x_a < -\sigma$ to $W(a,a)$; any $x_a - x_a < -\tau\sigma$ to $dW(a,a)$. By Axiom 1 and Definition 4, we have $W(a,a) \rightarrow \perp$ and $dW(a,a) \rightarrow \perp$. Following a similar argument as above, $\perp$ is τ-derivable from ϕ_i.

Therefore, in each case, $\perp$ is τ-derivable from ϕ_i. Since $\perp$ is τ-derivable from every disjunct ϕ_i, the formula ϕ is not τ-consistent. This contradicts that Σ is τ-consistent. $\square$

4.3 Soundness and Completeness of LEW^2_{fin}

To prove the soundness of the LEW^2_{fin} calculus, we show that for every formula ϕ in $L(LEW)$, if it is derivable using LEW^2_{fin}, then it is 2-valid. The proof of soundness is by an induction on the length of the derivation of ϕ. By Definitions 3 and 4, every instance of every axiom in LEW^2_{fin} is 2-valid and modus ponens preserves validity.

The proof of completeness (every 2-valid formula in $L(LEW)$ is derivable using LEW^2_{fin}) is similar to the completeness proof for LEW^τ in Section 4.2: let $\tau = 2$; instead of referring to AS 5, refer to Lemma 4. The proof of Lemma 4 is provided in Appendix A.

Lemma 4 *For any $n \in \mathbb{N}_{>1}$, if for any integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W)+2*number(dW) \geq number(\neg E)+2*number(\neg dE)$, then $\perp$ can be derived from $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ using LEW^2_{fin}.*

4.4 Soundness and Completeness of LEW^3_{fin}

To prove the soundness of the LEW^3_{fin} calculus, we show that for every formula ϕ in $L(LEW)$, if it is derivable using LEW^3_{fin}, then it is 3-valid. The proof of soundness is by an induction on the length of the derivation of ϕ. By Definitions 3 and 4, every instance of every axiom in LEW^3_{fin} is 3-valid and modus ponens preserves validity.

The proof of completeness (every 3-valid formula in $L(LEW)$ is derivable using LEW^3_{fin}) is similar to the completeness proof for LEW^τ in Section 4.2: let $\tau = 3$; instead of referring to AS 5, refer to Lemma 5. Its detailed proof is in Appendix A.

Lemma 5 *For any $n \in \mathbb{N}_{>1}$, if for any integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W)+3*number(dW) \geq number(\neg E)+3*number(\neg dE)$, then $\perp$ can be derived from $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ using LEW^3_{fin}.*

5. Non-Finite Axiomatisability of LEW for $\tau > 3$

Take an arbitrary integer $\tau > 3$. We will show that LEW is not finitely axiomatisable over 1D Euclidean space. For every $\tau \in \mathbb{N}_{>1}$, every $n \in \mathbb{N}_{>2}$, Lemma 6 below specifies an axiom in the LEW^τ calculus, under AS 5.

Lemma 6 *For every $\tau \in \mathbb{N}_{>1}$, every $n \in \mathbb{N}_{>2}$, the following expression A_n is an axiom in the LEW^τ calculus:*

$$W(a_0, a_1) \wedge W(a_1, a_2) \wedge \bigwedge_{0 \leq i < n} dW(b_i, b_{i+1}) \wedge \neg E(c_0, c_1) \wedge \neg E(c_1, c_2) \wedge \bigwedge_{0 \leq i < n} \neg dE(d_i, d_{i+1}) \to \perp$$

where $a_2 = b_0$, $b_n = c_0$, $c_2 = d_0$, $d_n = a_0$.

As shown in Figure 6, each edge in the graph represents a formula in $L(LEW)$. For example, the edge from a_0 to a_1 represents the formula $W(a_0, a_1)$, whose truth condition is $x_{a_0} - x_{a_1} < -\sigma$ by Definition 3. The axiom A_n states that, for every individual name $a_0, a_1, a_2, b_0, \ldots, b_n, c_0, c_1, c_2, d_0, \ldots, d_n$ in *Ind*, the formulas represented by all the edges in Figure 6 *cannot* be true at the same time in any 1D Euclidean model. This is because the

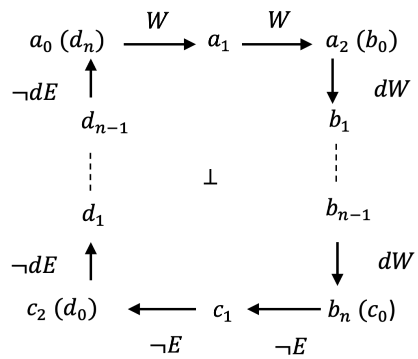

Figure 6: A graph illustration of A_n in Lemma 6

residue inequality of the loop over the sequence of the linear inequalities $x_{a_0} - x_{a_1} < -\sigma$, $x_{a_1} - x_{b_0} < -\sigma$, $x_{b_0} - x_{b_1} < -\tau\sigma$, $\ldots$, $x_{b_{n-1}} - x_{c_0} < -\tau\sigma$, $x_{c_0} - x_{c_1} \leq \sigma$, $x_{c_1} - x_{d_0} \leq \sigma$, $x_{d_0} - x_{d_1} \leq \tau\sigma$, $\ldots$, $x_{d_{n-1}} - x_{a_0} \leq \tau\sigma$ is $0 < 0$, i.e., this sequence has no solution over reals.

Definition 6 (Weighted directed graph model) *A weighted directed graph model M is a structure $(\mathcal{V}, \mathcal{E}, \mathcal{I})$, where $(\mathcal{V}, \mathcal{E})$ is a graph whose edges are directed and have weights, and $\mathcal{I}$ is an interpretation function which maps each individual name in Ind to a vertex in $\mathcal{V}$.*

Definition 7 (Truth definition) *By induction on the construction of a formula ϕ in $L(LEW)$ and Definition 4, we define the notion of $M \models \phi$ which is read as 'a formula ϕ in $L(LEW)$ is true in the weighted directed graph model M' or 'the weighted directed graph model M satisfies a formula ϕ in $L(LEW)$':*

$M \models dW(a,b)$ *iff* $(\mathcal{I}(a), \mathcal{I}(b), w_{dw}) \in \mathcal{E}$;

$M \models sW(a,b)$ *iff* $(\mathcal{I}(a), \mathcal{I}(b), w_{sw}) \in \mathcal{E}$;

$M \models nEW(a,b)$ *iff* $(\mathcal{I}(a), \mathcal{I}(b), w_{new}) \in \mathcal{E}$;

$M \models sE(a,b)$ *iff* $(\mathcal{I}(a), \mathcal{I}(b), w_{se}) \in \mathcal{E}$;

$M \models dE(a,b)$ *iff* $(\mathcal{I}(a), \mathcal{I}(b), w_{de}) \in \mathcal{E}$;

$M \models \neg\phi$ *iff* $M \not\models \phi$;

$M \models \phi \wedge \psi$ *iff* $M \models \phi$ *and* $M \models \psi$,

where a, b are individual names in Ind, ϕ, ψ are formulas in $L(LEW)$, w_{dw}, w_{sw}, w_{new}, w_{se} and w_{de} are different real numbers.

The formula F_n in Lemma 7 below is the negation of an instance of A_n in Lemma 6.

Lemma 7 *For every $\tau \in \mathbb{N}_{>3}$, every $n \in \mathbb{N}_{>2}$, there exists a weighted directed graph model satisfying the formula F_n below:*

$$W(a_0, a_1) \wedge W(a_1, a_2) \wedge \bigwedge_{0 \le i < n} dW(b_i, b_{i+1}) \ \wedge \neg E(c_0, c_1) \wedge \neg E(c_1, c_2) \wedge \bigwedge_{0 \le i < n} \neg dE(d_i, d_{i+1})$$

where $a_2 = b_0$, $b_n = c_0$, $c_2 = d_0$, $d_n = a_0$.

Proof. Take arbitrary integers $\tau > 3$, $n > 2$. We will construct a weighted directed graph model $M = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ and show that M satisfies F_n.

Let $\mathcal{V} = \{a^0, a^1, b^0, \dots, b^{n-1}, c^0, c^1, d^0, \dots, d^{n-1}\}$. Let $a^2 = b^0$, $b^n = c^0$, $c^2 = d^0$, and $d^n = a^0$. For every integer i such that $0 \le i \le 2$, let $\mathcal{I}(a_i) = a^i$, $\mathcal{I}(c_i) = c^i$. For every integer i such that $0 \le i \le n$, let $\mathcal{I}(b_i) = b^i$, $\mathcal{I}(d_i) = d^i$. For every individual name o in $Ind \setminus \{a_0, \dots, a_2, b_0, \dots, b_n, c_0, \dots, c_2, d_0, \dots, d_n\}$, let $\mathcal{I}(o) = a^0$.

The set of edges $\mathcal{E}$ is constructed by the steps below. Initially, it is empty.

1. For every pair of integers i, j such that $0 \le i < j \le 2$, add (a^i, a^j, w_{sw}) and (a^j, a^i, w_{se}) to $\mathcal{E}$.

2. For every pair of integers i, j such that $0 \le i < j \le n$, add (b^i, b^j, w_{dw}) and (b^j, b^i, w_{de}) to $\mathcal{E}$.

3. For every integer i such that $0 \le i < 2$, add (c^i, c^{i+1}, w_{new}) and (c^{i+1}, c^i, w_{new}) to $\mathcal{E}$; add (c^0, c^2, w_{se}) and (c^2, c^0, w_{sw}) to $\mathcal{E}$.

4. For every integer i such that $0 \le i < n$, add (d^i, d^{i+1}, w_{se}) and (d^{i+1}, d^i, w_{sw}) to $\mathcal{E}$. For every pair of integers i, j such that $0 \le i < j \le n$ and $j - i > 1$, add (d^i, d^j, w_{de}) and (d^j, d^i, w_{dw}) to $\mathcal{E}$.

5. For every pair of integers i, j such that $0 \le i < 2$ and $0 < j \le n$, add (a^i, b^j, w_{dw}) and (b^j, a^i, w_{de}) to $\mathcal{E}$.

6. For every pair of integers i, j such that $0 \le i < 2$ and $0 < j \le 2$, add (a^i, c^j, w_{dw}) and (c^j, a^i, w_{de}) to $\mathcal{E}$.

7. For every integer j such that $0 < j < n - 1$, add (a^1, d^j, w_{dw}) and (d^j, a^1, w_{de}) to $\mathcal{E}$; add (a^1, d^{n-1}, w_{sw}) and (d^{n-1}, a^1, w_{se}) to $\mathcal{E}$.

8. For every pair of integers i, j such that $0 \le i < n - 1$ and $0 < j \le 2$, add (b^i, c^j, w_{dw}) and (c^j, b^i, w_{de}) to $\mathcal{E}$. For every integer j such that $0 < j \le 2$, add (b^{n-1}, c^j, w_{sw}) and (c^j, b^{n-1}, w_{se}) to $\mathcal{E}$.

9. For every pair of integers i, j such that $0 \le i < n$ and $0 < j < n$, if $n - i - j > 1$, then add (b^i, d^j, w_{dw}) and (d^j, b^i, w_{de}) to $\mathcal{E}$; if $n - i - j = 1$, then add (b^i, d^j, w_{sw}) and (d^j, b^i, w_{se}) to $\mathcal{E}$; if $n - i - j = 0$, then add (b^i, d^j, w_{se}) and (d^j, b^i, w_{sw}) to $\mathcal{E}$; if $n - i - j < 0$, then add (b^i, d^j, w_{de}) and (d^j, b^i, w_{dw}) to $\mathcal{E}$.

10. For every pair of integers i, j such that $0 \le i < 2$ and $0 < j < n$, add (c^i, d^j, w_{de}) and (d^j, c^i, w_{dw}) to $\mathcal{E}$.

11. For every vertex $v \in \mathcal{V}$, add (v, v, w_{new}) to $\mathcal{E}$.

By Definition 7, M satisfies every conjunct of F_n, hence it satisfies F_n. $\square$

In the proof of Lemma 7, among all the steps taken to construct the set of edges $\mathcal{E}$, Step 9 is the most complicated. The cases in Step 9 are specified by comparing the number of dW and the number of $\neg dE$ involved in the formula $dW(b_i, b_{i+1}) \wedge \cdots \wedge dW(b_{n-1}, b_n) \wedge \neg E(c_0, c_1) \wedge \neg E(c_1, c_2) \wedge \neg dE(d_0, d_1) \wedge \cdots \wedge \neg dE(d_{j-1}, d_j)$. The number of dW is $n - i$, the number of $\neg dE$ is j. By Definitions 3 and 4, we have $x_{b_i} - x_{b_{i+1}} < -\tau\sigma, \ldots, x_{b_{n-1}} - x_{c_0} < -\tau\sigma$, $x_{c_0} - x_{c_1} \leq \sigma$, $x_{c_1} - x_{d_0} \leq \sigma$, $x_{d_0} - x_{d_1} \leq \tau\sigma, \ldots, x_{d_{j-1}} - x_{d_j} \leq \tau\sigma$; hence $x_{b_i} - x_{d_j} = (n-i)(-\tau\sigma) + 2\sigma + j\tau\sigma = (n-i-j)(-\tau\sigma) + 2\sigma$. Since $\tau > 3$, if $n - i - j > 1$, then $x_{b_i} - x_{d_j} < -\tau\sigma$, hence (b^i, d^j, w_{dw}) is added to $\mathcal{E}$. The other cases are similar.

The proof of Theorem 5 below is based on the intuition that an axiom over a small number of meta variables cannot rule out invalid formulas over some larger number of individual names. An axiom A_1 entails an axiom A_2, iff *any model* which satisfies all instances of A_1 satisfies all instances of A_2.

Theorem 5 *For every $\tau \in \mathbb{N}_{>3}$, there exists no finite sound axiomatisation of LEW which is complete for 1D Euclidean τ-models.*

Proof. Take an arbitrary integer $\tau > 3$. To show LEW is not finitely axiomatisable, we show that there is no LEW axiom $\mathcal{A}$ which entails all the axioms $A_n = (W(a_0, a_1) \wedge W(a_1, a_2) \wedge \bigwedge_{0 \leq i < n} dW(b_i, b_{i+1}) \wedge \neg E(c_0, c_1) \wedge \neg E(c_1, c_2) \wedge \bigwedge_{0 \leq i < n} \neg dE(d_i, d_{i+1}) \rightarrow \perp)$, where $n > 2$, $a_2 = b_0$, $b_n = c_0$, $c_2 = d_0$, $d_n = a_0$. Suppose such an axiom $\mathcal{A}$ exists. Then $\mathcal{A}$ is over some finite number of meta variables t. Counting 'equal' meta variables like a_2 and b_0 as one, A_n is over $2n + 4$ meta variables.

In **Step 1**, we construct a weighted directed graph model M satisfying F_n, which is an instance of $\neg A_n$ for some $2n + 4 > t$. The construction of M is described in the proof of Lemma 7.

In **Step 2**, we show that any $L(LEW)$ formula over at most t individual names which is true in M is also true in some 1D Euclidean τ-model. Hence all instances of $\mathcal{A}$ are true in M, because otherwise their negations would have been true in some 1D Euclidean τ-model (if an instance $\mathcal{F}$ of $\mathcal{A}$ is not true in M, then its negation $\neg\mathcal{F}$ is true in M. Since $\neg\mathcal{F}$ is over t individual names, it is true in some 1D Euclidean τ-model. This contradicts that every instance of $\mathcal{A}$ is τ-valid, i.e., true in all 1D Euclidean τ-models). Hence M satisfies all instances of $\mathcal{A}$ and an instance of $\neg A_n$: a contradiction with the assumption that $\mathcal{A}$ entails A_n.

Consider an arbitrary $L(LEW)$ formula ϕ over at most t individual names which is true in M. Let $names(F_n)$ denote the set of individual names involved in F_n. Clearly, $names(F_n)$ is of size $2n + 4$. Since $2n + 4 > t$, at least one individual name o in $names(F_n)$ is not involved in ϕ. Since ϕ is arbitrary, the individual name o could be *any* individual name in $names(F_n)$. By Definition 7 and the construction of M, for every pair of individual names a, b in $names(F_n)$, exactly one of $dW(a, b)$, $sW(a, b)$, $nEW(a, b)$, $sE(a, b)$ and $dE(a, b)$ is true in M. Let $\psi(a, b)$ be a function which takes a pair of individual names a, b in $names(F_n)$, and returns one of $dW(a, b)$, $sW(a, b)$, $nEW(a, b)$, $sE(a, b)$ and $dE(a, b)$ such that the returned formula is true in M. By Definitions 3 and 4, the formula $nEW(a, a)$ is

τ-valid for every individual name a in Ind. By the construction of M, every individual name in $\mathit{Ind} \setminus \mathit{names}(F_n)$ has the same interpretation as a_0. Hence, to show ϕ is true in some 1D Euclidean τ-model, it is sufficient to show that there exists a 1D Euclidean τ-model M_E such that for every pair of *different* individual names a, b in $\mathit{names}(F_n) \setminus \{o\}$, the formula $\psi(a, b)$ is true in M_E.

By Definitions 3 and 4, a set of linear inequalities S is constructed from $\psi(a, b)$ for every pair of different individual names a, b in $\mathit{names}(F_n)$. Initially, S is empty. For every pair of different individual names a, b in $\mathit{names}(F_n)$,

1. if $\psi(a, b)$ is $dW(a, b)$, then add $a - b < -\tau\sigma$ to S;
2. if $\psi(a, b)$ is $sW(a, b)$, then add $-\tau\sigma \leq a - b < -\sigma$ to S;
3. if $\psi(a, b)$ is $nEW(a, b)$, then add $-\sigma \leq a - b \leq \sigma$ to S;
4. if $\psi(a, b)$ is $sE(a, b)$, then add $\sigma < a - b \leq \tau\sigma$ to S;
5. if $\psi(a, b)$ is $dE(a, b)$, then add $\tau\sigma < a - b$ to S.

Since M satisfies F_n and F_n is not true in any 1D Euclidean model, the set S does not have any solution over reals.

We obtain $S_{\leq}$ by replacing every $<$ with $\leq$ for every linear inequality in S. Without loss of generality, let $\sigma = 1$. Then the following assignment I_1 provides a solution to $S_{\leq}$: $I_1(a_0) = 0$, $I_1(a_1) = 1$, $I_1(a_2) = I_1(b_0) = 2$, $I_1(b_1) = 2 + \tau$, $I_1(b_2) = 2 + 2\tau$, $\ldots$, $I_1(b_{n-1}) = 2 + (n-1)\tau$, $I_1(b_n) = I_1(c_0) = 2 + n\tau$, $I_1(c_1) = 1 + n\tau$, $I_1(c_2) = I_1(d_0) = n\tau$, $I_1(d_1) = (n-1)\tau$, $I_1(d_2) = (n-2)\tau$, $\ldots$, $I_1(d_{n-1}) = \tau$, $I_1(d_n) = I_1(a_0) = 0$.

Take an arbitrary individual name o in $\mathit{names}(F_n)$. We obtain S^o by removing o, as well as all the linear inequalities involving it, from S. Below we construct an assignment I_2 which provides a solution to S^o. First, a function next is introduced: for every integer i such that $0 \leq i \leq 1$, let $\mathit{next}(a_i) = a_{i+1}$, $\mathit{next}(c_i) = c_{i+1}$; for every integer i such that $0 \leq i \leq n-1$, let $\mathit{next}(b_i) = b_{i+1}$, $\mathit{next}(d_i) = d_{i+1}$. The function rank is defined as follows: let $\mathit{rank}(o) = 0$; for every individual name e in $\mathit{names}(F_n)$, if $\mathit{next}(e)$ is not o, then let $\mathit{rank}(\mathit{next}(e)) = (\mathit{rank}(e) + 1) \mod (2n+4)$. Let r_i denote the individual name in $\mathit{names}(F_n)$ whose rank is i, where $0 \leq i < 2n+4$. Then r_0 is o. The assignment I_2 is defined inductively as follows, where ϵ is a very small positive real number less than one: let $I_2(r_1) = I_1(r_1)$; for every individual name r_i such that $1 < i < 2n+4$, if r_i is a_j, where $0 < j \leq 2$, then let $I_2(r_i) = I_2(r_{i-1}) + 1 + \frac{\epsilon}{2^{(i-1)}}$; if r_i is b_j, where $0 < j \leq n$, let $I_2(r_i) = I_2(r_{i-1}) + \tau + \frac{\epsilon}{2^{(i-1)}}$; if r_i is c_j, where $0 < j \leq 2$, let $I_2(r_i) = I_2(r_{i-1}) - 1$; if r_i is d_j, where $0 < j \leq n$, let $I_2(r_i) = I_2(r_{i-1}) - \tau$. By the definitions of I_1 and I_2, for every individual name r_i such that $0 < i < 2n+4$, we have $0 \leq I_2(r_i) - I_1(r_i) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \ldots \frac{\epsilon}{2^{2n+3}} < \epsilon < 1$. It is verified that I_2 provides a solution to S^o (see Appendix B). Therefore, there exists a 1D Euclidean τ-model $M_E = (I_2, \sigma, \tau)$ such that for every pair of different individual names a, b in $\mathit{names}(F_n) \setminus \{o\}$, the formula $\psi(a, b)$ is true in M_E. A contradiction. $\Box$

6. Decidability and Complexity

We show that for every $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for a finite set of $L(LEW)$ formulas in a 1D Euclidean τ-model is NP-complete.

Lemma 8 *For every $\tau \in \mathbb{N}_{>1}$, let S be a set of linear inequalities obtained by applying the 'τ-σ-translation' function over $L(LEW)$ formulas as shown in Definition 5, where $\sigma = 1$; let $n > 0$ be the number of variables in S. If S is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the binary representation size of t is in $O(n)$.*

The proof of Lemma 8 is provided in Appendix C.

Definition 8 *Let ϕ be a formula in $L(LEW)$. Its size $s(\phi)$ is defined as follows:*

- $s(R(a,b)) = 3$, *where R is in $\{E, W, I_{ew}\}$;*
- $s(\neg\phi) = 1 + s(\phi)$*;*
- $s(\phi \wedge \psi) = 1 + s(\phi) + s(\psi)$,

where $a, b \in Ind$, ϕ, ψ are formulas in $L(LEW)$.

The combined size of $L(LEW)$ formulas in a set S is defined as the size of the conjunction of all formulas in S.

Theorem 6 *For every $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for a finite set of $L(LEW)$ formulas in a 1D Euclidean τ-model is NP-complete in the combined size of the formulas.*

Proof. Take an arbitrary integer $\tau > 1$. NP-hardness follows from that $L(LEW)$ includes standard logical operators $\neg$ and $\wedge$ in classical propositional logic and the propositional satisfiability problem is NP-complete. To prove that the satisfiability problem for a finite set of $L(LEW)$ formulas is in NP, we show that if a finite set of $L(LEW)$ formulas Σ is τ-satisfiable, then we can guess a 1D Euclidean τ-model for Σ and verify that this model satisfies Σ, both in time polynomial in the combined size of formulas in Σ. Let s denote the combined size of formulas in Σ, and n denote the number of individual names in Σ. By Definition 8, $n < s$. As σ is a scaling factor, if Σ is τ-satisfiable, then it is τ-satisfiable in a model where $\sigma = 1$.

We obtain a set Σ' by rewriting every $I_{ew}(a,b)$ in Σ as $\neg dW(a,b) \wedge \neg dE(a,b)$, obtain a formula ϕ which is the conjunction of all $L(LEW)$ formulas in Σ', then rewrite ϕ in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m > 0$ and every literal is of one of the forms: $W(a,b)$, $E(a,b)$, $dW(a,b)$, $dE(a,b)$ and their negations. Then Σ is τ-satisfiable, iff at least one of the disjuncts ϕ_i, where $1 \leq i \leq m$, is τ-satisfiable. We obtain a set of linear inequalities S_i by translating every literal in a disjunct ϕ_i by Definition 5. Then Σ is τ-satisfiable, iff there exists a set of linear inequalities S_i which is satisfiable. By Lemma 8, if S_i is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the representation size of t is in $O(n)$. Hence, for every individual name in Σ, we can guess such a rational number for it in $O(n)$. Thus we can guess a 1D Euclidean

τ-model M for Σ in $O(n^2)$. To verify that M satisfies Σ, we need to check every formula in Σ. For any formula $R(a, b)$, where R is in $\{E, W, I_{ew}\}$ and a, b are individual names in Ind, checking that $R(a, b)$ is true in M takes $O(n)$ time by Definition 3 and applying bit operations. Hence checking all formulas in Σ takes time polynomial in s. □

An alternative decidability and complexity proof could use reduction to a finite set of disjunctive linear relations (DLRs) (Jonsson & Bäckström, 1998): the satisfiablity problem for a set of DLRs is NP-complete.

It is possible to decide satisfiability of a special class of $L(LEW)$ formulas in polynomial time. As defined by Koubarakis and Skiadopoulos (2000), a $UTVPI^{\neq}$ constraint is a linear inequality of the form $\pm x \leq c$, $\pm x \neq c$, $\pm x \pm y \leq c$ or $\pm x \pm y \neq c$, where x, y are rational variables, c is a rational number. A disequation is of the form $\pm x \neq c$ or $\pm x \pm y \neq c$. A linear inequality of the form $x - y < c$ can be rewritten as $x - y \leq c$ *and* $x - y \neq c$. Different from the linear inequalities studied in this paper, $UTVPI^{\neq}$ constraints are over rationals rather than reals. The decidability and complexity results, as well as efficient algorithms, were presented for $UTVPI^{\neq}$ constraints (Koubarakis & Skiadopoulos, 2000): the satisfiability problem for a set of $UTVPI^{\neq}$ constraints (i.e., whether a set of $UTVPI^{\neq}$ constraints has a solution in rationals) is decidable in $O(n^3 + d)$ time, where n is the number of variables and d is the number of disequations in the set. By Lemma 8, these results are applicable to the satisfiability problem for a set of linear inequalities over reals obtained by applying the 'τ-σ-translation' function over $L(LEW)$ formulas as shown in Definition 5.

More recently, Ostuni et al. (2021) proposed a faster polynomial-time algorithm to solve a finite set of inequalities of the form $(x - y) \sim c$, where x, y are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. The time complexity of the algorithm is $O(nn')$, where n is the number of real variables and n' is the number of inequalities in the finite set. This result is also applicable to the satisfiability problem for a set of linear inequalities over reals obtained by Definition 5.

Consider any $L(LEW)$ formula in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m > 0$ and every literal is of one of the forms: $W(a, b)$, $E(a, b)$, $dW(a, b)$, $dE(a, b)$ and their negations. Then, by Koubarakis and Skiadopoulos (2000), the satisfiability problem for any disjunct ϕ_i, where $1 \leq i \leq m$, is decidable in $O(n^3)$ time; more precisely, by Ostuni et al. (2021), it is in $O(nn')$, where n is the number of variables, n' is the number of inequalities.

7. Discussion

All the soundness, completeness, non-finite axiomatisability, decidability and complexity results for LEW are applicable to LNS (a logic of north and south), LD (a logic of directions) and higher-dimensional extensions of LEW, since every primitive direction relation (e.g., E) is defined with respect to one dimension only. For instance, the counterpart of Theorem 3 for LD would be stated as follows.

Theorem 7 *Assume that LD^3_{fin} is the calculus which contains the LEW^3_{fin} calculus and the LNS^3_{fin} calculus. Then the LD^3_{fin} calculus is sound and complete for 2D Euclidean 3-models.*

Though $L(LEW)$ is a subset of the language of first-order logic, the non-finite axiomatisability theorem (Theorem 5) is *not* applicable to first-order logic.

Different from the axiomatisation-based approach taken by this work, several qualitative spatial or temporal calculi have been studied by taking a relation-algebraic approach (Düntsch, Wang, & McCloskey, 2001; Ligozat, 2012; Hirsch, Jackson, & Kowalski, 2019), where the inverse and composition operations over relations are defined, and composition tables are constructed. Recall that if R and S are binary relations over a set U, then their composition is $R \circ S =_{def} \{(x, y) \in U \times U \mid \exists z \in U$ such that $(x, z) \in R$ and $(z, y) \in S\}$. Such a composition (represented as one cell in a composition table) can be translated into an axiom over at most three meta variables. For example, for any $\tau \in \mathbb{N}_{>1}$, the corresponding axiom of $dW \circ dW = dW$ is $dW(a, b) \wedge dW(b, c) \rightarrow dW(a, c)$. Though such a composition can *not* be explicitly stated in any axiomatisation (LEW^{τ}, LEW^2_{fin} or LEW^3_{fin}) because the composition symbol $\circ$ is not in the language of LEW, any instance of its corresponding axiom (as above not containing $\circ$) can be derived using the axiomatisation, since LEW^{τ}, LEW^2_{fin} and LEW^3_{fin} are complete. An additional difference between the composition tables and the axiomatisations LEW^{τ}, LEW^2_{fin} and LEW^3_{fin} is that the axiomatisations contain axioms over more than three meta variables, e.g., Axiom 8.

The development of finite sound and complete axiomatisations is useful for developing rule-based reasoners (Du et al., 2015). The derivation process in an axiomatisation-based consistency checking is integrated with a truth maintenance system (Forbus & de Kleer, 1993), such that minimal sets of formulas for deriving a logical contradiction can be generated as explanations. Such explanations are useful for understanding how a logical contradiction is derived, based on which, actions (e.g., remove or change formulas) can be taken to restore the consistency. The source code of the reasoners based on LEW^2_{fin} and LEW^3_{fin} is publicly available[2], which will be presented in a separate publication.

Below we examine the relationship between LEW and the INDU calculus, which extends Allen's intervals with a comparison of their lengths (Pujari et al., 1999). In total, INDU defines 25 atomic relations between two intervals. In the propositional closure of the INDU calculus (Wolter & Lee, 2016), a formula is a Boolean combination of relations within the INDU calculus. It is worth noting that the models of LEW use quantitative thresholds $\pm\sigma$ and $\pm\tau\sigma$, whilst the models of the propositional closure of the INDU calculus are scale-invariant. Following the convention by Pujari et al. (1999), let X^b, X^e and X^d denote the start point, the end point and the duration, respectively, of an interval X. We show that the satisfiability problem for $L(LEW)$ can be translated to that of the INDU calculus.

Proposition 1 *For any $\sigma \in \mathbb{R}_{>0}$, any $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for a finite set of $L(LEW)$ formulas can be translated to the satisfiability problem for a finite set of formulas in the propositional closure of the INDU calculus over 1D Euclidean space.*

Proof. Take arbitrary $\sigma \in \mathbb{R}_{>0}$, $\tau \in \mathbb{N}_{>1}$. By the truth definition of INDU relations (Pujari et al., 1999), the 'before and duration equal' relation $b^{=}(X, Y)$ holds in 1D Euclidean space, iff $X^b < X^e < Y^b < Y^e$ and $X^d = Y^d$. Suppose that the durations of all intervals are equal, and without loss of generality, they are equal to σ. Then, as shown in Figure 7, the relation $b^{=}(X, Y)$ holds in 1D Euclidean space, iff $X^e - Y^b < 0$, iff $X^b - Y^b < -\sigma$, iff $X^e - Y^e < -\sigma$. By Definition 3, $W(X^b, Y^b)$ and $b^{=}(X, Y)$ are equisatisfiable over

2. `https://github.com/Can-ZHOU/Spatial-Logic`

1D Euclidean space. By Definitions 3 and 4, $dW(X_0^b, X_\tau^b)$ and $\bigwedge_{0 \leq i < \tau} W(X_i^b, X_{i+1}^b)$ are equisatisfiable over 1D Euclidean space.

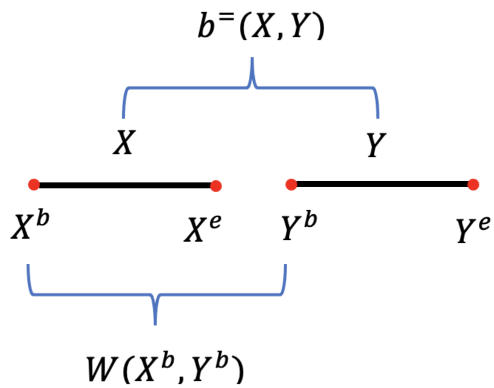

Figure 7: The lengths of intervals X, Y are both σ. $W(X^b, Y^b)$ and $b^=(X, Y)$ are equisatisfiable over 1D Euclidean space.

A finite set of $L(LEW)$ formulas Σ can be rewritten as a formula ϕ which is the conjunction of all the formulas in Σ. The function f below translates a formula ϕ in $L(LEW)$ to a formula $f(\phi)$ in the propositional closure of the INDU calculus such that ϕ and $f(\phi) \wedge \bigwedge_{X^b, Y^b \in names(\phi)} =(X, Y)$ are equisatisfiable over 1D Euclidean space, where $names(\phi)$ is the set of individual names involved in ϕ and the 'duration equal' relation $=(X, Y)$ is equivalent to $b^=(X, Y) \vee m^=(X, Y) \vee o^=(X, Y) \vee eq^=(X, Y) \vee oi^=(X, Y) \vee mi^=(X, Y) \vee bi^=(X, Y)$.

- $f(W(X^b, Y^b)) = b^=(X, Y)$;
- $f(E(X^b, Y^b)) = f(W(Y^b, X^b))$;
- $f(dW(X_0^b, X_\tau^b)) = f(\bigwedge_{0 \leq i < \tau} W(X_i^b, X_{i+1}^b))$;
- $f(dE(X^b, Y^b)) = f(dW(Y^b, X^b))$;
- $f(I_{ew}(X^b, Y^b)) = f(\neg dW(X^b, Y^b) \wedge \neg dE(X^b, Y^b))$;
- $f(\neg\phi) = \neg f(\phi)$;
- $f(\phi \wedge \psi) = f(\phi) \wedge f(\psi)$.

Since the satisfiability problem for a finite set of atomic formulas in the propositional closure of INDU is decidable using polynomial algorithms for solving Horn disjunctive linear relations (Jonsson & Bäckström, 1998; Wolter & Lee, 2016), the satisfiability problem for a finite set of $L(LEW)$ formulas is NP-complete. $\square$

The complexity result of LEW obtained above is consistent with the result presented in Section 6.

8. Conclusion

We have introduced a new qualitative logic of east and west (LEW) for reasoning about directions in Euclidean spaces. The logic incorporates a margin of error and a level of indeterminacy in directions $\tau \in \mathbb{N}_{>1}$, which together allow the logic to be used to compare and reason about not perfectly aligned representations of the same spatial objects in different datasets (for example, hand sketches or crowd-sourced digital maps). For every $\tau \in \mathbb{N}_{>1}$, we have shown LEW^{τ} to be sound and complete, and that the satisfiability problem of $L(LEW)$ formulas over 1D Euclidean space is NP-complete. The finite axiomatisability of LEW depends on τ: if $\tau = 2$ or $\tau = 3$, then there exists a finite sound and complete axiomatisation; if $\tau > 3$, then it is not finitely axiomatisable. While there have been many spatial calculi previously proposed, LEW is unique in allowing indeterminate directions which we believe are crucial in practice. Moreover, many previous spatial calculi have not been treated to the same theoretical analysis that we do here (i.e., the soundness, completeness, finite axiomatisability, decidability and complexity results in this paper). In future work, we plan to develop new qualitative direction logics to reason about regions or sets of points, and combine the logics for qualitative distances (Du et al., 2013; Du & Alechina, 2016) and qualitative directions.

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Appendix A. Proofs of Lemmas 4 and 5

Lemmas 4 and 5 are proved by cases which are specified in Lemmas 9 and 10.

For a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, we refer to $R_j(a_{j-1}, a_j)$ and $R_{j+1}(a_j, a_{j+1})$ as *neighbours*, where $1 \leq j < n$ and $a_n = a_0$. The conjuncts $R_1(a_0, a_1)$ and $R_n(a_{n-1}, a_0)$ are also referred to as *neighbours*.

Lemma 9 *For every $\tau \in \mathbb{N}_{>1}$, every $n \in \mathbb{N}_{>1}$, let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$. If there exists an R_i in F_n such that R_i is in $\{\neg E, \neg dE\}$, then there exist conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_n such that they are neighbours and one of the two cases holds:*

1. *R_s is in $\{W, dW\}$ and R_t is in $\{\neg E, \neg dE\}$;*

2. *R_s is in $\{\neg E, \neg dE\}$ and R_t is in $\{W, dW\}$.*

Proof. Let us prove by contradiction. Take arbitrary integers $\tau > 1$, $n > 1$. Suppose for every pair of conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_n, if they are neighbours, then neither Case 1 nor Case 2 holds, this is, they are both in $\{W, dW\}$ or both in $\{\neg E, \neg dE\}$. Since there exists an R_i in F_n such that R_i is in $\{\neg E, \neg dE\}$, all of $R_1, \ldots, R_n$ are in $\{\neg E, \neg dE\}$. This contradicts $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$. □

The proof of Lemma 10 is similar, hence omitted.

Lemma 10 *For every $\tau \in \mathbb{N}_{>1}$, every $n \in \mathbb{N}_{>1}$, let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. Then there exist conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_n such that they are neighbours and one of the two cases holds:*

1. *R_s is in $\{W, dW\}$ and R_t is in $\{\neg E, \neg dE\}$;*
2. *R_s is in $\{\neg E, \neg dE\}$ and R_t is in $\{W, dW\}$.*

Lemma 4 is presented in Section 4.3 and used to prove the completeness of LEW^2_{fin}.

Lemma 4 *For any $n \in \mathbb{N}_{>1}$, if for any integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$, then $\perp$ can be derived from $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ using LEW^2_{fin}.*

Proof. For $n = 1$, let F_n denote a formula of the form $W(a, a)$ or $dW(a, a)$. For any $n > 1$, let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. We will show that for any $n > 0$, $\perp$ can be derived from F_n using LEW^2_{fin} by mathematical induction.

Base case When $n = 1$, by Axiom 1 and Definition 4, $\perp$ can be derived.

When $n = 2$, since R_i is in $\{W, dW, \neg E, \neg dE\}$ and $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$, then $\{R_1, R_2\} = \{W, \neg E\}$, $\{R_1, R_2\} = \{dW, \neg E\}$, $\{R_1, R_2\} = \{dW, \neg dE\}$, or R_1, R_2 are both in $\{W, dW\}$. If $\{R_1, R_2\} = \{W, \neg E\}$, then by Axiom 2, $\perp$ can be derived. If $\{R_1, R_2\} = \{dW, \neg E\}$, by Axioms 7, 2 and 1, $\perp$ can be derived. If $\{R_1, R_2\} = \{dW, \neg dE\}$, then by Definition 4, Axioms 2 and 3, we have $dW(a, b) \leftrightarrow dE(b, a)$, hence $\perp$ can be derived. If R_1, R_2 are both in $\{W, dW\}$, then by Definition 4, Axioms 6, 2 and 1, $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from $F_1, F_2, \ldots, F_n$ using LEW^2_{fin}, where $n \geq 2$, we will show $\perp$ can be derived from F_{n+1}. If every R_i in F_{n+1} is W or dW, then by Definition 4, Axioms 6, 3, 2, and 1, $\perp$ can be derived from F_{n+1}.

Otherwise, there exists at least one R_i in F_{n+1} which is $\neg E$ or $\neg dE$. By Lemma 9, there exist conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_{n+1} such that they are neighbours and one of the two cases holds:

Case 1 R_s is in $\{W, dW\}$ and R_t is in $\{\neg E, \neg dE\}$;

Case 2 R_s is in $\{\neg E, \neg dE\}$ and R_t is in $\{W, dW\}$.

Let us proceed by cases. Since $n+1>2$, in addition to $R_s(a,b)$, $R_t(b,c)$ has another neighbour $R_k(c,d)$.

1. If R_s is W and R_t is $\neg E$, then
   (a) if R_k is W, then by Axiom 8, $W(a,b)\wedge\neg E(b,c)\wedge W(c,d)\rightarrow E(d,a)$ is 2-valid; by Axiom 2, $E(d,a)\rightarrow W(a,d)$ is 2-valid. Hence $W(a,b)\wedge\neg E(b,c)\wedge W(c,d)\rightarrow W(a,d)$ is 2-valid.
   (b) if R_k is dW, then by Axiom 7, $\neg E(b,c)\wedge dW(c,d)\rightarrow E(d,b)$ is 2-valid; by Axiom 2, $E(d,b)\rightarrow W(b,d)$ is 2-valid; by Axiom 6, $W(a,b)\wedge W(b,d)\rightarrow dE(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $dE(d,a)\rightarrow dW(a,d)$ is 2-valid. Hence $W(a,b)\wedge\neg E(b,c)\wedge dW(c,d)\rightarrow dW(a,d)$ is 2-valid.
   (c) if R_k is $\neg E$, then by Axiom 7, $\neg E(b,c)\wedge\neg E(c,d)\rightarrow\neg dW(d,b)$ is 2-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,b)\rightarrow\neg dE(b,d)$ is 2-valid; by Axiom 6, $W(a,b)\wedge\neg dE(b,d)\rightarrow\neg W(d,a)$ is 2-valid; by Axiom 2, $\neg W(d,a)\rightarrow\neg E(a,d)$ is 2-valid. Hence $W(a,b)\wedge\neg E(b,c)\wedge\neg E(c,d)\rightarrow\neg E(a,d)$ is 2-valid.
   (d) if R_k is $\neg dE$, then by Axiom 9, $W(a,b)\wedge\neg E(b,c)\wedge\neg dE(c,d)\rightarrow\neg dW(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a)\rightarrow\neg dE(a,d)$ is 2-valid. Hence $W(a,b)\wedge\neg E(b,c)\wedge\neg dE(c,d)\rightarrow\neg dE(a,d)$ is 2-valid.

   In each case, we replace $R_s(a,b)\wedge R_t(b,c)\wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of W and the number of $\neg E$ are reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, we have $\mathit{number}(W)+2*\mathit{number}(dW)\geq\mathit{number}(\neg E)+2*\mathit{number}(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

2. If R_s is W and R_t is $\neg dE$, then by Axiom 6, $W(a,b)\wedge\neg dE(b,c)\rightarrow\neg W(c,a)$ is 2-valid; by Axiom 2, $\neg W(c,a)\rightarrow\neg E(a,c)$ is 2-valid. Hence $R_s(a,b)\wedge R_t(b,c)\rightarrow\neg E(a,c)$ is 2-valid. We replace $R_s(a,b)\wedge R_t(b,c)$ in F_{n+1} with $\neg E(a,c)$ to obtain a formula F'. Since the number of W and the number of $\neg dE$ are reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged, we have $\mathit{number}(W)+2*\mathit{number}(dW)\geq\mathit{number}(\neg E)+2*\mathit{number}(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

3. If R_s is dW and R_t is $\neg E$, then by Axiom 7, $dW(a,b)\wedge\neg E(b,c)\rightarrow E(c,a)$ is 2-valid; by Axiom 2, $E(c,a)\rightarrow W(a,c)$ is 2-valid. Hence $R_s(a,b)\wedge R_t(b,c)\rightarrow W(a,c)$ is 2-valid. We replace $R_s(a,b)\wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$ to obtain a formula F'. Since the number of dW and the number of $\neg E$ are reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged, we have $\mathit{number}(W)+2*\mathit{number}(dW)\geq\mathit{number}(\neg E)+2*\mathit{number}(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

4. If R_s is dW and R_t is $\neg dE$, then
   (a) if R_k is W, then by Axiom 6, $\neg dE(b,c)\wedge W(c,d)\rightarrow\neg W(d,b)$ is 2-valid; by Axiom 2, $\neg W(d,b)\rightarrow\neg E(b,d)$ is 2-valid; by Axiom 7, $dW(a,b)\wedge\neg E(b,d)\rightarrow E(d,a)$ is 2-valid; by Axiom 2, $E(d,a)\rightarrow W(a,d)$ is 2-valid. Hence $dW(a,b)\wedge\neg dE(b,c)\wedge W(c,d)\rightarrow W(a,d)$ is 2-valid.

(b) if R_k is dW, then by Axiom 11, $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \to dE(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 2-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \to dW(a,d)$ is 2-valid.

(c) if R_k is $\neg E$, then by Axiom 10, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \to \neg W(d,a)$ is 2-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 2-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$ is 2-valid.

(d) if R_k is $\neg dE$, then by Axiom 12, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \to \neg dW(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 2-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$ is 2-valid.

In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of dW and the number of $\neg dE$ are reduced by 1, the number of W and the number of $\neg E$ are unchanged, we have $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

5. If R_s is $\neg E$ and R_t is W, then

(a) if R_k is W, then by Axiom 6, $W(b,c) \wedge W(c,d) \to dE(d,b)$ is 2-valid; by Definition 4, Axioms 2 and 3, $dE(d,b) \to dW(b,d)$ is 2-valid; by Axiom 7, $\neg E(a,b) \wedge dW(b,d) \to E(d,a)$ is 2-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 2-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \to W(a,d)$ is 2-valid.

(b) if R_k is dW, then by Axiom 10, $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \to dE(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 2-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \to dW(a,d)$ is 2-valid.

(c) if R_k is $\neg E$, then by Axiom 8, $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg W(d,a)$ is 2-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 2-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$ is 2-valid.

(d) if R_k is $\neg dE$, then by Axiom 6, $W(b,c) \wedge \neg dE(c,d) \to \neg W(d,b)$ is 2-valid; by Axiom 2, $\neg W(d,b) \to \neg E(b,d)$ is 2-valid; by Axiom 7, $\neg E(a,b) \wedge \neg E(b,d) \to \neg dW(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 2-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$ is 2-valid.

In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of $\neg E$ and the number of W are reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

6. If R_s is $\neg E$ and R_t is dW, then by Axiom 7, $\neg E(a,b) \wedge dW(b,c) \to E(c,a)$ is 2-valid; by Axiom 2, $E(c,a) \to W(a,c)$ is 2-valid. Hence $R_s(a,b) \wedge R_t(b,c) \to W(a,c)$ is 2-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$ to obtain a formula F'. Since the number of $\neg E$ and the number of dW are reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged, we have $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

7. If R_s is $\neg dE$ and R_t is W, then by Axiom 6, $\neg dE(a,b) \wedge W(b,c) \to \neg W(c,a)$ is 2-valid; by Axiom 2, $\neg W(c,a) \to \neg E(a,c)$ is 2-valid. Hence $R_s(a,b) \wedge R_t(b,c) \to$

$\neg E(a,c)$ is 2-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $\neg E(a,c)$ to obtain a formula F'. Since the number of $\neg dE$ and the number of W are reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged, we have $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

8. If R_s is $\neg dE$ and R_t is dW, then
   (a) if R_k is W, then by Axiom 9, $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow E(d,a)$ is 2-valid; by Axiom 2, $E(d,a) \rightarrow W(a,d)$ is 2-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow W(a,d)$ is 2-valid.
   (b) if R_k is dW, then by Axiom 12, $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \rightarrow dW(a,d)$ is 2-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$ is 2-valid.
   (c) if R_k is $\neg E$, then by Axiom 7, $dW(b,c) \wedge \neg E(c,d) \rightarrow E(d,b)$ is 2-valid; by Axiom 2, $E(d,b) \rightarrow W(b,d)$ is 2-valid; by Axiom 6, $\neg dE(a,b) \wedge W(b,d) \rightarrow \neg W(d,a)$ is 2-valid; by Axiom 2, $\neg W(d,a) \rightarrow \neg E(a,d)$ is 2-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$ is 2-valid.
   (d) if R_k is $\neg dE$, then by Axiom 11, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$ is 2-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \rightarrow \neg dE(a,d)$ is 2-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$ is 2-valid.

   In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of $\neg dE$ and the number of dW are reduced by 1, the number of W and the number of $\neg E$ are unchanged, we have $number(W) + 2 * number(dW) \geq number(\neg E) + 2 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}.

Therefore, for any $n \in \mathbb{N}_{>0}$, $\perp$ can be derived from F_n using LEW^2_{fin}. □

Lemma 5 stated in Section 4.4 is used to prove the completeness of LEW^3_{fin}. Lemma 5 is proved by proving Lemmas 11 and 12, where $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$ and $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$, respectively. Similar to Lemma 4, Lemmas 11 and 12 are proved using mathematical induction. The proof of Lemma 12 refers to Lemma 11.

Lemma 11 *For any $n \in \mathbb{N}_{>1}$, if for any integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$, then $\perp$ can be derived from $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ using LEW^3_{fin}.*

Proof. For any integer $n > 1$, let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$. We will show that for any $n > 1$, $\perp$ can be derived from F_n using LEW^3_{fin} by mathematical induction.

Base case When $n = 2$, since R_i is in $\{W, dW, \neg E, \neg dE\}$, $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$, then $\{R_1, R_2\} = \{W, \neg E\}$ or $\{R_1, R_2\} = \{dW, \neg dE\}$.

If $\{R_1, R_2\} = \{W, \neg E\}$, then by Axiom 2, $\perp$ can be derived. Otherwise, by Definition 4, Axioms 2 and 3, $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from $F_2, \ldots, F_n$ using LEW^3_{fin}, where $n \geq 2$, we will show $\perp$ can be derived from F_{n+1}. By Lemma 10, there exist conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_{n+1} such that they are neighbours and one of the two cases holds:

Case 1 R_s is in $\{W, dW\}$ and R_t is in $\{\neg E, \neg dE\}$;

Case 2 R_s is in $\{\neg E, \neg dE\}$ and R_t is in $\{W, dW\}$.

Let us proceed by cases. Since $n + 1 > 2$, in addition to $R_s(a, b)$, $R_t(b, c)$ has another neighbour $R_k(c, d)$.

1. If R_s is W and R_t is $\neg E$, then
   (a) if R_k is W, then by Axiom 8, $W(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to E(d, a)$ is 3-valid; by Axiom 2, $E(d, a) \to W(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to W(a, d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 16, $W(a, b) \wedge \neg E(b, c) \wedge dW(c, d) \to dE(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d, a) \to dW(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge dW(c, d) \to dW(a, d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 15, $W(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to \neg W(d, a)$ is 3-valid; by Axiom 2, $\neg W(d, a) \to \neg E(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to \neg E(a, d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 9, $W(a, b) \wedge \neg E(b, c) \wedge \neg dE(c, d) \to \neg dW(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d, a) \to \neg dE(a, b)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge \neg dE(c, d) \to \neg dE(a, d)$ is 3-valid.
2. If R_s is W and R_t is $\neg dE$, then
   (a) if R_k is W, then by Axiom 13, $W(a, b) \wedge \neg dE(b, c) \wedge W(c, d) \to \neg W(d, a)$ is 3-valid; by Axiom 2, $\neg W(d, a) \to \neg E(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg dE(b, c) \wedge W(c, d) \to \neg E(a, d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 17, $W(a, b) \wedge \neg dE(b, c) \wedge dW(c, d) \to E(d, a)$ is 3-valid; by Axiom 2, $E(d, a) \to W(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg dE(b, c) \wedge dW(c, d) \to W(a, d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 19, $W(a, b) \wedge \neg dE(b, c) \wedge \neg E(c, d) \to \neg dW(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d, a) \to \neg dE(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg dE(b, c) \wedge \neg E(c, d) \to \neg dE(a, d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then no axiom in LEW^3_{fin} is applied.
3. If R_s is dW and R_t is $\neg E$, then
   (a) if R_k is W, then by Axiom 17, $dW(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to dE(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d, a) \to dW(a, d)$ is 3-valid. Hence $dW(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to dW(a, d)$ is 3-valid.
   (b) if R_k is dW, then no axiom in LEW^3_{fin} is applied.
   (c) if R_k is $\neg E$, then by Axiom 14, $dW(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to E(d, a)$ is 3-valid; by Axiom 2, $E(d, a) \to W(a, d)$ is 3-valid. Hence $dW(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to W(a, d)$ is 3-valid.

   (d) if R_k is $\neg dE$, then by Axiom 18, $dW(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \to \neg E(a,d)$ is 3-valid.

4. If R_s is dW and R_t is $\neg dE$, then
   (a) if R_k is W, then by Axiom 16, $dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \to E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \to W(a,d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 11, $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \to dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \to dW(a,d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 10, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 12, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \to \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$ is 3-valid.

5. If R_s is $\neg E$ and R_t is W, then
   (a) if R_k is W, then by Axiom 15, $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \to E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \to W(a,d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 10, $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \to dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \to dW(a,d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 8, $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 17, $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \to \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$ is 3-valid.

6. If R_s is $\neg E$ and R_t is dW, then
   (a) if R_k is W, then by Axiom 19, $\neg E(a,b) \wedge dW(b,c) \wedge W(c,d) \to dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge dW(b,c) \wedge W(c,d) \to dW(a,d)$ is 3-valid.
   (b) if R_k is dW, then no axiom in LEW^3_{fin} is applied.
   (c) if R_k is $\neg E$, then by Axiom 14, $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \to E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \to W(a,d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 16, $\neg E(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \to \neg E(a,d)$ is 3-valid.

7. If R_s is $\neg dE$ and R_t is W, then

(a) if R_k is W, then by Axiom 13, $\neg dE(a,b) \wedge W(b,c) \wedge W(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge W(b,c) \wedge W(c,d) \to \neg E(a,d)$ is 3-valid.

(b) if R_k is dW, then by Axiom 18, $\neg dE(a,b) \wedge W(b,c) \wedge dW(c,d) \to E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge W(b,c) \wedge dW(c,d) \to W(a,d)$ is 3-valid.

(c) if R_k is $\neg E$, then by Axiom 16, $\neg dE(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge W(b,c) \wedge \neg E(c,d) \to \neg dE(a,d)$ is 3-valid.

(d) if R_k is $\neg dE$, then no axiom in LEW^3_{fin} is applied.

8. If R_s is $\neg dE$ and R_t is dW, then

(a) if R_k is W, then by Axiom 9, $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \to E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \to W(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \to W(a,d)$ is 3-valid.

(b) if R_k is dW, then by Axiom 12, $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \to dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \to dW(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \to dW(a,d)$ is 3-valid.

(c) if R_k is $\neg E$, then by Axiom 17, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \to \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \to \neg E(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$ is 3-valid.

(d) if R_k is $\neg dE$, then by Axiom 11, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \to \neg dW(d,a)$ is 3-valid. By Definition 4, Axioms 2 and 3, $\neg dW(d,a) \to \neg dE(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$ is 3-valid.

In the cases 2.d, 3.b, 6.b and 7.d, no axiom in LEW^3_{fin} is applied. Let us call *all the other cases* above 'valid'. Then for *every* three conjuncts $R_s(a,b), R_t(b,c), R_k(c,d)$ in F_{n+1}, at least one of these valid cases holds; otherwise, $R_s, R_t, R_k \in \{W, dW\}$, *or* $R_s, R_t, R_k \in \{\neg E, \neg dE\}$, *or* exactly one of R_s, R_t, R_k is W and the rest are $\neg dE$, *or* exactly one of R_s, R_t, R_k is $\neg E$ and the rest are dW, which contradicts with $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$. In each valid case, we obtain a formula of the form $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \to R_x(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_x(a,d)$ to obtain a formula F', where $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

Therefore, for any $n > 1$, $\perp$ can be derived from F_n using LEW^3_{fin}. □

Lemma 12 *For any $n \in \mathbb{N}_{>1}$, if for any integer i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$, then $\perp$ can be derived from $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ using LEW^3_{fin}.*

Proof. For $n = 1$, let F_n denote a formula of the form $W(a,a)$ or $dW(a,a)$. For any $n > 1$, let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$, where for every integer i such that $1 \leq i \leq n$, R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + 3 * number(dW) >$

$number(\neg E) + 3 * number(\neg dE)$. We will show that for any $n > 0$, $\bot$ can be derived from F_n using LEW^3_{fin} by mathematical induction.

Base case When $n = 1$, by Axiom 1 and Definition 4, $\bot$ can be derived.

When $n = 2$, since R_i is in $\{W, dW, \neg E, \neg dE\}$, and $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$, then R_1, R_2 is in $\{W, dW\}$ or $\{R_1, R_2\} = \{dW, \neg E\}$. If R_1, R_2 is in $\{W, dW\}$, then by Definition 4, Axioms 15, 1 and 2, $\bot$ can be derived (by Axiom 15, $W(a, b) \wedge W(b, a) \wedge \neg E(a, a) \wedge \neg E(a, a) \to \bot$ is 3-valid; by Axioms 1 and 2, $\neg E(a, a)$ is 3-valid; hence $W(a, b) \wedge W(b, a) \to \bot$ is 3-valid). Otherwise, by Axioms 14, 1 and 2, $\bot$ can be derived.

Inductive step Suppose $\bot$ can be derived from any of $F_1, F_2, \ldots, F_n$ using LEW^3_{fin}, where $n \geq 2$, we will show that $\bot$ can be derived from F_{n+1}. If every R_i in F_{n+1} is W or dW, then by Definition 4, Axioms 15, 1 and 2, $\bot$ can be derived (by Axiom 15, $W(a, b) \wedge W(b, c) \wedge \neg E(c, a) \wedge \neg E(a, a) \to \bot$ is 3-valid; by Axioms 2 and 1, $\neg E(a, a)$ is 3-valid; by Axiom 2, $E(c, a) \to W(a, c)$ is 3-valid; hence $W(a, b) \wedge W(b, c) \to W(a, c)$ is 3-valid).

Otherwise, there exists at least one R_i which is $\neg E$ or $\neg dE$. By Lemma 9, there exist conjuncts $R_s(a, b)$ and $R_t(b, c)$ in F_{n+1}, such that they are neighbours and one of the following cases holds:

Case 1 R_s is in $\{W, dW\}$ and R_t is in $\{\neg E, \neg dE\}$;

Case 2 R_s is in $\{\neg E, \neg dE\}$ and R_t is in $\{W, dW\}$.

Let us proceed by cases. Since $n + 1 > 2$, in addition to $R_s(a, b)$, $R_t(b, c)$ has another neighbour $R_k(c, d)$.

1. If R_s is W and R_t is $\neg E$, then
   (a) if R_k is W, then by Axiom 8, $W(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to E(d, a)$ is 3-valid; by Axiom 2, $E(d, a) \to W(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge W(c, d) \to W(a, d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 16, $W(a, b) \wedge \neg E(b, c) \wedge dW(c, d) \to dE(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d, a) \to dW(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge dW(c, d) \to dW(a, d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 15, $W(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to \neg W(d, a)$ is 3-valid; by Axiom 2, $\neg W(d, a) \to \neg E(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge \neg E(c, d) \to \neg E(a, d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 9, $W(a, b) \wedge \neg E(b, c) \wedge \neg dE(c, d) \to \neg dW(d, a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d, a) \to \neg dE(a, d)$ is 3-valid. Hence $W(a, b) \wedge \neg E(b, c) \wedge \neg dE(c, d) \to \neg dE(a, d)$ is 3-valid.

   In each case, we replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$ to obtain a formula F'. Since the number of W and the number of $\neg E$ are reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$. By inductive hypothesis, $\bot$ can be derived from F', hence from F_{n+1}.

2. If R_s is W and R_t is $\neg dE$, then by Axiom 19, $W(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,c) \rightarrow \neg dW(c,a)$ is 3-valid; by Axioms 1 and 2, $\neg E(c,c)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(c,a) \rightarrow \neg dE(a,c)$ is 3-valid. Hence $W(a,b) \wedge \neg dE(b,c) \rightarrow \neg dE(a,c)$ is 3-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $R_t(a,c)$ to obtain a formula F'. Since the number of W is reduced by 1, the number of $\neg E$, the number of dW and the number of $\neg dE$ are unchanged, we have $\mathit{number}(W) + 3 * \mathit{number}(dW) \geq \mathit{number}(\neg E) + 3 * \mathit{number}(\neg dE)$. If $\mathit{number}(W) + 3 * \mathit{number}(dW) = \mathit{number}(\neg E) + 3 * \mathit{number}(\neg dE)$, then by Lemma 11, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence in either case, $\perp$ can be derived from F_{n+1}.

3. If R_s is dW and R_t is $\neg E$, by Axiom 14, $dW(a,b) \wedge \neg E(b,c) \wedge \neg E(c,c) \rightarrow E(c,a)$ is 3-valid; by Axioms 1 and 2, $\neg E(c,c)$ is 3-valid; by Axiom 2, $E(c,a) \rightarrow W(a,c)$ is 3-valid. Hence $dW(a,b) \wedge \neg E(b,c) \rightarrow W(a,c)$ is 3-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$ to obtain a formula F'. Since the the number of dW and the number of $\neg E$ are reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged, we have $\mathit{number}(W) + 3 * \mathit{number}(dW) \geq \mathit{number}(\neg E) + 3 * \mathit{number}(\neg dE)$. If $\mathit{number}(W) + 3 * \mathit{number}(dW) = \mathit{number}(\neg E) + 3 * \mathit{number}(\neg dE)$, then by Lemma 11, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence in either case, $\perp$ can be derived from F_{n+1}.

4. If R_s is dW and R_t is $\neg dE$, then
   (a) if R_k is W, then by Axiom 16, $dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \rightarrow E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \rightarrow W(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \rightarrow W(a,d)$ is 3-valid.
   (b) if R_k is dW, then by Axiom 11, $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \rightarrow dW(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$ is 3-valid.
   (c) if R_k is $\neg E$, then by Axiom 10, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \rightarrow \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \rightarrow \neg E(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$ is 3-valid.
   (d) if R_k is $\neg dE$, then by Axiom 12, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \rightarrow \neg dE(a,d)$ is 3-valid. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$ is 3-valid.

   In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of dW and the number of $\neg dE$ are reduced by 1, the number of W and the number of $\neg E$ are unchanged, we have $\mathit{number}(W) + 3 * \mathit{number}(dW) > \mathit{number}(\neg E) + 3 * \mathit{number}(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

5. If R_s is $\neg E$ and R_t is W, then
   (a) if R_k is W, then by Axiom 15, $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \rightarrow W(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow W(a,d)$ is 3-valid.

(b) if R_k is dW, then by Axiom 10, $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \rightarrow dW(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$ is 3-valid.

(c) if R_k is $\neg E$, then by Axiom 8, $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \rightarrow \neg E(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$ is 3-valid.

(d) if R_k is $\neg dE$, then by Axiom 17, $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \rightarrow \neg dE(a,d)$ is 3-valid. Hence $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$ is 3-valid.

In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of $\neg E$ and the number of W are reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

6. If R_s is $\neg E$ and R_t is dW, by Axiom 14, $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,c) \rightarrow E(c,a)$ is 3-valid; by Axioms 1 and 2, $\neg E(c,c)$ is 3-valid; by Axiom 2, $E(c,a) \rightarrow W(a,c)$ is 3-valid. Hence $\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$ is 3-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$ to obtain a formula F'. Since the the number of dW and the number of $\neg E$ are reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged, we have $number(W) + 3 * number(dW) \geq number(\neg E) + 3 * number(\neg dE)$. If $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$, then by Lemma 11, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence in either case, $\perp$ can be derived from F_{n+1}.

7. If R_s is $\neg dE$ and R_t is W, then by Axiom 16, $\neg dE(a,b) \wedge W(b,c) \wedge \neg E(c,c) \rightarrow \neg dW(c,a)$ is 3-valid; by Axioms 1 and 2, $\neg E(c,c)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(c,a) \rightarrow \neg dE(a,c)$ is 3-valid. Hence $\neg dE(a,b) \wedge W(b,c) \rightarrow \neg dE(a,c)$ is 3-valid. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $R_s(a,c)$ to obtain a formula F'. Since the number of W is reduced by 1, the number of $\neg E$, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + 3 * number(dW) \geq number(\neg E) + 3 * number(\neg dE)$. If $number(W) + 3 * number(dW) = number(\neg E) + 3 * number(\neg dE)$, then by Lemma 11, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence in either case, $\perp$ can be derived from F_{n+1}.

8. If R_s is $\neg dE$ and R_t is dW, then

(a) if R_k is W, then by Axiom 9, $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow E(d,a)$ is 3-valid; by Axiom 2, $E(d,a) \rightarrow W(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow W(a,d)$ is 3-valid.

(b) if R_k is dW, then by Axiom 12, $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $dE(d,a) \rightarrow dW(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$ is 3-valid.

(c) if R_k is $\neg E$, then by Axiom 17, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \rightarrow \neg W(d,a)$ is 3-valid; by Axiom 2, $\neg W(d,a) \rightarrow \neg E(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$ is 3-valid.

(d) if R_k is $\neg dE$, then by Axiom 11, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$ is 3-valid; by Definition 4, Axioms 2 and 3, $\neg dW(d,a) \rightarrow \neg dE(a,d)$ is 3-valid. Hence $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$ is 3-valid.

In each case, we replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$ to obtain a formula F'. Since the number of $\neg dE$ and the number of dW are reduced by 1, the number of W and the number of $\neg E$ are unchanged, we have $number(W) + 3 * number(dW) > number(\neg E) + 3 * number(\neg dE)$. By inductive hypothesis, $\perp$ can be derived from F', hence from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}.

Therefore, for any $n > 0$, $\perp$ can be derived from F_n using LEW^3_{fin}. $\square$

Appendix B. Proof Details of Theorem 5

This section verifies that I_2 provides a solution to S^o. Recall that S^o is obtained by removing an arbitrary individual name o from S. There are four cases: o is a_x, where $0 \leq x \leq 2$; o is b_x, where $0 < x < n$; o is c_x, where $0 \leq x \leq 2$; o is d_x, where $0 < x < n$. Below we provide verification details for the second case. The other cases are similar and simpler.

Since o is b_x, where $0 < x < n$, by the definition of I_2, we have:

- $I_2(b_{x+1}) = 2 + (x+1)\tau$;
- for every integer i such that $x + 1 < i \leq n$, $I_2(b_i) = 2 + i\tau + \frac{\epsilon}{2} + \cdots + \frac{\epsilon}{2^{(i-1-x)}}$;
- for every integer i such that $0 \leq i \leq 2$, $I_2(c_i) = I_2(b_n) - i$;
- for every integer i such that $0 \leq i \leq n$, $I_2(d_i) = I_2(c_2) - i\tau$;
- $I_2(a_0) = I_2(d_n)$, $I_2(a_1) = I_2(a_0) + 1 + \frac{\epsilon}{2^{(2n-x+2)}}$, $I_2(a_2) = I_2(a_0) + 2 + \frac{\epsilon}{2^{(2n-x+2)}} + \frac{\epsilon}{2^{(2n-x+3)}}$;
- for every integer i such that $0 \leq i \leq x-1$, $I_2(b_i) = I_2(a_0) + 2 + i\tau + \frac{\epsilon}{2^{(2n-x+2)}} + \cdots + \frac{\epsilon}{2^{(2n-x+3+i)}}$.

Note that $0 \leq I_2(a_0) \leq \frac{\epsilon}{2} + \cdots + \frac{\epsilon}{2^{(n-1-x)}} < 1$. Referring to the items 1-11 in the proof of Lemma 7, below we verify that I_2 provides a solution to S^{b_x}.

1. $I_2(a_0) - I_2(a_1) = -1 - \frac{\epsilon}{2^{(2n-x+2)}} \in (-2,-1)$, $I_2(a_1) - I_2(a_2) = -1 - \frac{\epsilon}{2^{2n-x+3}} \in (-2,-1)$, $I_2(a_0) - I_2(a_2) = -2 - \frac{\epsilon}{2^{(2n-x+2)}} - \frac{\epsilon}{2^{2n-x+3}} \in (-3,-2)$. Since $\tau > 3$, for every pair of integers i, j such that $0 \leq i < j \leq 2$, by Definitions 3 and 4, the corresponding linear inequalities of $sW(a_i, a_j)$ and $sE(a_j, a_i)$ in S^{b_x} are satisfied.
2. For every pair of integers i, j such that $0 \leq i < j \leq x-1$ or $x+1 \leq i < j \leq n$, we have $I_2(b_i) - I_2(b_j) < -\tau$. For every pair of integers i, j such that $0 \leq i \leq x-1$ and $x+1 \leq j \leq n$, we have $I_2(b_i) - I_2(b_j) \leq I_2(b_{x-1}) - I_2(b_{x+1}) < -\tau$. Hence for every pair of integers i, j such that $0 \leq i < j \leq n$, $i \neq x$ and $j \neq x$, we have $I_2(b_i) - I_2(b_j) < -\tau$; by Definitions 3 and 4, the corresponding linear inequalities of $dW(b_i, b_j)$ and $dE(b_j, b_i)$ in S^{b_x} are satisfied.

3. $I_2(c_0) - I_2(c_1) = 1$, $I_2(c_1) - I_2(c_2) = 1$, $I_2(c_0) - I_2(c_2) = 2$. Since $\tau > 3$, by Definitions 3 and 4, the corresponding linear inequalities of $nEW(c_0, c_1)$, $nEW(c_1, c_0)$, $nEW(c_1, c_2)$, $nEW(c_2, c_1)$, $sE(c_0, c_2)$ and $sW(c_2, c_0)$ in S^{b_x} are satisfied.

4. For every pair of integers i, j such that $0 \leq i < j \leq n$, we have $I_2(d_i) - I_2(d_j) = (j - i)\tau$. By Definitions 3 and 4, if $j = i + 1$, the corresponding linear inequalities of $sE(d_i, d_j)$ and $sW(d_j, d_i)$ in S^{b_x} are satisfied; and if $j > i+1$, the corresponding linear inequalities of $dE(d_i, d_j)$ and $dW(d_j, d_i)$ in S^{b_x} are satisfied.

5. For every pair of integers i, j such that $0 \leq i < 2$, $0 < j \leq n$ and $j \neq x$, if $1 \neq x$ (i.e., b_1 is not b_x), we have $I_2(a_i) - I_2(b_j) \leq I_2(a_1) - I_2(b_1) < -\tau$; otherwise, we have $I_2(a_i) - I_2(b_j) \leq I_2(a_1) - I_2(b_2) < -\tau$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $dW(a_i, b_j)$ and $dE(b_j, a_i)$ in S^{b_x} are satisfied.

6. For every pair of integers i, j such that $0 \leq i < 2$ and $0 < j \leq 2$, we have $I_2(a_i) - I_2(c_j) \leq I_2(a_1) - I_2(c_2) < -\tau$, since $n > 2$ and $\tau > 3$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $dW(a_i, c_j)$ and $dE(c_j, a_i)$ in S^{b_x} are satisfied.

7. For every integer j such that $0 < j < n - 1$, we have $I_2(a_1) - I_2(d_j) \leq I_2(a_1) - I_2(d_{n-2}) < -\tau$, as $n > 2$ ands $\tau > 3$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $dW(a_1, d_j)$ and $dE(d_j, a_1)$ in S^{b_x} are satisfied. Since $\tau > 3$, we have $I_2(a_1) - I_2(d_{n-1}) \in (-\tau, -1)$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $sW(a_1, d_{n-1})$ and $sE(d_{n-1}, a_1)$ in S^{b_x} are satisfied.

8. For every pair of integers i, j such that $0 \leq i < n - 1$, $i \neq x$ and $0 < j \leq 2$, if $n - 2 \neq x$, then we have $I_2(b_i) - I_2(c_j) \leq I_2(b_{n-2}) - I_2(c_2) < -\tau$; otherwise, we have $I_2(b_i) - I_2(c_j) \leq I_2(b_{n-3}) - I_2(c_2) < -\tau$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $dW(b_i, c_j)$ and $dE(c_j, b_i)$ in S^{b_x} are satisfied. If $n - 1 \neq x$, then for every integer j such that $0 < j \leq 2$, $I_2(b_{n-1}) - I_2(c_j) \leq I_2(b_{n-1}) - I_2(c_2) \in (-\tau, -1)$, as $\tau > 3$. Hence by Definitions 3 and 4, the corresponding linear inequalities of $sW(b_{n-1}, c_j)$ and $sE(c_j, b_{n-1})$ in S^{b_x} are satisfied.

9. For every pair of integers i, j such that $0 \leq i < n$, $i \neq x$ and $0 < j < n$, we have $I_2(b_i) - I_2(d_j) < 2 + i\tau + 1 - (n - j)\tau = -(n - i - j)\tau + 3$ and $I_2(b_i) - I_2(d_j) > 2 + i\tau - (n-j)\tau - 1 = -(n-i-j)\tau + 1$. Hence if $n - i - j > 1$, then $I_2(b_i) - I_2(d_j) < -\tau$, by Definitions 3 and 4, the corresponding linear inequalities of $dW(b_i, d_j)$ and $dE(d_j, b_i)$ in S^{b_x} are satisfied; if $n - i - j = 1$, then $I_2(b_i) - I_2(d_j) \in (-\tau, -1)$, by Definitions 3 and 4, the corresponding linear inequalities of $sW(b_i, d_j)$ and $sE(d_j, b_i)$ in S^{b_x} are satisfied; if $n - i - j = 0$, then $I_2(b_i) - I_2(d_j) \in (1, \tau)$, by Definitions 3 and 4, the corresponding linear inequalities of $sE(b_i, d_j)$ and $sW(d_j, b_i)$ in S^{b_x} are satisfied; if $n - i - j < 0$, then $I_2(b_i) - I_2(d_j) > \tau$, by Definitions 3 and 4, the corresponding linear inequalities of $dE(b_i, d_j)$ and $dW(d_j, b_i)$ in S^{b_x} are satisfied.

10. For every pair of integers i, j such that $0 \leq i < 2$ and $0 < j < n$, we have $I_2(c_i) - I_2(d_j) \geq I_2(c_1) - I_2(d_1) > \tau$, by Definitions 3 and 4, the corresponding linear inequalities of $dE(c_i, d_j)$ and $dW(d_j, c_i)$ in S^{b_x} are satisfied.

11. For every individual name e in *Ind*, if e is not b_x, then $I_2(e) - I_2(e) = 0$. By Definitions 3 and 4, the corresponding linear inequality of $nEW(e, e)$ in S^{b_x} is satisfied.

Therefore, I_2 provides a solution to S^{b_x}.

Appendix C. Proof of Lemma 8

Lemma 8 *For every $\tau \in \mathbb{N}_{>1}$, let S be a set of linear inequalities obtained by applying the 'τ-σ-translation' function over $L(LEW)$ formulas as shown in Definition 5, where $\sigma = 1$; and let $n > 0$ be the number of variables in S. If S is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the binary representation size of t is in $O(n)$.*

Proof. Take an arbitrary integer $\tau > 1$. Suppose that S is satisfiable. By Definition 5, every inequality in S is of the form $(x_1 - x_2) \sim c$, where x_1, x_2 are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. Let G be a graph for S. By Corollary 1, the graph G has no infeasible simple loop. By extending the proof of Theorem 4 (Shostak, 1981) (pp. 777 and 778), which is for non-strict inequalities only, to include both strict and non-strict inequalities, a solution to S can be constructed as follows. Let $v_1, \ldots, v_{n-1}$ be the variables of S other than v_0 (the zero variable). The *residue inequality* of an admissible path P is denoted as $(a_p x + b_p y) \sim c_p$, where $\sim$ is $\leq$ or $<$, and x, y are the first and last vertices of P. We construct a sequence of reals $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{n-1}$ as a solution to S and a sequence of graphs $G_0, G_1, \ldots, G_{n-1}$ inductively.
1. Let $\hat{v}_0 = 0$ and $G_0 = G$.
2. If $\hat{v}_i$ and G_i have been determined for $0 \leq i < j < n$, let

$$\sup_j = \min\{\tfrac{c_p}{a_p} \mid P \text{ is an admissible path from } v_j \text{ to } v_0 \text{ in } G_{j-1} \text{ and } a_p > 0\},$$

$$\inf_j = \max\{\tfrac{c_p}{b_p} \mid P \text{ is an admissible path from } v_0 \text{ to } v_j \text{ in } G_{j-1} \text{ and } b_p < 0\},$$

where $\min \emptyset = \infty$ and $\max \emptyset = -\infty$. The range of $\hat{v}_j$ is obtained as follows.

- If there is an admissible path P from v_j to v_0 in G_{j-1} such that the residue inequality of P is $a_p v_j < c_p$, where $a_p > 0$, and $\frac{c_p}{a_p} = \sup_j$, then $\hat{v}_j < \sup_j$, otherwise, $\hat{v}_j \leq \sup_j$.
- If there is an admissible path P from v_0 to v_j in G_{j-1} such that the residue inequality of P is $b_p v_j < c_p$, where $b_p < 0$, and $\frac{c_p}{b_p} = \inf_j$, then $\hat{v}_j > \inf_j$, otherwise, $\hat{v}_j \geq \inf_j$.

Instead of letting $\hat{v}_j$ be any real number in the range (Shostak, 1981), we assign a value to $\hat{v}_j$ as follows:

- if there exists an integer within the range of $\hat{v}_j$, we assign an integer to $\hat{v}_j$;
- otherwise, we assign $\frac{\inf_j + \sup_j}{2}$ to $\hat{v}_j$.

The graph G_j is obtained from G_{j-1} by adding two new edges from v_j to v_0, labelled $v_j \leq \hat{v}_j$ and $v_j \geq \hat{v}_j$, respectively.

To ensure that $\hat{v}_j$ and G_j are well defined, we prove the following two claims:
1. For every integer j such that $1 \leq j < n$, the range of $\hat{v}_j$ is not empty.
2. For every integer j such that $0 \leq j < n$, the graph G_j has no infeasible simple loop.

We prove them by induction on j, similar to the proof presented by Shostak (1981).
Base case $j = 0$. 1 holds vacuously; 2 holds since $G_0 = G$.
Inductive step Suppose the claims hold for $j-1$ such that $0 \leq j-1 < n-1$. We will show the claims hold for j.
For 1, suppose, to the contrary, that the range of $\hat{v}_i$ is empty. Then in G_{j-1}, there exist an admissible path P_1 from v_j to v_0, where $a_p > 0$, and an admissible path P_2 from v_0 to v_j, where $b_p < 0$. Then P_1 and P_2 form an admissible loop. By the construction of the range of $\hat{v}_i$ described above, if this range is empty, then the admissible loop formed by P_1 and P_2 is infeasible, which contradicts the inductive hypothesis that G_{j-1} has no infeasible simple loop.
For 2, suppose G_j has an infeasible simple loop P. Since G_{j-1} has no such loop, and the loop formed by the two new edges added to G_{j-1} to obtain G_j is not infeasible, then P (or its reverse) is of the form $P'E$, where E is one of the two new edges (say the one labelled $v_j \leq \hat{v}_j$; the other case is handled similarly), and P' is a path from v_0 to v_j in G_{j-1}. If P' is strict, then by the definition of infeasible loop of P, we have $\hat{v}_j \leq \frac{c_{p'}}{b_{p'}}$, which contradicts $\hat{v}_j > \frac{c_{p'}}{b_{p'}}$ (if $\inf_j = \frac{c_{p'}}{b_{p'}}$, then $\hat{v}_j > \inf_j$; otherwise, $\inf_j > \frac{c_{p'}}{b_{p'}}$, $\hat{v}_j \geq \inf_j$); if P' is not strict, then $\hat{v}_j < \frac{c_{p'}}{b_{p'}}$, which contradicts $\hat{v}_j \geq \frac{c_{p'}}{b_{p'}}$, since $\hat{v}_j \geq \inf_j$ and $\inf_j \geq \frac{c_{p'}}{b_{p'}}$. Q.E.D.

Now, it remains to show that $\hat{v}_j$ satisfies S. Let $ax + by \leq c$ be an inequality in S. We will show that $a\hat{x} + b\hat{y} \leq c$. We present the case where $a > 0$ and $b < 0$. The other cases are similar. Let E be the edge labelled $ax + by \leq c$ in G_{n-1}. Then, where E_1 is the edge labelled $\hat{x} \leq x$ in G_{n-1} and E_2 is the one labelled $y \leq \hat{y}$, the edges E_1, E and E_2 form an admissible loop E_1EE_2. Since G_{n-1} has no infeasible loop, the loop E_1EE_2 is feasible. Hence we have $a\hat{x} + b\hat{y} \leq c$. The proof for inequalities of the form $ax + by < c$ is similar.

By Definition 5, we have $-n\tau \leq c_p \leq n\tau$, $a_p = 1$ for $\sup_j$, $b_p = -1$ for $\inf_j$. Therefore, $\sup_j \leq n\tau$, $\inf_j \geq -n\tau$. Hence every $\hat{v}_j$ $(0 < j < n)$ is a rational number in $[-n\tau, n\tau]$.

Now, we show that the representation size of $\hat{v}_j$ $(0 < j < n)$ is polynomial in the size of n. By the construction described above, $\hat{v}_j$ is either an integer in $[-n\tau, n\tau]$ or obtained by applying the 'average operation' $\hat{v}_j = \frac{\inf_j + \sup_j}{2}$. Since τ is a natural number and $\sigma = 1$, $\inf_1$ and $\sup_1$ are integers in $[-n\tau, n\tau]$. Also, since $0 < j < n$, the number of 'average operations' applied to obtain a $\hat{v}_j$ is at most n. Hence the largest denominator of the values of $\hat{v}_j$ is 2^n. Therefore, $\hat{v}_j$ can be represented in a binary notation of size $\log(2n\tau * 2^n)$, which is in $O(n)$. $\square$

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