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Boolean Observation Games

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Abstract

We introduce *Boolean Observation Games*, a subclass of multi-player finite strategic games with *incomplete information* and *qualitative objectives*. In Boolean observation games, each player is associated with a finite set of propositional *variables* of which only it can observe the value, and it controls whether and to whom it can *reveal* that value. It does not control the given, fixed, value of variables. Boolean observation games are a generalization of Boolean games, a well-studied subclass of strategic games but with complete information, and wherein each player controls the value of its variables.

In Boolean observation games, player goals describe multi-agent knowledge of variables. As in classical strategic games, players choose their strategies simultaneously and therefore observation games capture aspects of both imperfect and incomplete information. They require reasoning about sets of outcomes given sets of indistinguishable valuations of variables. An outcome relation between such sets determines what the Nash equilibria are. We present various outcome relations, including a qualitative variant of ex-post equilibrium. We identify conditions under which, given an outcome relation, Nash equilibria are guaranteed to exist. We also study the complexity of checking for the existence of Nash equilibria and of verifying if a strategy profile is a Nash equilibrium. We further study the subclass of Boolean observation games with 'knowing whether' goal formulas, for which the satisfaction does not depend on the value of variables. We show that each such Boolean observation game corresponds to a Boolean game and vice versa, by a different correspondence, and that both correspondences are precise in terms of existence of Nash equilibria.

1. Introduction

Reasoning about strategic agents is an important problem in the theory of multi-agent systems and game-theoretic models and techniques are often used as a tool in such analysis. Strategic games (Osborne & Rubinstein, 1994) is a classic and well-studied framework that models one-shot multi-player games where agents make their choice simultaneously. It forms a simple and intuitive formalism to analyse and reason about the strategic behaviour of agents. From the perspective of computer science and artificial intelligence, one of the main drawbacks of strategic games is that the explicit representation of the payoff (or utility) function is exponential in the number of players and the strategies available for each player. In many applications, compact representation of the underlying game model is highly desirable.

Various approaches have been suggested to achieve compact representation of games and these mainly involve imposing restrictions on the payoff functions. For instance, constraining the payoff functions to be pairwise separable (Janovskaya, 1968; Cai & Daskalakis, 2011) results in the well-studied class of games with a compact representation, called *polymatrix games*. *Additively separable hedonic games* (Aziz & Savani, 2016) form another subclass of strategic games with pairwise separable payoff functions which can be used to analyse coalition formation in multi-agent systems. It is also possible to achieve compact representation by explicitly restricting the dependency of payoff functions to a "small" number of other agents (or neighbourhood) as done in *graphical games* (Kearns, Littman, & Singh, 2001).

An alternative approach to imposing quantitative constraints on payoffs is to restrict the payoffs to qualitative outcomes which are presented as logical formulas. For example, "extensive" games played on graphs where the goal formulas can specify the evolution of play with a combination of temporal and epistemic specifications. Although originally defined as two-player perfect information games motivated by questions in automata theory and logic, these models are now sophisticated to reason about multi-player games and imperfect information (Chatterjee, Doyen, Henzinger, & Raskin, 2007; Apt & Grädel, 2011; Gutierrez, Murano, Perelli, Rubin, & Wooldridge, 2017). Boolean games (Harrenstein, van der Hoek, Meyer, & Witteveen, 2001), a subclass of strategic games with complete information where objectives are expressed as Boolean formulas, is also a well-studied framework with such qualitative outcomes.

In Boolean games, each player controls a disjoint subset of propositional variables where their strategies correspond to choosing values for these variables and each player's goal is specified by a Boolean formula over the set of all variables. While the model was originally defined to analyse two-player games, the framework has been extended in many directions.

Multi-player, non-zero-sum Boolean games are studied in (Harrenstein, 2004; Bonzon, Lagasquie-Schiex, Lang, & Zanuttini, 2006). In (Harrenstein et al., 2001; Harrenstein, 2004) Boolean games are modelled as imperfect information games by taking the uncertainty over the other player's actions as an information set, as in (van Benthem, 2001). In (Dunne & van der Hoek, 2004; Bonzon et al., 2006; Dunne & Wooldridge, 2012) the computational properties of Boolean games are adressed, in (Bonzon, Lagasquie-Schiex, & Lang, 2009) graphical dependency structures for Boolean games and their implications for various structural and computational properties, and in (Ianovski & Ong, 2014) mixed strategy Nash equilibria and related computational questions. The issue of equilibrium selection is considered in (Ågotnes, Harrenstein, van der Hoek, & Wooldridge, 2013a). Iterated Boolean games (Gutierrez, Harrenstein, & Wooldridge, 2015; Gutierrez, Harrenstein, Perelli, & Wooldridge, 2016) model repeated interaction between players with temporal goals specified in linear time temporal logic (LTL). Partial ordering of the run-time events in terms of a dependency graph on propositions is studied in (Bradfield, Gutierrez, & Wooldridge, 2016).

Epistemic Boolean games, wherein goal formulas may be epistemic, were proposed in (Ågotnes, Harrenstein, van der Hoek, & Wooldridge, 2013b; Herzig, Lorini, Maffre, & Schwarzentruber, 2016). Both works combine the control of variables with the observation of variables (or formulas), where some of this is strategic and some is given with the game. This hybrid setting allows the authors to continue to analyse these epistemic Boolean games as complete information strategic form games. Realizing epistemic objectives depends on the valuation of variables *resulting* from strategic action.

In this paper, we introduce Boolean observation games as a qualitative model to analyse and reason about a subclass of strategic games with *incomplete information*. In Boolean observation games, *players control whether and to whom they reveal (announce) the value of propositional variables that can only be observed by them.* This constitutes a multi-player game model with concise representation where players have (qualitative) epistemic objectives. It is incomplete information because realizing the objectives depends on a given fixed valuation that the players cannot control. Players do not know what that valuation is and therefore do not know what game they play. Realizing epistemic objectives depends on the unknown valuation of variables that is independent from strategic action. (We should note that such incomplete games of imperfect information can also be modelled as complete games of imperfect information by assuming an initial random move of a player 'nature' determining the valuation.)

Since Boolean observation games define a subclass of strategic games, they form an ideal framework to analyse interactive situations that incorporate aspects of both imperfect as well as incomplete information games. Please consider the following examples.

Example 1 (A West Side Story)**.** *Tony and Maria (or was Romeo and Juliet? or Shanbo and Jingtai?) are in love with each other. But they have not declared their love to each other yet. This is risky business, as they are both uncertain about the feelings of the other one. Surely, given that they both love each other, their objective is to get to know that. But they consider it possible that the other person does not love them, in which case they might prefer not to declare their love. Their personalities are different in that respect. What Tony wants to know, depends on how his feelings (being in love / not being in love) relate to the other person's: if they match, he wants the other person to know, otherwise, he doesn't. Whereas what Maria wants to know only depends on the other person's feelings: if the other one is in love, she wants the other one to know her true feelings and otherwise not.*

Given their state of mind and their personalities, should they declare their love to each other?

Let Tony be player 1 and Maria be player 2, and let p_1 represent 'Tony is in love' and p_2 represent 'Maria is in love'. Propositions p_1 and p_2 are both true and remain so forever after. They cannot be controlled. The objectives (goals) denoted γ_i for player i, and where $K_i p_j$ means 'player i knows p_j', are:

$$\begin{array}{rcllll} \gamma_1 = \gamma_2 & = & p_1 \wedge p_2 & \to & K_1 p_2 \wedge K_2 p_1 & \wedge \\ & & p_1 \wedge \neg p_2 & \to & K_1 \neg p_2 \wedge \neg K_2 p_1 & \wedge \\ & & \neg p_1 \wedge p_2 & \to & \neg K_1 p_2 \wedge \neg K_2 \neg p_1 & \wedge \\ & & \neg p_1 \wedge \neg p_2 & \to & \neg K_1 \neg p_2 \wedge K_2 \neg p_1 & \end{array}$$

They each have two strategies: declare their feelings (revealing the value of p_i), or not. We succinctly explain that in this game, whatever the facts are, there is a strategy profile in which both players win by satisfying their goal formulas, but that they can never know that they win. It is not so clear whether there (hopefully) is an equilibrium strategy profile allowing them to declare their love to each other. As p_1 and p_2 are true, it is an equilibrium when they both announce that, as $K_1 p_2 \wedge K_2 p_1$ is then true and they both win (the other three strategy profiles result in both losing, including another equilibrium namely when both don't declare). But Tony considers it possible that $\neg p_2$ in which case announcing p_1, and Maria's behaviour being equal, goal $K_1 \neg p_2 \wedge \neg K_2 p_1$ will fail. In that case he should have kept his mouth shut to have them win. Given the uncertainty over the game he has to reason *about not two but* **four** *strategies for Maria: depending on whether she is in love or not, whether she would show her feelings or not. What he will do given this information set of two indistinguishable outcomes, also depends on his risk aversity. If he's an optimist, he might still go for it. But if he's a pessimist, maybe better not. Maria's considerations are not dissimilar, but recall that she has a different personality (the goals are a different function from their value of p_i, in other words, permuting all occurrences of 1 and 2 in the goal results in a different goal). Example 11 on page 321 will reveal it all.*

Example 2 (A game of pennies that do not match)**.** *Consider two players Odd and Even both having a penny. They also both have a dice cup wherein they put their penny, shake the cup, and then put it on the table and watch privately whether their penny is heads or tails. Now they decide whether to inform the other player of the result, or not. If they both do or if they both don't, Even wins. So, Even wins if either 2 players know or 0 players know, so that one might therefore say that their state of knowledge is 'even'. Otherwise, Odd wins. For the outcome it only matters whether they* **know** *that the penny is heads or tails, it does not matter whether it* **is** *heads or tails. What should they do?*

We let Odd be player 1 and Even be player 2, and we let p_1 represent 'Odd's penny is heads', whereas p_2 stands for 'Even's penny is heads'. The goals are therefore (where Kw_ip_j abbreviates $K_ip_j \vee K_i\neg p_j$ and means 'player i knows whether p_j):

$$\begin{array}{rcl} \gamma_1 & = & Kw_1p_2 \leftrightarrow \neg Kw_2p_1 \\ \gamma_2 & = & Kw_1p_2 \leftrightarrow Kw_2p_1 \end{array}$$

On first sight it seems quite straightforward what they should do, as the outcome does not depend on the valuation of p_1 and p_2. If Odd and Even both announce the result of their throw with the penny, Odd would then have done better not to make that announcement. But if that were to have happened, Even would have done better not to announce either. And so on. There is no equilibrium. Or is there? Yes, there is. And it is pure. Example 12 on page 321 will reveal it all.

Our framework of Boolean observation games clearly builds upon (Ågotnes et al., 2013b; Herzig et al., 2016) but a main difference is that these are complete information games whereas ours are incomplete information games. Thus we have very different strategies. Players do not control the *values* of variables, but they control whether they *reveal* the fixed values of variables that only they can observe. In that respect our framework also builds on the public announcement games of (Ågotnes & van Ditmarsch, 2011; Ågotnes, van Benthem, van Ditmarsch, & Minica, 2011). They only allow strategies that are public announcements wherein the same information is revealed to all players. However, they permit announcing any epistemic formula, not merely propositional variables. A more detailed comparison with all these approaches is only possible after having given our framework in detail and is therefore in a later Section 3.3.

Our games are strictly qualitative and thus abstract from truly Bayesian approaches (Harsanyi, 1968) with probabilities. To determine equilibrium we compare information sets, called 'expected outcomes'. As the expected outcome may not be a value and the relation may not be a total order, our work is therefore in ordinal game theory (Durieu, Haller, Quérou, & Solal, 2008; Cruz & Simaan, 2000; Amor, Fargier, & Sabbadin, 2017).

Our Contributions. We analyse structural and computational properties of Boolean observation games. We define Boolean observation games as incomplete and imperfect information games, a novel perspective in Boolean games. We show that Boolean observation games form a fragment of strategic games with compact representation. We determine equilibria based on four different profitable deviations from information sets, namely defined as: the worst outcome is better, the best outcome is better, the expected outcome is better, and the outcome without uncertainty is better (the outcome is better even if all information sets are singleton, so that the game is one of complete information). We also provide existence results for such equilibria, which highly depends on what is considered a profitable deviation. We identify various fragments of Boolean observation games including one where your goal may be to keep others ignorant but not to keep yourself ignorant, the self-positive goals, and another one where the goals are 'knowing whether formulas' of which the realization does not depend on the valuation. The latter we call knowing-whether games. We provide an embedding of the standard Boolean games into a fragment of the knowing-whether games, and we also provide an embedding of the knowing-whether games into the Boolean games. Employing these embeddings we show that the knowing-whether games correspond to Boolean games in terms of existence of equilibrium outcomes. We also provide complexity results for the natural questions of verification and checking of emptiness of equilibrium outcomes in Boolean observation games, for most of the profitable deviations considered, and stretching the results as much as possible to also include fragments with ignorance goals. An overview of these complexity results is found in the conclusions in Table 3.

Overview of Contents. Section 2 provides technical preliminaries needed to define Boolean observation games, that are then defined in the subsequent Section 3, of which the final Subsection 3.3 compares our proposal to other epistemic Boolean games. Section 4 presents the correspondence between Boolean games and Boolean observation games. Section 5 provides various results for the existence of Nash equilibria and Section 6 contains the results on the computational complexity of determining whether a strategy profile is an equilibrium, and whether equilibria exist.

2. Preliminaries

In this section we introduce an auxiliary notion that is a complete information strategic game, which is played with strategies that are epistemic actions, that has epistemic formulas as goals and for which we propose a greatly simplified epistemic logic, and where outcomes are the truth values of those goals. Boolean observation games, that are incomplete information strategic games with more complex strategies and outcomes, will then be defined in the next section. The logic is simple in order to ensure a compact representation allowing to obtain complexity results comparable to those for Boolean games. Some logical details that are fairly elementary but that might distract from the game theoretical content that is our focus, are deferred to the Appendix.

2.1 Strategies Consisting of Players Revealing Observations

Let $N = \{1, \dots, n\}$ be a finite set of *players* i and P a finite set of (propositional) *variables* such that $(P_i)_{i \in N}$ defines a partition of P. The set P_i is the set of variables p_i *observed* by player i (that is, of which player i, and only player i, observes the value). A *valuation* is a subset $v \subseteq P$, where $p_i \in v$ means that p_i is true and $p_i \notin v$ means that p_i is false. The set $\mathcal{P}(P)$ of all valuations is denoted V.

A *strategy* for player i is a function $s_i : N \to \mathcal{P}(P_i)$ that assigns to each player j the set $s_i(j) \subseteq P_i$ of variables that player i reveals (announces) to player j. We require that $s_i(i) = P_i$. Let S_i denote the set of all strategies of player i. A *strategy profile* is a member s of $S = S_1 \times \cdots \times S_n$. The set $P_i(s) = \{p_j \in P \mid p_j \in s_j(i)\}$ consists of the variables revealed to i in s. As $s_i(i) \subseteq P_i$, $P_i \subseteq P_i(s)$. For $i \in N$, we denote the n-tuple s as (s_i, s_{-i}) where s_{-i} represents the $(n-1)$-tuple of the strategies of other players. Strategy $s_i^\emptyset$ is such that for all $j \in N$ with $j \neq i$, $s_i(j) = \emptyset$. This means that player i does not reveal anything to anyone. Strategy $s_i^\forall$ is such that for all $i, j \in N$, $s_i(j) = P_i$. This means that player i reveals everything and to everyone.

Given $i \in N$ and strategy profile s, the *observation relation* $\sim_i^s$ on V is defined as, for $v, w \in V$:

$$v \sim_i^s w \quad \text{iff} \quad v \cap P_i(s) = w \cap P_i(s).$$

Observation relation $\sim_i^s$ encodes the informative effect of s. For $\sim_i^{s^\emptyset}$ we write $\sim_i$. This is the *initial observation relation.* We further note that $P_i(s^\forall) = V$ for any player i, so that $\sim_i^{s^\forall}$ is the the identity relation $=$. A $\sim_i^s$ equivalence class, defined as $[v]_i^s := \{w \in V \mid w \sim_i^s v\}$ (where $[v]_i^{s^\emptyset}$ is denoted $[v]_i$), is also called an *information set* (of player i given valuation v and observation relation $\sim_i^s$).

Inasfar as strategies consist of each player i selecting a subset P_i' of her variables P_i, these are like the strategies in Boolean games. However we interpret this differently: player i does not make the variables in P_i' true, but reveals the value of the variables in P_i' according to a fixed valuation v. Another departure (or generalization) from Boolean games is that different variables are revealed to different agents. This is because we felt that more interesting game theoretical results could be obtained for such a generalization, and because more interesting communicative scenarios could then be treated with the game theoretical machinery.

Example 3. *We assume a strategy profile to take place in some instantaneous, synchronous, fashion, such as, when $s_1(2) = \{p_1, q_1\}$, $s_1(3) = \{p_1, q_1\}$, and $s_1(4) = \emptyset$, player 1 informing player 2 and player 3 that p_1 and q_1 are both true, and such that player 4 observes this without being party to the message content (for example, 1 whispering to 2 and 3). In other words, player 4 knows that player 1 informs player 2 and player 3* **whether** *p_1 and q_1, but player 4 remains uncertain of the value of p_1 and q_1, so does not know that 1 informs 2 and 3* **that** *p_1 and q_2.*[1]

Now consider s_1' that is like s_1 except that $s_1(4) = \{p_1, q_1\}$ as well. This is the public announcement of p_1 and q_1 by player 1 to all players.

What if for example $s_1''(2) = \{p_1\}$ but $s_1''(3) = \{p_1, p_2\}$? And what about s_2, s_3 and s_4? This cannot be done instantaneously. But we can ensure independence: all players commit

1. In a different semantics for strategies, less informative to the players, each player only learns what variables have been revealed by others to herself, and what variables she reveals to others. Applied to Example 3, this would also leave player 4 uncertain whether player 1 has informed player 2 and player 3. See Appendix A.3.

to their s_i before they execute it, and not after they see what variables are revealed to other players before it is their turn to reveal. Instead of whispering we can all have prepared closed envelopes adressed to all others on which is written for example, 'from player 1 to player 2: contains the truth about p_1 and p_2'. All envelopes are collected blindly and then put on the table for all to see and are then handed out.

Such forms of communication are known as semi-public announcement (van Ditmarsch, 2002), see Appendix A.2 on dynamic epistemic logic for details.

2.2 Goals that are Epistemic Formulas

The *language of epistemic logic* is defined as follows, where $i \in N$ and $p_i \in P_i$.

$$L^K \ni \qquad \alpha := p_i \mid \neg\alpha \mid \alpha \vee \alpha \mid K_i\alpha$$

Here, $\neg$ is negation, $\vee$ is disjunction, and $K_i\varphi$ stands for 'player i knows φ.' Other propositional connectives are defined by abbreviation, and also $\hat{K}_i\alpha := \neg K_i \neg\alpha$ (player i considers α possible), and $Kw_i\alpha := K_i\alpha \vee K_i\neg\alpha$ (player i knows whether α). The members of L^K are *goals* and may as well be called, suiting our purposes *formulas*.

The following fragments of L^K also play a role, where $i, j \in N$ and $p_i \in P_i$.

$$\begin{array}{ll}
L^B \ni & \alpha := p_i \mid \neg\alpha \mid \alpha \vee \alpha \\
L^K_{\mathsf{nnf}} \ni & \alpha := p_i \mid \neg p_i \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid K_i\alpha \mid \hat{K}_i\alpha \\
L^+ \ni & \alpha := p_i \mid \neg p_i \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid K_i\alpha \\
L^{Kw} \ni & \alpha := Kw_j p_i \mid \neg\alpha \mid \alpha \vee \alpha \\
L^{Kw}_{\mathsf{nnf}} \ni & \alpha := Kw_j p_i \mid \neg Kw_j p_i \mid \alpha \vee \alpha \mid \alpha \wedge \alpha
\end{array}$$

The language L^B of the *Booleans* is the fragment of L^K without K_i modalities. In the language L^{Kw} of *knowing whether formulas* (*Kw formulas*) the constructs $Kw_j p_i$ play the role of propositional variables. The fragments L^K_{nnf} and L^{Kw}_{nnf} are those of the *negation normal form* (nnf) of respectively L^K and L^{Kw}, where the language L^+ of the *positive formulas* is the fragment of L^K_{nnf} without $\hat{K}_i$ modalities (corresponding to a universal fragment of first-order logic). Note that L^{Kw} and L^{Kw}_{nnf} are really propositional languages, not modal languages. A goal is *guarded* if it has shape $\gamma_i = K_i\alpha$.

Apart from the above fragments yet another fragment plays a role in our contribution, namely that of the self-positive goals. The *self-positive goal formulas* are defined as $L^{\mathsf{self}+} := \bigcup_{j \in N} L^{j+}$, where each L^{j+} is given by the following BNF, wherein $i, k \in N$ and $k \neq j$.

$$L^{j+} \ni \quad \alpha_j \quad ::= \quad p_i \mid \neg p_i \mid \alpha_j \wedge \alpha_j \mid \alpha_j \vee \alpha_j \mid K_j\alpha_j \mid K_k\alpha_j \mid \hat{K}_k\alpha_j$$

Here, α_j is the goal for player j. Note that L^+ is a fragment of L^{j+}, namely the fragment where all occurrences of K_k are positive, and that L^{j+} is a fragment of L^K_{nnf}, namely the fragment wherein all occurrences of K_j are positive. In a self-positive goal for agent j, j's objective is to (get to) know others' variables **and others' knowledge and ignorance**, although other players may either know or remain ignorant of j's knowledge. This implies that j's goal also cannot be for others to know j's ignorance. A larger number of communicative scenarios seem to have self-positive goals than merely positive goals: it seems fairly typical that you wish others to remain ignorant even when you are only interested in obtaining (factual) knowledge.

The inductively defined semantics of L^K formulas are relative to a valuation v and a strategy profile s, where $i \in N$ and $p_i \in P_i$.

$$\begin{array}{lll} v, s \models p_i & \text{iff} & p_i \in v \\ v, s \models \neg\alpha & \text{iff} & v, s \not\models \alpha \\ v, s \models \alpha_1 \vee \alpha_2 & \text{iff} & v, s \models \alpha_1 \text{ or } v, s \models \alpha_2 \\ v, s \models K_i\alpha & \text{iff} & w, s \models \alpha \text{ for all } w \text{ such that } v \sim_i^s w \end{array}$$

For $v, s^\emptyset \models \alpha$ we write $v \models \alpha$. This is a bit sneaky: by definition this represents what players know after the strategy profile is executed wherein nobody reveals anything, but we can therefore just as well let it stand for what players initially know, before anything has been revealed.

We let $s \models \alpha$ denote "for all $v \in V$, $v, s \models \alpha$," and $\models \alpha$ denote "for all $s \in S$, $s \models \alpha$". In our semantics, $K_i p_i$, $K_i \neg p_i$, and $Kw_i p_i$ are always true (equivalent to the trivial assertion $\top$). We therefore informally assume that they do not occur in goal formulas.

We note that our epistemic semantics is not the usual one for the epistemic language, interpreted on arbitrary Kripke models, but a greatly simplified epistemic semantics dedicated to reason about strategies that are joint revelations of observed variables. We do not even use the word 'model'. And we do not allow announcements (revelations) of other information than variables. In Appendix A.2 we show how (valuation, strategy) pairs induce multi-agent Kripke models. All these simplifications are in order to obtain a smooth comparison with Boolean games and with comparable complexities, unlike the higher complexities common in multi-agent epistemic reasoning.

We continue with some elementary properties of this simple logical semantics, in the form of propositions.

Proposition 4. *Each formula in L^K is equivalent to a formula in L^K_{nnf}. Similarly, each formula in L^{Kw} is equivalent to a formula in L^{Kw}_{nnf}.*

Proof. This well-known result in modal logic for L^K is shown by induction on formula structure, using the equivalences $\neg\neg\alpha \leftrightarrow \alpha$, $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$ and $\neg K_i\alpha \leftrightarrow \hat{K}_i\neg\alpha$. For L^{Kw}, as this is essentially a propositional and not a modal language, we only need to use the first equivalence. $\square$

Proposition 5. *For all $\varphi \in L^{Kw}$, valuations v, and strategy profiles s: $v, s \models \varphi$ iff $s \models \varphi$.*

The basic but lengthy proof of this proposition is in Appendix A.1. Prop. 5 says in other words, that if $v, s \models \varphi$ for *some* $v \in V$, then $v, s \models \varphi$ for *all* $v \in V$.

Proposition 6. *For any $\alpha \in L^{Kw}$, $\models \alpha \leftrightarrow K_i\alpha$.*

Proof. Let valuation v and strategy profile s be given.

Assume $v, s \models \alpha$. Then from Prop. 5 it follows that for all $w \in V$, $w, s \models \alpha$. Therefore, in particular, $w, s \models \alpha$ for all $w \sim_i^s v$, which is by definition $v, s \models K_i\alpha$.

Now assume $v, s \models K_i\alpha$. From $v \sim_i^s v$ and the semantics of knowledge now directly follows $v, s \models \alpha$.

As v and s were arbitrary, we have shown $\models \alpha \leftrightarrow K_i\alpha$. $\square$

As a consequence each formula in the fragment $Kw_i p_j \mid \neg\alpha \mid \alpha \vee \alpha \mid K_i\alpha$ is equivalent to a formula in L^{Kw}, in other words, knowledge can then be eliminated. This explains why we defined the fragment L^{Kw} without an inductive clause for knowledge.

Knowledge cannot generally be eliminated from a language with knowing whether variables. For example, Anne (1) may know whether Bill (2) passed the exam (p_2), but Bill may be uncertain *whether* she knows. So we have $Kw_1p_2 \wedge \neg K_2 Kw_1 p_2$. Props. 5 and 6 (and the subsequent Prop. 7) do not hold for knowing whether fragments on arbitrary Kripke models.

Proposition 7. *For all $i, j, k \in N$: $\models Kw_i Kw_j p_k$.*

Proof. Formula $Kw_i Kw_j p_k$ is by definition equivalent to $K_i Kw_j p_k \vee K_i \neg Kw_j p_k$. From Prop. 6 it follows that this is equivalent to $Kw_j p_k \vee \neg Kw_j p_k$ which is a tautology. □

Therefore, in our very simple epistemic logic it is common knowledge whether a player knows a variable. This reflects the dynamics of revealing variables. Suppose all players hold cards named p_1, q_1, p_2, ... on the back side and the value 0 or 1 on the front (face) side. You may not know *that* your neighbour has shown to your other neighbour that the value of the card p_1 is 1 (true). But you know *whether* your neighbour has shown card p_1 to your other neighbour. You saw it happen.

2.3 Pointed Boolean Observation Games

A *pointed Boolean observation game* (pointed observation game) is a pair (G, v), denoted $G(v)$, where $v \in V$ and where G is a triple $(N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$, where all $\gamma_i \in L^K$. The players' strategies in the pointed observation game are the strategies $s_i \in S_i$. The players' goals in the pointed observation game are the $\gamma_i \in L^K$. Given $i \in N$, the *outcome function* $u_i : V \times S \to \{0, 1\}$ of a pointed observation game is defined as:

$$u_i(v, s) = 1 \text{ if } v, s \models \gamma_i \text{ and } u_i(v, s) = 0 \text{ if } v, s \not\models \gamma_i.$$

A strategy profile s is a *Nash equilibrium* of $G(v)$ iff for all $i \in N$ and $s'_i \in S_i$ we have $u_i(v, s) \geq u_i(v, (s'_i, s_{-i}))$. That is, no player has a *profitable deviation* from s in $G(v)$, which would therefore be a $s'_i \in S_i$ such that $u_i(v, s) < u_i(v, (s'_i, s_{-i}))$. Observe that a player can only make a profitable deviation from s if her goal is not satisfied in s. Let $NE(G(v))$ denote the set of Nash equilibria of $G(v)$.

The pointed observation game is an auxiliary notion, matching the intuition that after revealing variables a player wins when her goal has become true. The game is one of complete information because the valuation is known to you, the reader. But the valuation is typically not known to the players. It already uses the parameters of the Boolean observation game that we will now define in the next section.

Example 8. *We recall Example 1. We summarily describe a pointed Boolean observation game and its equilibria, where a fuller development is only given in Example 11. Consider pointed game $G(v)$ with $G = (\{1, 2\}, (\{p_1\}, \{p_2\}), (\{\gamma_1, \gamma_2\})$ where γ_1, γ_2 are as in Example 1, and where valuation $v = \{p_1, p_2\}$ (both are in love). The strategies are to reveal nothing or to reveal all, that is: $s_1^\emptyset$, $s_1^\forall$, $s_2^\emptyset$, and $s_2^\forall$.*

The strategy profile $(s_1^\forall, s_2^\forall)$ is an equilibrium strategy profile of the pointed game $G(v)$, with outcome 1 *for both players. This is the only way to make $K_1 p_2 \wedge K_2 p_1$ true. However, both players not announcing their variable is also an equilibrium with outcomes* 0*.*

The pointed game $G(w)$ for valuation $w = \{p_1\}$ (only Tony is in love) has equilibrium $(s_1^\emptyset, s_2^\forall)$. We now need to make $K_1 \neg p_2 \wedge \neg K_2 p_1$ true. (Another equilibrium $(s_1^\forall, s_2^\emptyset)$ is when both get outcome 0*.)*

3. Defining Boolean Observation Games

We will now define the *Boolean observation game*. A Boolean observation game is an incomplete information strategic form game with **uniform strategies** (uniform functions from valuations to strategies) and **expected outcomes** (information sets of outcomes), whereas the auxiliary notion of a pointed observation game is a complete information strategic form game with **strategies** and with (Boolean-valued) **outcomes**.

3.1 Boolean Observation Games

This section contains the crucial game theoretical notions of our contribution.

Boolean Observation Game. A *Boolean observation game* (or *observation game*) is a triple $G = (N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$, where all $\gamma_i \in L^K$. Formula γ_i is the *goal* (*objective*) of player i. It is played with *uniform strategies* and the payoffs are *expected outcomes*. Both will now be defined.

Uniform Strategy. A *uniform strategy* for player $i \in N$ is a function $\mathbf{s}_i : V \to S_i$ such that for all v, w with $v \sim_i w$, $\mathbf{s}_i(v) = \mathbf{s}_i(w)$. It is *globally uniform* iff for all $v, w \in V$, $\mathbf{s}_i(v) = \mathbf{s}_i(w)$.

So, uniform means the same for all indistinguishable valuations, which is different from globally uniform, which means the same for all valuations. Let $\mathbf{S}_i$ denote the set of uniform strategies of player i, and $\mathbf{S} = \mathbf{S}_1 \times \cdots \times \mathbf{S}_n$ the set of *uniform strategy profiles*. Let $\mathbf{S}_i^g$ and $\mathbf{S}^g$ denote the set of globally uniform strategies of player i and the set of globally uniform strategy profiles respectively. Given a valuation v, a uniform strategy profile $\mathbf{s}$ determines a strategy profile $\mathbf{s}(v) = (\mathbf{s}_1(v), \dots, \mathbf{s}_n(v))$. Note that $(\mathbf{s}(v)_i, \mathbf{s}(v)_{-i}) = (\mathbf{s}_i, \mathbf{s}_{-i})(v)$. For $i \in N$ and $s_i \in S_i$, we define $\dot{s_i} \in \mathbf{S}_i^g$ as: for all $v \in V$, $\dot{s_i}(v) = s_i$. Similarly for $s \in S$ we define $\dot{s} \in \mathbf{S}^g$ as the globally uniform strategy profile such that for all $v \in V$, $\dot{s}(v) = s$. It follows from the definition that every globally uniform strategy profile $\mathbf{s} \in \mathbf{S}^g$ is of the form $\dot{s}$ for some strategy profile $s \in S$.

Expected Outcome. Given $i \in N$, the *expected outcome function* is a function $\mathbf{u}_i : V \times \mathbf{S} \to \{0,1\}^*$ that is uniform in V, and defined as $\mathbf{u}_i(v, \mathbf{s}) = (u_i(w, \mathbf{s}(w)))_{w \sim_i v}$. So, *expected outcome* $\mathbf{u}_i(v, \mathbf{s})$ is a vector of outcomes $u_i(w, \mathbf{s}(w))$ for each valuation w in the information set of player i. In our setting where outcomes are 0 (lose) or 1 (win) this vector is a bitstring.

As far as nomenclature is concerned, we are putting the reader on the wrong foot, as a uniform strategy is not a kind of strategy (as defined in the previous section), nor is expected outcome a kind of (binary valued) outcome. However, we are in good company: an artificial brain is not a brain, and a cable car is not a car. So we hope the reader will allow us this slight abuse of language.

Outcome Relation and Nash Equilibrium. To define the notion of an equilibrium in observation games, we need to first define a comparison relation between uniform strategy profiles. Note that unlike in classical strategic games, the expected outcome function in observation games generates a vector of outcomes. Therefore, there is no canonical definition for the comparison relation. We define an *outcome relation* $>$ over vectors of outcomes and write $\mathbf{u}_i(v, \mathbf{s}) > \mathbf{u}_i(v, \mathbf{s}')$ for "player i prefers $\mathbf{s}_i$ over $\mathbf{s}'_i$ in the information set containing v"; we also say that $\mathbf{s}_i$ is a profitable deviation from $\mathbf{s}'_i$.

This outcome relation may not be a total order. We therefore prefer not to use notation $\leq$ to compare the bitstrings that are outcome sets, as it is ambiguous whether $x \leq y$ means ($x < y$ or $x = y$) or $x \not> y$ (and even when defined as either one or the other, it seems unkind to the reader).

Given an outcome relation $>$, a uniform strategy profile is a Nash equilibrium if no player has a profitable deviation.

> A uniform strategy profile $\mathbf{s}$ is a *Nash equilibrium* of G iff for all $i \in N$, $\mathbf{s}'_i \in \mathbf{S}_i$ and $v \in V$, we have that $\mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i})) \not> \mathbf{u}_i(v, \mathbf{s})$.

Given an observation game G, $NE(G)$ denotes its Nash equilibria, and among those $NE^g(G)$ denotes the globally uniform Nash equilibria.

Also, a uniform strategy $\mathbf{s}_i \in \mathbf{S}_i$ is *dominant* if for all $\mathbf{s} \in \mathbf{S}$ with $\mathbf{s} = (\mathbf{s}_i, \mathbf{s}_{-i})$, for all $\mathbf{s}'_i \in \mathbf{S}_i$, and for all v, $\mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i})) \not> \mathbf{u}_i(v, \mathbf{s})$.[2]

Four Outcome Relations. It remains to define the outcome relation. We propose four.

$$\begin{array}{llll}
\text{optimist}: & \mathbf{u}_i(v, \mathbf{s}) >^{\mathsf{opt}} \mathbf{u}_i(v, \mathbf{s}') & \text{iff} & \max \mathbf{u}_i(v, \mathbf{s}) > \max \mathbf{u}_i(v, \mathbf{s}') \\
\text{pessimist}: & \mathbf{u}_i(v, \mathbf{s}) >^{\mathsf{pess}} \mathbf{u}_i(v, \mathbf{s}') & \text{iff} & \min \mathbf{u}_i(v, \mathbf{s}) > \min \mathbf{u}_i(v, \mathbf{s}') \\
\text{realist}: & \mathbf{u}_i(v, \mathbf{s}) >^{\mathsf{real}} \mathbf{u}_i(v, \mathbf{s}') & \text{iff} & \Sigma \mathbf{u}_i(v, \mathbf{s}) > \Sigma \mathbf{u}_i(v, \mathbf{s}') \\
\text{maximal}: & \mathbf{u}_i(v, \mathbf{s}) >^{\mathsf{max}} \mathbf{u}_i(v, \mathbf{s}') & \text{iff} & u_i(w, \mathbf{s}(w)) > u_i(w, \mathbf{s}'(w)) \text{ for some } w \sim_i v
\end{array}$$

The optimist, pessimist and realist outcome relations are (strict) total orders, as it suffices to assign a number to the information set constituting an expected outcome. The maximal outcome relation is not a total order.

We let $NE_{\mathsf{pess}}(G)$, $NE_{\mathsf{opt}}(G)$, $NE_{\mathsf{real}}(G)$, and $NE_{\mathsf{max}}(G)$ denote the Nash equilibria under the pessimist, optimist, realist and maximal outcome relation, respectively. The optimist, pessimist and realist outcome relations are (strict) total orders, as it suffices to assign a number to the information set constituting an expected outcome. The maximal outcome relation is not a total order as illustrated in Example 9. However, defining this relation is useful since $NE_{\mathsf{max}}(G)$ has an interesting interpretation which we discuss below.

Example 9. *Let us consider an abstract example where a player has to choose between expected outcomes (bitstrings) 00, 10, 01, 11. We then get (where clustered bitstrings means equally preferred):*

$$\begin{array}{l}
\{01, 10, 11\} >^{\mathsf{opt}} 00 \\
11 >^{\mathsf{pess}} \{00, 01, 10\} \\
11 >^{\mathsf{real}} \{01, 10\} >^{\mathsf{real}} 00 \\
ij >^{\mathsf{max}} kl \;\; \textit{iff} \;\; i > k \textit{ or } j > l
\end{array}$$

2. This is weak dominance of the kind 'always at least as good' where we emphasize that we do not define it as 'always at least as good and sometimes strictly better', which is also common in game theory.

The $>^{\mathsf{max}}$ relation is neither antisymmetric nor transitive. For instance, in Example 9 we have that $10 >^{\mathsf{max}} 01$ but also $01 >^{\mathsf{max}} 10$, and it is not transitive because $01 >^{\mathsf{max}} 10 >^{\mathsf{max}} 01$ however $01 \not>^{\mathsf{max}} 01$. Thus the $>^{\mathsf{max}}$ relation is neither a total order nor a preorder. However, it has a maximum and a minimum: the expected outcome where the player always wins is preferred over all other expected outcomes, and the expected outcome where the player always loses is less preferred than all other expected outcomes.

The maximal outcome relation also satisfies the important property that all outcomes can be compared and therefore, the notion of a Nash equilibrium is well-defined. If $\mathbf{u}_i(v, \mathbf{s}) \neq \mathbf{u}_i(v, \mathbf{s}')$, then $\mathbf{u}_i(v, \mathbf{s}) >^{\mathsf{max}} \mathbf{u}_i(v, \mathbf{s}')$ or $\mathbf{u}_i(v, \mathbf{s}') >^{\mathsf{max}} \mathbf{u}_i(v, \mathbf{s})$. The disjunction in the consequent is inclusive, both may hold (we recall that $10 >^{\mathsf{max}} 01$ as well as $01 >^{\mathsf{max}} 10$, as in Example 9). To require this property is common in ordinal game theory (Durieu et al., 2008).

The outcome relations that we have proposed are qualitative versions of well-known criteria in decision theory and Bayesian reasoning. None assume a probability distribution, however, all assume a strictly positive probability for each valuation.

- The optimist outcome relation is the max instantiation (as there is only one maximal value) of the *minimax regret* decision criterion (Savage, 1951). With respect to the highest possible outcome in the information set, a lower possible outcome in the information set (which can only be 0 instead of 1) would cause regret if this were to happen.

- The pessimist outcome relation is the min instantiation of the *maximin* or *Wald* decision criterion (Wald, 1945). We then choose the information set with the best worst outcome. This outcome relation has been used to model uncertainty in voting (with similar considerations involving Nash equilibria and dominance) in (Conitzer, Walsh, & Xia, 2011; van Ditmarsch, Lang, & Saffidine, 2013; Bakhtiari, van Ditmarsch, & Saffidine, 2019).

- The realist outcome relation is a qualititative version (lack of justification to rule out any outcome) of a random decision in Bayesian terms, also known as the insufficient reason or Laplace decision criterion, or as the principle of indifference (Keynes, 1921, Chapter IV).

  Instead of taking the sum of the outcomes in the information set we could of course have normalized this so it adds up to 1, suggesting an even distribution of probability mass. Such scaling is irrelevant for our purposes of determining Nash equilibria and dominance, wherein we only need to compare outcomes. That comparison relation remains the same.

  This outcome relation was used in (Ågotnes & van Ditmarsch, 2011; Ågotnes et al., 2011) to determine equilibria of similar incomplete information games, but where more complex formulas than mere variables could be 'revealed' (however, they could only be publicly announced). An issue for the realist outcome relation is whether bisimilar game states (that therefore satisfy the same goals for all players) should be counted once or twice.[3] On the one hand, if two game states are bisimilar this is

3. Personal communication by Martin Otto.

justification / sufficient reason to rule out one of them, according to Laplace. On the other hand these bisimilar game states might have originated from playing strategy profiles (executing epistemic actions) in initial game states that were non-bisimilar. It is relevant to observe this as we note that this phenomenon **cannot** occur in our simpler setting involving observation relations.

- The notion of Nash equilibrium for the maximal outcome relation has an interesting interpretation. A maximal Nash equilibrium is a uniform strategy profile where no player has a profitable deviation even if the player has complete information about the game. There is an equivalent formulation of maximal Nash equilibrium as a qualitative version of *ex-post equilibrium* (Apt, 2011), which we show in Proposition 10.

Various of the above outcome relations have also been considered in (Parikh, Tasdemir, & Witzel, 2013).

Proposition 10. *A uniform strategy profile* $\mathbf{s}$ *is a maximal Nash equilibrium for* G *iff for all* $v \in V$, $\mathbf{s}(v)$ *is a Nash equilibrium for* $G(v)$.

Proof. Suppose $\mathbf{s} \notin NE_{\mathsf{max}}(G)$. Then there exist $v \in V$, $i \in N$, and $\mathbf{s}'_i \in \mathbf{S}_i$ such that $\mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i})) >^{\mathsf{max}} \mathbf{u}_i(v, \mathbf{s})$. It follows that there is $w \sim_i v$ such that $u_i(w, (\mathbf{s}'_i, \mathbf{s}_{-i})(w)) > u_i(w, \mathbf{s}(w))$, so $u_i(w, (\mathbf{s}'_i, \mathbf{s}_{-i})(w)) = 1$ and $u_i(w, \mathbf{s}(w)) = 0$. Therefore $\mathbf{s}(w) \notin NE(G(w))$.

Suppose $\mathbf{s}(w) \notin NE(G(w))$ for some valuation w. Then there exist $i \in N$, $s'_i \in S_i$ such that $u_i(w, (s'_i, \mathbf{s}_{-i}(w)) > u_i(w, \mathbf{s}(w))$. Let $\mathbf{s}'_i \in \mathbf{S}_i$ be the uniform strategy such that for all $v \sim_i w$, $\mathbf{s}'_i(v) = s'_i$ (so in particular, $\mathbf{s}'_i(w) = s'_i$), and for all $v \not\sim_i w$, $\mathbf{s}'_i(v) = \mathbf{s}_i(v)$. By the maximal relation, from $u_i(w, (\mathbf{s}'_i, \mathbf{s}_{-i})(w)) = u_i(w, (s'_i, \mathbf{s}(w)_{-i}) > u_i(w, \mathbf{s}(w))$ it follows that $\mathbf{u}_i(w, (\mathbf{s}'_i, \mathbf{s}_{-i})) >^{\mathsf{max}} \mathbf{u}_i(w, \mathbf{s})$. Therefore $\mathbf{s} \notin NE_{\mathsf{max}}(G)$. □

In the remaining sections we focus on optimist, pessimist and maximal Nash equilibrium and not on realist Nash equilibrium. We use the operational definition of maximal Nash equilibrium given by the correspondence in Proposition 10. It is easy to see that a maximal Nash equilibrium is also an optimist, pessimist, and realist Nash equilibrium. In that sense the maximal outcome relation is the *strongest* notion, resulting in the smallest number of equilibria for a game (if any).

3.2 Various Classes of Observation Games

With all the technical tools now at our disposal, very different observation games are of specific interest. We can distinguish them by which outcome relation they employ, and independently by the shape of the epistemic goals. Concerning goals it is useful to distinguish the following.

- In *two-player zero-sum games*, $\gamma_i = \neg\gamma_j$ and in *two-player symmetric games* $\gamma_i = \gamma_j$, where $|N| = 2$, $i \neq j$, and $i, j \in N$. In *cooperative games* $\bigwedge_{i \in N} \gamma_i$ is consistent. Example 11 below is symmetric, and Example 12 is zero-sum (and therefore not consistent). Communicative scenarios obeying the Gricean cooperative principle are clearly consistent observation games (and might still be considered games inasfar as people want to outdo each other in being informative). Whereas security protocol settings with eavesdroppers (consider observing an SMS code that you were sent to confirm a bank transfer) tend to be zero-sum; that is, a generalization of zero-sum: the objectives of the principals are the opposite of those of the eavesdroppers.

We do not have theoretical results for zero-sum or symmetric games.

- In *knowing-whether observation games* (*knowing-whether games, Kw games*) all goals γ_i are in L^{Kw}. In knowing-whether games the outcome does not depend on the valuation. Whether some $Kw_i p_j$ is true only depends on player j revealing p_j to player i, and does not depend on the valuation, because the truth of p_j does not depend on the value of p_j.

  Section 4 is entirely devoted to knowing-whether games, and Section 5 contains results on existence of equilibria. They relate well to the usual Boolean game. Not surprisingly, as the outcome does not depend on the valuation, they also score better on the computational complexity of determining whether a uniform strategy profile is a Nash equilibrium, or whether Nash equilibria exist, than other classes of observation game. That will be investigated in Section 6.3.

- In observation games with *guarded* goals (of shape $\gamma_i = K_i\alpha$, see Subsection 2.2) the players know whether they have achieved their objective after playing the game. Whereas in games where the goals are not guarded they may not and need an oracle to inform them of the outcome (such as, when standing in front of an ATM teller, the bank's interface informing them). If goals are guarded, Nash equilibria always exists for the optimist and the pessimist outcome relation, as formulated and shown in Theorem 28 in Section 5.

- In games where all γ_i are *positive formulas* (in the fragment L^+ where negations do not bind K_j modalities, see Subsection 2.2), a player's goal is never to remain ignorant of a fact, or even for other players to remain ignorant. Under such circumstances revealing all you know is a dominant strategy. This is therefore rather restricted.

- More interesting than positive goals are the observation games with *self-positive goals* wherein your goal is to become less ignorant yourself although you may wish to keep other players ignorant (see again Subsection 2.2). We provide a result for self-positive goals in Corollary 29 in Section 5.

For all these, results on existence of equilibria and complexity also depend on which outcome relation is used, as already occasionally listed above.

Last but not least one can consider *iterated observation games* with temporal eventuality goals, where players successively reveal more and more of their observed variables. An example are (successive) *question-answer games* wherein the strategic aspect is what variable(s) to ask another player(s) to reveal, which seems of particular interest for strategic negotiation (if you give me this, I'll give you that, and so on). All these come with specific questions on compact representation and existence of equilibria.

We defer the investigation of iterated games and question-answer games to future research. In this work we focus on knowing-whether games and their relation to Boolean games, on the existence of equilibria for various outcome relations (where the realist outcome relation plays no role), and on complexity results for some of our variations.

We now continue with some detailed examples.

Example 11. *Recall Example 1 (page 309) and Example 8. We now give full details.*

Consider the observation game G where $N = \{1, 2\}$, $P_1 = \{p_1\}$, $P_2 = \{p_2\}$ and the (symmetric) goals:

$$\begin{array}{rclcll} \gamma_1 = \gamma_2 & = & p_1 \wedge p_2 & \to & K_1 p_2 \wedge K_2 p_1 & \wedge \\ & & p_1 \wedge \neg p_2 & \to & K_1 \neg p_2 \wedge \neg K_2 p_1 & \wedge \\ & & \neg p_1 \wedge p_2 & \to & \neg K_1 p_2 \wedge \neg K_2 \neg p_1 & \wedge \\ & & \neg p_1 \wedge \neg p_2 & \to & \neg K_1 \neg p_2 \wedge K_2 \neg p_1 & \end{array}$$

As there are only two players and each player observes a single variable the strategies are to reveal nothing or to reveal all, that is: $s_1^{\emptyset}$, $s_1^{\forall}$, $s_2^{\emptyset}$, and $s_2^{\forall}$.

For each valuation v the pointed observation game $G(v)$ has an equilibrium where both players get outcome 1. For example, if p_1 and p_2 are both true, then both players revealing (announcing) that is an equilibrium with outcome 1 for both players. However, both players not announcing their variable is also an equilibrium with outcome 0.

Let us now determine equilibria for G, with uniform strategies instead of strategies, and let us consider the different outcome relations.

- **pessimist.** *Player 1 cannot distinguish between the valuations $\{p_1, p_2\}$ and $\{p_1\}$. Thus, for all $\mathbf{s} \in \mathbf{S}$ and for all $v \in V$, $\min \mathbf{u}_1(v, \mathbf{s}) = 0$. The situation is symmetric for player 2. Therefore, for all $\mathbf{s} \in \mathbf{S}$, $\mathbf{s} \in NE_{\mathsf{pess}}(G)$.*
- **optimist.** *Similarly, for all $\mathbf{s} \in \mathbf{S}$, for all $v \in V$ and for all $i \in \{1, 2\}$, $\max \mathbf{u}_i(v, \mathbf{s}) = 1$. Therefore, for all $\mathbf{s} \in \mathbf{S}$, $\mathbf{s} \in NE_{\mathsf{opt}}(G)$.*
- **realist.** *In this example, whatever the valuation v, $\Sigma \mathbf{u}_i(v, \mathbf{s}) = \max \mathbf{u}_i(v, \mathbf{s}) = 1$, so that also, for all $\mathbf{s} \in \mathbf{S}$, $\mathbf{s} \in NE_{\mathsf{real}}(G)$.*
- **maximal.** *$NE_{\mathsf{max}}(G) = \emptyset$. There are no maximal Nash equilibria, because every information set for both players always contains a win and a lose, so if they were to know the real valuation, one of those is not an equilibrium for the pointed game.*

Possibly, the equilibria depend on what we called the 'personalities of Tony and Maria', that is on the shape of the goals? We considered two different personalities that therefore allow four different goals, but (the reader can check that) none makes a difference for any of the four outcome relations, as the property that each information set contains a win and a lose persists throughout such transformations. The best is always win, and the worst is always lose. However for other 'personalities' (for lack of a better term) this need not be, for example, change $\neg K_1 p_2 \wedge \neg K_2 \neg p_1$ in the third conjunct into $\neg K_1 p_2 \wedge K_2 \neg p_1$ (we removed one negation symbol). It is now dominant for player 1 to announce the value of p_1 in the information set wherein p_1 is false.

Example 12. *Recall Example 2 on page 310 about the pennies that do not match. We can now model this as a knowing-whether Boolean observation game G where $N = \{1, 2\}$, $P = P_1 \cup P_2$ with $P_1 = \{p_1\}$, $P_2 = \{p_2\}$, and*

$$\begin{array}{rcrcr} \gamma_1 & = & Kw_1 p_2 & \leftrightarrow & \neg Kw_2 p_1 \\ \gamma_2 & = & Kw_1 p_2 & \leftrightarrow & Kw_2 p_1 \end{array}$$

For $i = 1, 2$, player i has strategy $s_i^{\emptyset}$ wherein she reveals nothing ('hide p_i') and strategy $s_i^{\forall}$ wherein she reveals the value of p_i. Irrespective of the valuation, in the strategy profiles $(s_1^{\emptyset}, s_2^{\emptyset})$ and $(s_1^{\forall}, s_2^{\forall})$, player 1 has a profitable deviation in the corresponding pointed observation game. Similarly, in $(s_1^{\emptyset}, s_2^{\forall})$ and $(s_1^{\forall}, s_2^{\emptyset})$, player 2 has a profitable deviation. Thus it can be verified that $NE_{\mathsf{max}}(G) = \emptyset$. Also, within the set of all globally uniform strategy profiles, G does not have a Nash equilibrium for the pessimist and optimist outcome relation.

However, this game has a Nash equilibrium with uniform strategies that are not globally uniform, for the pessimist and for the optimist outcome relation. Consider the uniform strategy profile $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$ where in $\mathbf{s}_1$, player 1 reveals p_1 to 2 when p_1 is true and hides p_1 from 2 when p_1 is false, and in $\mathbf{s}_2$, player 2 reveals p_2 to 1 when p_2 is true and hides p_2 from 1 when p_2 is false. Thus $\min \mathbf{u}_1(v, \mathbf{s}) = \min \mathbf{u}_2(v, \mathbf{s}) = 0$. It can then be verified that no player has profitable deviation from $\mathbf{s}$ and therefore $\mathbf{s} \in NE_{\mathsf{pess}}(G)$. Similarly, it can be noted that $\max \mathbf{u}_1(v, \mathbf{s}) = \max \mathbf{u}_2(v, \mathbf{s}) = 1$. Therefore, $\mathbf{s} \in NE_{\mathsf{opt}}(G)$.

3.3 Comparison to Related Work

In this section we compare in more detail our epistemic Boolean games to the two prior proposals in the literature known to us (Ågotnes et al., 2013b; Herzig et al., 2016), that were already mentioned in the introductory section. We recall that these are imperfect information games (they feature epistemic objectives), however they are not incomplete information games. We also succinctly compare our proposal to an incomplete information game, that is however not a Boolean game (Ågotnes & van Ditmarsch, 2011).

Comparison to 'Boolean Games with Epistemic Goals'. In 'Boolean games with epistemic goals' (Ågotnes et al., 2013b) the set of variables P is partitioned into $|N| = n$ mutually disjoint subsets of variables P_i, for $i \in N$, such that the variables in P_i can only be controlled by player i. This is as usual in Boolean games, and therefore the strategies played are also as usual, so that a strategy profile is a valuation of all variables. However, the goals are different from the usual in Boolean games, and like ours: these are not merely Boolean goals (formulas in the language L^B) whose satisfaction depends on this valuation but these are epistemic goals (the language L^K) whose satisfaction depends on what the players know about this valuation. This is where another parameter of their games comes into play: apart from a set P_i of 'controlled variables' each player i also has a finite 'visibility set' consisting of Boolean formulas, that is, some finite subset of the language L^B: those are the propositions whose value that player can observe of the outcome valuation. Such Booleans may involve variables not controlled by player i but by other players j. Already, this seems to beg some questions on logical closure, for example if $p \wedge q$ is in the visibility set but neither variable p nor variable q (where we note that the epistemic goal formulas have the usual compositional semantics, so $K_i(p \wedge q)$ is true if and only if $K_i p$ and $K_i q$ are true). However, a special case is when the visibility set consists of variables only, which (Ågotnes et al., 2013b) call *atomic* games, and this suffices for a comparison with our results. The visibility set determines what is known by the players and thus which epistemic goals are satisfied in a valuation. Because the players altogether control the value of all variables the game is not one of incomplete information (strategies do not depend on an unknown initial valuation) although it is one of imperfect information (over the outcome valuation). The authors then determine that model checking goal formulas is PSPACE-complete and that the existence of

Nash equilibria is in PSPACE, although they do not show a lower bound. They also provide an interesting embedding of their epistemic Boolean games into the standard Boolean games by observing that an epistemic goal corresponds to an exponentially larger Boolean goal that is the disjunction of all valuations over which the epistemic goal is uncertain. For example, in some given game, $K_i p$ may abbreviate $(p \wedge \neg q) \vee (p \wedge q)$. This is therefore a rather different embedding from our embedding of knowing-whether Boolean observation games into Boolean games wherein the goals remain the same but the set of variables (and thus valuations) is larger: we recall that a Kw game G for variables p_i is transformed into a Boolean game B_G for variables $Kw_j p_i$: the knowing-whether formulas are now considered atomic propositions. The goals remain the same in our approach, because knowing-whether goals are Booleans in the language wherein $Kw_j p_i$ are atomic propositions.

Comparison to 'Epistemic Boolean Games Based on a Logic of Visibility and Control'. The authors of this work (Herzig et al., 2016) propose a very expressive logical language and semantics for players controlling the value of propositional variables or observing the value of propositional variables. They also axiomatize this logic. They then use the logic to formalize game theoretical primitives, in particular the existence of equilibria, in an epistemic extension of Boolean games. This formalization allows them to determine the complexity of these games. The problems of determining whether a profile is Nash equilibrium as well as the existence of Nash equilibrium are both in PSPACE.

Their language extension includes knowledge, common knowledge, and for control or observation of propositional variables they propose additional propositional variables. We not only have, for example, a variable p, but also $S_i p$, for 'player i observes the value of p' and $C_i p$ for 'player i controls the value of p'. But also variables like $C_j S_i p$, for 'player j controls whether player i observes p', and so on for any stack of C_j or S_i predicates. The interest of these complex propositional variables is that they induce relational Kripke models or can be used to formalize strategies in Boolean games.

In the epistemic Boolean games of (Herzig et al., 2016) the **strategies** assign values to variables that are stacks of S_i binding some atom p (so without any C_i or K_i), as in $S_i S_j p$, saying that i can see whether j can see the value of p, whereas the **goals** are epistemic formulas in the language for such atoms p (so without S_i or C_i), as in $K_i p \wedge \neg K_j p$.[4] One might say that their epistemic Boolean games essentially remain *Boolean* games, because the players still only control the value of variables, but this is only by a (quite smart) stretch of the modeling imagination, because their Boolean variables hard-code arbitrarily complex higher-order multi-agent observations. However, these are not games of incomplete information.

The focus of (Herzig et al., 2016) is the axiomatization of their logic of visibility and control (it also contains program modalities with primitive operations assigning values to variables). The game-theorical contribution is mainly 'proof of concept'.

4. As S_i stacks are arbitrarily long, there is an infinite set of such atoms to consider. However, the partition among players controlling variables is of a finite subset only of that infinite set. This permits $S_i p$ but not p to be in that finite subset, which would rule out to determine the value of a goal $K_i p$ (as no player gives a value to p, that is, no player controls p). In their accompanying examples, the finite subset jointly controlled by all players is always subformula closed. This therefore seems an omitted requirement.

Comparison to 'Public Announcement Games'. 'Public announcement games' which are studied in (Ågotnes & van Ditmarsch, 2011) and the related 'question answer games' (Ågotnes et al., 2011) also present incomplete games of imperfect information. Expected outcomes are compared with the realist outcome relation. The value of variables is not controlled in any way in (Ågotnes & van Ditmarsch, 2011; Ågotnes et al., 2011), the valuations are fixed. Public announcement games are not Boolean games, because the players' strategies are revelations of any formula, not merely of Booleans. Of course one could consider a class of public announcement games wherein the strategies are restricted to announcing propositional variables only. However, we recall that public announcements are revelations of the same information to all players simultaneously, so this is not as general as our proposal.

4. Knowing-Whether Boolean Observation Games

In this section we show a correspondence between knowing-whether Boolean observation games (Kw games) and Boolean games. We provide polynomial time reductions that convert a Boolean game to a Kw game and vice-versa.

We first recall the definition of Boolean game. We then show that every Boolean game defines a Kw Boolean observation game, and that every Kw Boolean observation game defines a Boolean game. These embeddings are different, the first is not the converse of the second.

We further show a utility preserving equivalence between strategies in Boolean games and equivalence classes of globally uniform strategies in Kw games (Lemmas 18, 22). As a consequence, we prove a correspondence between the existence of Nash equilibria in Boolean games and the existence of maximal Nash equilibria in Kw games, and for both reductions (Theorems 19, 24).

We finally show that there always exists a pessimist equilibrium for 2-player Kw games, but not for Kw games in general: we give an 8-player Kw game without a Nash equilibrium (where we do not know if such games exist for between 3 and 7 players).

Recall that for any $v \in V$ and $\alpha \in L^{Kw}$, $v, s \models \alpha$ iff $s \models \alpha$ (Prop. 5). This justifies writing $u_i(s)$ for the outcome $u_i(v, s)$ of a pointed Kw game. Now consider a globally uniform strategy profile $\dot{s} \in \mathbf{S}^g$. As $\mathbf{u}_i(v, \dot{s}) = u_i(v, \dot{s}(v)) = u_i(v, s)$, this justifies writing $\mathbf{u}_i(\dot{s})$ for the expected outcome of a such a Kw game.

4.1 Boolean Games

Boolean games have the same parameters as Boolean observation games but simpler strategies. A Boolean game is denoted B to distinguish it from a Boolean observation game G.

A *Boolean game* is a tuple $B = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ where all $\gamma_i \in L^B$ (all goals are Boolean). For $i \in N$, a *strategy* v_i for player i is a *(local) valuation* $v_i \subseteq P_i$, where, slightly abusing notation, we identify a *strategy profile* $v = (v_1, \ldots, v_n)$ with a valuation $v = (v_1 \cup \ldots \cup v_n) \in V$. For Boolean games, the *outcome function* is denoted u^B to distinguish it from the outcome function u of pointed Boolean observation games. We define $u_i^B(v) = 1$ if $v \models \gamma_i$ and $u_i^B(v) = 0$ if $v \not\models \gamma_i$. Equilibrium is as for pointed observation games: a strategy profile $v \in V$ is a *Nash equilibrium* in B if for all $i \in N$ and $v_i' \subseteq P_i$, $u_i^B(v) \geq u_i^B(v_i', v_{-i})$. Given B, its Nash equilibria are denoted $NE(B)$.

Let us emphasize the difference between Boolean games and Boolean observation games. In Boolean observation games, as in Boolean games, a player i selects a subset v_i of her local variables P_i. However, in Boolean observation games this subset may be a different subset $s_i(j) \subseteq P_i$ for each other player j. Also, in Boolean games, excuting strategy v_i means that the $p_i \in v_i$ *become true* whereas the $p_i \in P_i \setminus v_i$ become false. Whereas in Boolean observation games, executing strategy with component $s_i(j)$ means that the $p_i \in s_i(j)$, that already have an observed truth value, are *revealed* (to j).

4.2 Boolean Games to Knowing-Whether Games

We construct a *Kw* game denoted G_B from a Boolean game B as follows. Let $B = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$. Then $G_B := (N, (P_i)_{i \in N}, (\beta_i)_{i \in N})$ where each $\beta_i := \lambda(\gamma_i)$ is defined as follows. Let $i^+ := i + 1$ for $i = 1, \ldots, n-1$ and $n^+ := 1$. Then $\lambda : L^B \to L^{Kw}$ is inductively defined as: for all i, $p_i \in P_i$, $\lambda(p_i) := Kw_{i^+} p_i$, and (trivially) $\lambda(\neg\alpha) := \neg\lambda(\alpha)$ and $\lambda(\alpha_1 \vee \alpha_2) := \lambda(\alpha_1) \vee \lambda(\alpha_2)$. Note that B and G_B are defined for the same players and variables.

Given a strategy profile $v \in V$ for B, we define globally uniform strategy profile $\dot{s^v} \in \mathbf{S}^g$ for G_B such that for all $i \in N$ and $p_i \in P_i$: $p_i \in s_i^v(i^+)$ if $p_i \in v$; $s_i^v(i) = P_i$; and for all $j \in N$ with $j \neq i, i^+$, $s_i^v(j) = \emptyset$. Note that for all valuations w, including v, $\dot{s^v}(w) = s^v$. Notation s^v is therefore not to be confused with notation $\mathbf{s}(v)$ for uniform profiles $\mathbf{s}$. In this section we will show how the v strategy for B corresponds to the $\dot{s^v}$ strategy for G_B.

Note that for all $i \in N$, we have $|\beta_i| = \mathcal{O}(|\gamma_i|)$ where $|\beta_i|$ and $|\gamma_i|$ denote the size of (number of symbols in) β_i and γ_i respectively. Thus given B, the associated *Kw* game G_B can be constructed in polynomial time.

Example 13. *We illustrate how to construct a Kw game G_B from a Boolean game B. (We will not analyze the equilibria of the game, if any.) Consider*

$$B = (\{1, 2, 3\}, (\{p_1\}, \{p_2\}, \{p_3\}), (p_1 \leftrightarrow p_3, p_3 \to p_1, \neg p_1 \to p_2))$$

Then G_B has the same variables p_1, q_1, p_2, p_3 but different goals, namely $Kw_2 p_1 \leftrightarrow Kw_1 p_3$ for player 1, $Kw_1 p_3 \to Kw_2 p_1$ for player 2, and $\neg Kw_2 p_1 \to Kw_3 p_2$ for player 3.

In the Boolean game, for player 1 to obtain her goal $\gamma_1 = p_1 \leftrightarrow p_3$, player 1 has to make p_1 true, it does not matter whether player 2 makes p_2 true or false, and player 3 has to make p_3 true. In the Kw game, in order to achieve the goal $\beta_1 = Kw_2 p_1 \leftrightarrow Kw_1 p_3$, player 1 has to reveal p_1 to player 2, it does not matter whether player 2 reveals p_2 to player 3, and player 3 has to reveal p_3 to player 1, and all three do this independently from the valuation. Because in fact, for example, player 1 reveals the value *of p_1 to player 2, but what the value is does not matter as the outcome of a Kw game is independent from the valuation. So they players execute globally uniform strategies. More precisely, in order to ensure β_1 globally uniform strategy $\dot{s}$ is required such that $s_1(2) = \{p_1\}$, $s_3(2)$ does not matter, and $s_3(1) = \{p_3\}$.*

Lemma 14. *Let G_B be the Kw game associated with the Boolean game B. For all $i \in N$, for all $w \in V$, $s^w \models \lambda(\gamma_i)$ iff $w \models \gamma_i$.*

Proof. This is shown by induction where only the base case is not-trivial. For that, we have that $s^w \models Kw_{i^+}p_i$ iff $w \models p_i$ by definition of the embedding. $\square$

Therefore, for any $i \in N$ and $v \in V$: $\mathbf{u}_i(v, \dot{s^w}) = u_i(v, s^w) = u_i(s^w) = u_i^B(w)$. This correspondence allows us to relate Nash equilibria in B to Nash equilibria in G_B. The result uses an interesting property of NE_{max} in Kw games (this property does not hold for, e.g., the pessimist outcome relation).

Lemma 15. *Let Kw game G be given. Let $\mathbf{s} \in NE_{\mathsf{max}}(G)$ and $v \in V$. Let $s = \mathbf{s}(v)$. Then $\dot{s} \in NE_{\mathsf{max}}(G)$.*

Proof. Consider an arbitrary $v \in V$ and let $s = \mathbf{s}(v)$. Suppose that $\mathbf{s} \in NE_{\mathsf{max}}(G)$, we claim that the globally uniform strategy profile $\dot{s} \in NE_{\mathsf{max}}(G)$. Suppose not. Then there exists $i \in N$, $w \in V$ and $s'_i \in S_i$ such that $u_i(w, (s'_i, \dot{s}_{-i}(w))) > u_i(w, \dot{s}(w))$. This implies that $w, (s'_i, \dot{s}_{-i}(w)) \models \gamma_i$ and $w, (\dot{s}(w)) \not\models \gamma_i$. Since for all $j \in N$, $\dot{s}_j(w) = \mathbf{s}_j(v)$, we have $w, (s'_i, \mathbf{s}_{-i}(v)) \models \gamma_i$. Since $\gamma_i \in L^{Kw}$, we have $v, (s'_i, \mathbf{s}_{-i}(v)) \models \gamma_i$. Also, since $w, \dot{s}(w) \not\models \gamma_i$, we have $v, \dot{s}(w) \not\models \gamma_i$ and by definition of $\dot{s}$, we have $v, \mathbf{s}(v) \not\models \gamma_i$. Therefore, $\mathbf{s} \notin NE_{\mathsf{max}}(G)$ which gives the required contradiction. $\square$

Corollary 16. *Let Kw game G be given. If $NE_{\mathsf{max}}(G) \neq \emptyset$ then $NE^g_{\mathsf{max}}(G) \neq \emptyset$.*

An Equivalence Relation over Global Strategy Profiles. Recall that every $\mathbf{s} \in \mathbf{S}^g$ is of the form $\dot{s}$ where $s \in S(G_B)$. We define an equivalence relation over $\mathbf{S}^g$ in G_B as follows. For $i \in N$, $\dot{s}_i \equiv_i \dot{t}_i$ iff $s_i(i^+) = t_i(i^+)$. For $\dot{s}, \dot{t} \in \mathbf{S}^g$, we define $\dot{s} \equiv \dot{t}$ iff for all $i \in N$, $\dot{s}_i \equiv_i \dot{t}_i$. Let $\mathbf{S}^g/\!\equiv$ denote the set of equivalence classes and $[\mathbf{s}]$ denote the equivalence class containing $\mathbf{s} \in \mathbf{S}^g$.

Lemma 17. *Given $\mathbf{s} \in \mathbf{S}^g$, for all $\mathbf{t} \in [\mathbf{s}]$, for all $i \in N$, for all $v \in V$, $u_i(v, \mathbf{s}(v)) = u_i(v, \mathbf{t}(v))$.*

Proof. Let $\mathbf{s} = \dot{s}$ and $\mathbf{t} = \dot{t}$. For all $i \in N$, since $\mathbf{t} \in [\mathbf{s}]$, we have $s_i(i^+) = t_i(i^+)$. By induction of the structure of γ_i, we can prove the following: for all $v \in V$, for all $i \in N$ and for all $\gamma_i \in L^{Kw}$, we have $v, (\dot{s}(v)) \models \gamma_i$ iff $v, (\dot{t}(v)) \models \gamma_i$. This implies that for all $i \in N$, for all $v \in V$, $u_i(v, \dot{s}(v)) = u_i(v, \dot{t}(v))$. $\square$

An Outcome Preserving Bijection. We now show that there is an outcome preserving bijection χ between strategy profiles in B and equivalence classes in $\mathbf{S}^g/\!\equiv$. For a Boolean game B, and $v \in V$, $\chi(v) = [\dot{s^v}]$.

Lemma 18. *Given a Boolean game B, the function $\chi : V \to \mathbf{S}^g/\!\equiv$ is a bijection.*

Proof. Given $\dot{s} \in \mathbf{S}^g$, consider $v \in V$ defined as follows: for all $i \in N$ and $p_i \in P_i$, $p_i \in v$ iff $p_i \in s_i(i^+)$. We then have $\chi(v) = [\dot{s}]$ and therefore χ is onto. For $v, w \in V$ such that $v \neq w$, there exists $i \in N$, there exists $p_i \in P_i$ such that $p_i \in v$ and $p_i \notin w$. Thus, for $\chi(v) = [\dot{s}]$ and $\chi(w) = [\dot{t}]$, we have $\dot{s} \not\equiv_i \dot{t}$, which implies that $\dot{s} \not\equiv \dot{t}$. Therefore, χ is a bijection. $\square$

Consequently, we can prove a correspondence between Nash equilibria existence.

Theorem 19. *Let B be a Boolean game. Then $NE_{\mathsf{max}}(G_B) \neq \emptyset$ iff $NE(B) \neq \emptyset$.*

Proof. ($\Leftarrow$) We argue that if $w \in NE(B)$ then $\dot{s^w} \in NE_{\mathsf{max}}(G_B)$. Suppose not, then there exists $i \in N$, $v \in V$ and $t_i \in S_i$ such that $u_i(v, (t_i, \dot{s^w}_{-i}(v))) > u_i(v, \dot{s^w}(v))$. Let $w' = \chi^{-1}([\dot{t_i}, \mathbf{s}_{-i}])$ From Lemmas 14, 17 and 18 it follows that $u_i^B(w') = u_i(v, (\dot{t_i}, \mathbf{s}_{-i})(v)) > u_i(v, \dot{s^w}(v)) = u_i^B(w)$ for all $v \in V$. Therefore $w \notin NE(B)$ which is a contradiction.

($\Rightarrow$) Suppose $NE_{\mathsf{max}}(G_B) \neq \emptyset$. By Lemma 15 there exists $\mathbf{s} \in \mathbf{S}^g$ such that $\mathbf{s} \in NE_{\mathsf{max}}(G_B)$. Let $w = \chi^{-1}([\mathbf{s}])$. We claim that $w \in NE(B)$. Suppose not, then there exists $i \in N$ and w_i' such that $u_i^B(w_i', w_{-i}) > u_i^B(w)$. Let $w' = (w_i', w_{-i})$. Note that by definition $w \neq w'$. From Lemma 14 it follows that for all v we have $u_i^B(w') = u_i(v, \dot{s^{w'}}(v))$. From Lemmas 14, 17 and 18 it follows that $u_i^B(w) = u_i(v, \mathbf{s}(v))$ for all $v \in V$. Therefore for all $v \in V$, $u_i(v, \dot{s^{w'}}(v)) = u_i^B(w') > u_i^B(w) = u_i(v, \mathbf{s}(v))$ which contradicts the fact that $\mathbf{s} \in NE_{\mathsf{max}}(G_B)$. □

Other ways to get a Kw Game from a Boolean Game. Let us imagine our n players sitting round a table numbered in clockwise fashion. In the embedding $\lambda : L^B \to L^{Kw}$ with basic clause

$$\lambda(p_i) := Kw_{i^+}p_i,$$

every player i reveals the value of her observed variable p_i to her left neighbour (while other players observe her doing that). There are many other embeddings that would serve equally well to obtain our results. For example, every player i could reveal her variable to her **right** neighbour. This would be a λ' with basic clause

$$\lambda'(p_i) := Kw_{i^-}p_i$$

where i^- is $i-1$ except for $1^- := n$. A more interesting embedding would be every player publicly announcing p_i to all other players. We then have a λ'' for which

$$\lambda''(p_i) := \bigwedge_{j \in N} Kw_j p_i.$$

4.3 Knowing-Whether Games to Boolean Games

A Kw Game to a Boolean Game. We now construct a Boolean game denoted B_G from a knowing-whether Boolean observation game G. Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$. Assume that the goals γ_i do not contain trivial constituents $Kw_i p_i$.[5] Then $B_G := (N, (Q_i)_{i \in N}, (\gamma_i)_{i \in N})$ where for all $i \in N$, $Q_i = \{Kw_j p_i \mid p_i \in P_i, i \neq j\}$. We view $Kw_j p_i$, for each i and j with $i \neq j$, as atomic propositions in B_G. Let $Q = \bigcup_{i \in N} Q_i$.

Observation. Both G and B_G are defined over the same set of players and goal formulas. The number of variables in B_G for each $i \in N$ is $|Q_i| = (n-1)|P_i|$. Thus given G, the associated Boolean game B_G can be constructed in polynomial time. Also, note that $B_{G_B} \neq B$ and $G_{B_G} \neq G$, the constructions are unrelated. Let us give an example of that.

5. As such $Kw_i p_i$ are always true, this would otherwise cause a problem in the translation, because the players in the constructed Boolean game would then be able to control the value of propositional variables $Kw_i p_i$ (which is undesirable), unlike the players in the given Kw game. One can also address this formally, without assumptions, with an inductively defined translation mapping $Kw_i p_i$ to $\top$.

Example 20. *As an illustration to construct a Boolean game from a Kw game, let us take the Kw game just constructed in Example 13. We recall that*

$$G_B = (\{1,2,3\}, (\{p_1, q_1\}, \{p_2\}, \{p_3\}), (\gamma_1, \gamma_2, \gamma_3))$$

where

- $\gamma_1 = Kw_2 p_1 \leftrightarrow Kw_1 p_3$,
- $\gamma_2 = Kw_1 p_3 \rightarrow Kw_2 p_1$,
- $\gamma_3 = \neg Kw_2 p_1 \rightarrow Kw_3 p_2$.

The Boolean game B_{G_B} constructed from that has the same goals but has more variables, namely $Kw_i p_j$ for all $i, j \in N$ with $i \neq j$ and for all $p_j \in P_i$, that is: $Kw_1 p_2$, $Kw_1 p_3$, $Kw_2 p_1$, $Kw_2 q_1$, $Kw_2 p_3$, $Kw_3 p_1$, $Kw_3 q_1$, $Kw_3 p_2$. Therefore B_{G_B} has more variables than B. The constructions are not each other's converse. However, in order to realize the goals of B_{G_B} the players only need to assign a value to variables $Kw_i p_j$ occurring in the goal formulas, so with respect to playing **this game** *the extra variables do not play a role. After replacing $Kw_2 p_1$ by p_1, etcetera for other variables ocurring in goal formulas, we recover the original Boolean game for, however, far more variables that are not used in goals.*

Let $W = \mathcal{P}(Q)$ be the set of valuations over Q. We define a function $\eta : \mathbf{S}^g \rightarrow W$ and argue that it is a bijection which is outcome equivalent. Given $\dot{s} \in \mathbf{S}^g$, define $w = \eta(\dot{s})$ as follows: for $i \in N$, $Kw_j p_i \in \eta(\dot{s})_i$ iff $p_i \in s_i(j)$.

Lemma 21. *Let B_G be the Boolean game associated with the Kw game G. For all $i \in N$, for all $s \in S$ and for all γ_i, $s \models \gamma_i$ iff $\eta(\dot{s}) \models \gamma_i$.*

Proof. This is shown by induction using as the base case that $s \models Kw_j p_i$, iff $\eta(\dot{s}) \models Kw_j p_i$ The other cases are trivial. □

It therefore also follows, similarly to the above, that $\mathbf{u}_i(\dot{s}) = u_i(s) = u_i^B(w^s)$.

Lemma 22. *Given a Kw game G, let B_G be the associated Boolean game. The function $\eta : \mathbf{S}^g \rightarrow W$ is a bijection.*

Proof. For an arbitrary $w \in W$, consider $\dot{s} \in \mathbf{S}^g$ defined as follows. For all $i \in N$, and for all $p_i \in P_i$, $p_i \in s_i(j)$ iff $i = j$ or $Kw_j p_i \in w_i$. By definition, $\eta(\dot{s}) = w$ and thus η is onto.

Consider $\dot{s}, \dot{t} \in \mathbf{S}^g$ where $\dot{s} \neq \dot{t}$. Then there exists $i, j \in N$ with $i \neq j$ and there exists $p_i \in P_i$ such that $p_i \in s_i(j)$ and $p_i \notin t_i(j)$. This implies that $Kw_j p_i \in \eta(\dot{s})_i$ and $Kw_j p_i \notin \eta(\dot{t})_i$. Therefore η is a bijection. □

Non-global Uniform Strategies as Mixed Strategies for Boolean Games. We allow ourselves a little detour. We can straightforwardly adjust the function η mapping globally uniform strategy profiles of the *Kw* game to valuations that are strategy profiles of the Boolean game, to a function mapping arbitrary uniform strategy profiles of the *Kw* game G to *mixed strategy profiles* of the Boolean game B_G. We simply define the 'revised η' on the level of strategy profiles $s \in S$. Given a uniform strategy profile $\mathbf{s} \in \mathbf{S}$, for each $s \in S$ such that $\mathbf{s}(v) = s$ for some $v \in V$, we let $\pi(s) := |\{v \in V \mid \mathbf{s}(v) = s\}|/2^{|P|}$. Note that

$2^{|P|} = |V|$. So $\pi(s)$ is the probability that a valuation is mapped in s, given $\mathbf{s}$. We can now define a mixed strategy profile $w^{\mathbf{s}}$ of the Boolean game B_G as the one executing each $s \in S$ with probability $\pi(s)$. We defer the investigation of embeddings into a mixed equilibrium to future research and for now restrict ourselves to a relevant example, finally closing the loop with matching pennies.

Example 23. *Once more we recall Example 2 on page 310 about the pennies that do not match, already further developed in Example 12, wherein it was shown that this game does not have a Nash equilibrium with globally uniform strategies for the pessimist and optimist outcome relation, but has a Nash equilibrium with uniform strategies that are not globally uniform: the uniform strategy profile $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$ where in $\mathbf{s}_1$, player 1 reveals p_1 to 2 when p_1 is true and hides p_1 from 2 when p_1 is false, and in $\mathbf{s}_2$ player 2 reveals p_2 to 1 when p_2 is true and hides p_2 from 1 when p_2 is false.*

When translating this game into a Boolean game, we can now observe that this uniform strategy $\mathbf{s}$ becomes a mixed strategy $\eta(\mathbf{s})$ where player 1 randomly chooses between revealing or hiding her propositional variable Kw_2p_1 and where player 2 randomly chooses between revealing or hiding his propositional variable Kw_1p_2. To realize that this is indeed random it is important to observe that in Example 2 the probability of observing p_1 or $\neg p_1$ was determined by Odd flipping its penny before privately watching the outcome under the dice cup, and similarly for p_2 and Even. So after all, for those who may have wondered, there was a reason for setting up the experiment just like that.

We continue with a relevant result for the maximal outcome relation.

Theorem 24. *Let G be a Kw game. Then $NE(B_G) \neq \emptyset$ iff $NE_{\mathsf{max}}(G) \neq \emptyset$.*

Proof. ($\Leftarrow$) Suppose $NE_{\mathsf{max}}(G_B) \neq \emptyset$. By Lemma 15 there exists $\mathbf{s} \in \mathbf{S}^g$ such that $\mathbf{s} \in NE_{\mathsf{max}}(G_B)$. Let $w = \eta(\mathbf{s})$, we argue that w is a Nash equilibrium. Suppose not, there exists $i \in N$, there exists w'_i such that $u_i^B(w'_i, w_{-i}) > u_i^B(w_i, w_{-i})$. Consider the globally uniform strategy profile $\mathbf{t} = \eta^{-1}(w'_i, w_{-i})$ (this is well defined by Lemma 22). From Lemmas 21 and 22 it follows that $u_i(v, \mathbf{t}(v)) = u_i^B(w'_i, w_{-i}) > u_i^B(w) = u_i(v, \mathbf{s}(v))$. This implies that $\mathbf{s} \notin NE_{\mathsf{max}}(G_B)$ which is a contradiction.

($\Rightarrow$) Suppose $w \in NE(B_G)$. Let $\mathbf{s} = \eta^{-1}(w)$, we claim that $\mathbf{s} \in NE_{\mathsf{max}}(G)$. Suppose not, then there exists $i \in N$, $v \in V$ and $t_i \in S_i$ such that $u_i(v, (t_i, \mathbf{s}_{-i}(v))) > u_i(v, \mathbf{s}(v))$. Let $w' = \eta(\dot{t}_i, \mathbf{s}_{-i})$. From Lemmas 21 and 22 it follows that $u_i^B(w') = u_i(v, (\dot{t}_i, \mathbf{s}_{-i})(v)) > u_i(v, \mathbf{s}(v)) = u_i^B(w)$. This implies that $w \notin NE(B_G)$ which is a contradiction. □

5. Existence of Nash Equilibrium

In this section we focus on the question of existence of Nash equilibria for observation games and identify various subclasses in which a Nash equilibrium is guaranteed to exist.

5.1 Existence of Pessimist Nash Equilibrium in Knowing-Whether Games

Example 12 shows that in the *Kw* fragment a maximal Nash equilibrium is not guaranteed to exist even for two-player games. It is natural to ask if a similar observation holds for pessimist Nash equilibrium. We first show that for two-player *Kw* games, a pessimist Nash equilibrium always exists (Proposition 25). However, for general Kw games, existence is not guaranteed. Example 26 gives an 8-player Kw game without a Nash equilibrium.

Proposition 25. *All two-player Kw games have a pessimist Nash equilibrium.*

Proof. We construct a uniform strategy profile $(\mathbf{s}_1^*, \mathbf{s}_2^*)$ as follows. For $i \in \{1,2\}$, let $\bar{\imath}$ denote the player such that $\bar{\imath} \neq i$. If $i \in \{1,2\}$ has a uniform strategy s_i that is dominant, then set $\mathbf{s}_i^*(v) = s_i$ for all $v \in V$ and let $\mathbf{s}_{\bar{\imath}}^*$ be the best response to $\mathbf{s}_i^*$. It can be verified that $\mathbf{s}^*$ as defined above is a Nash equilibrium.

If neither player has a uniform strategy that is dominant, then we have the following

- For all $s_1 \in S_1$, there exists $s_2 \in S_2$ such that $v, (s_1, s_2) \not\models \gamma_1$ for all $v \in V$.
- For all $t_2 \in S_2$, there exists $t_1 \in S_1$ such that $v, (t_1, t_2) \not\models \gamma_2$ for all $v \in V$.

For two player games there is a bijection between the set of strategies S_i and the set of (local) valuations V_i (one can think of a strategy as deciding for each proposition whether to reveal to the other player). Therefore for each t_1 and s_2 as described above, we can set $\mathbf{s}_1^*(v^1) = t_1$ and $\mathbf{s}_2^*(v^2) = s_2$ appropriately for some v^1 and v^2.

To see that $(\mathbf{s}_1^*, \mathbf{s}_2^*)$ is a Nash equilibrium, note that for all $i \in \{1,2\}$ and for all $v \in V$, $\min \mathbf{u}_i(v, \mathbf{s}^*) = 0$. Also, for all $v \in V$, $\min \mathbf{u}_i(v, (\mathbf{s}_i', \mathbf{s}_{-i}^*)) = 0$ due to the above condition. $\square$

However, for more than two players a Kw game need not have a pessimist Nash equilibrium. We present a counterexample for eight players.

Example 26. *Consider the observation game G where $N = \{1,2,\ldots,8\}$ and $P_i = \{p_i\}$ for $i \in N$. Player 8 acts as an "observer" whose goal $\gamma_8 = \top$. To specify the goals of the other players we use the following formulas.*

$$\begin{array}{ll} A = Kw_3 p_1 \land Kw_4 p_1 & B = Kw_3 p_1 \land \neg Kw_4 p_1 \\ C = \neg Kw_3 p_1 \land Kw_4 p_1 & D = \neg Kw_3 p_1 \land \neg Kw_4 p_1 \end{array}$$

The main idea is to exploit that player 1 controls a single variable. Therefore in any uniform strategy player 1 can choose to satisfy at most two of A, B, C, D. E.g., "when p_1 is true reveal p_1 to 3 and 4 (A), when p_1 is false reveal p_1 to 4 but not to 3 (C).

Definition of the Goal Formulas. *For players $\{2,\ldots,7\}$ the goal formulas are as follows.*

$$\begin{array}{rl}
\gamma_2 = & ((C \lor A) \to Kw_8 p_2) \land (\neg(C \lor A) \to \neg Kw_8 p_2), \\
\gamma_3 = & (((B \lor D) \to Kw_8 p_3) \land (\neg(B \lor D) \to \neg Kw_8 p_3)) \lor \\
& ((A \lor D) \land Kw_8 p_2 \land \neg Kw_8 p_3), \\
\gamma_4 = & ((D \lor C) \to Kw_8 p_4) \land (\neg(D \lor C) \to \neg Kw_8 p_4) \land \\
& (((B \lor C) \land Kw_8 p_5) \to \neg Kw_8 p_4), \\
\gamma_5 = & (((A \lor B) \to Kw_8 p_5) \land (\neg(A \lor B) \to \neg Kw_8 p_5)) \lor \\
& ((A \lor C) \land Kw_8 p_7 \land \neg Kw_8 p_5), \\
\gamma_6 = & (((A \lor D) \to Kw_8 p_6) \land (\neg(A \lor D) \to \neg Kw_8 p_6)) \lor \\
& ((A \lor B) \land (Kw_8 p_3 \lor Kw_8 p_7) \land \neg Kw_8 p_6), \\
\gamma_7 = & ((B \lor C) \to Kw_8 p_7) \land (\neg(B \lor C) \to \neg Kw_8 p_7)
\end{array}$$

A	$2, 5, 6$	D, C, B
B	$3, 5, 7$	A, C, A
C	$2, 4, 7$	D, B, A
D	$3, 4, 6$	A, B, B

Table 1: Uniform strategies for player 1. Explanations are given in the text.

The goal of player 1 *is defined as* $\gamma_1 := \bigvee_{j=1}^{6} \alpha_j$ *where* $\alpha_1 = Kw_8 p_2 \wedge D$, $\alpha_2 = Kw_8 p_3 \wedge A$, $\alpha_3 = Kw_8 p_4 \wedge B$, $\alpha_4 = Kw_8 p_5 \wedge C$, $\alpha_5 = Kw_8 p_6 \wedge B$ *and* $\alpha_6 = Kw_8 p_7 \wedge A$. *We will now verify that* $NE_{\mathsf{pess}}(G) = \emptyset$.

The goals of the players (except 8) involve assertions about whether players $2, \dots, 7$ *reveal the proposition that they control to player 8 along with whether 1 reveals* p_1 *to players 3 and 4. For the purpose of this example, note that for all* $j \in \{2, \dots, 7\}$, *for all* $s_j \in S_j$ *and* $k \neq 8$, *the value of* $s_j(k)$ *is irrelevant.*

We now argue that $NE_{\mathsf{pess}}(G) = \emptyset$. *To simplify the presentation, we split the reasoning into two parts. First we argue that no globally uniform strategy of player 1 can be part of a pessimist Nash equilibrium in* G. *In the second part we extend this to cover all uniform strategy profiles.*

Globally Uniform Strategies of Player 1. *We show that for all uniform strategy profiles* $\mathbf{s} \in \mathbf{S}$, *if* $\mathbf{s}_1$ *is globally uniform then* $\mathbf{s} \notin NE_{\mathsf{pess}}(G)$. *In other words, no uniform strategy profile in* G *with a globally uniform strategy for player* 1 *can be a pessimist Nash equilibrium.*

Consider an arbitrary uniform strategy profile $\mathbf{s} \in \mathbf{S}$ *where* $\mathbf{s}_1 = \dot{s}_1 \in \mathbf{S}_1^g$. *The uniform strategy* $\dot{s}_1$ *satisfies exactly one of the formulas* A, B, C, D. *From the goal formulas we can see that there exists a non-empty subset of players* $X \subseteq \{2, \dots, 7\}$ *such that for all* $j \in X$, $\min \mathbf{u}_j(v, (\dot{s}_1, \dot{s}_j^{\forall}, \mathbf{s}_{N-\{1,j\}})) = 1$ *for all* v. *Thus if* $\mathbf{s} \in NE_{\mathsf{pess}}(G)$ *then* $\mathbf{u}_j(v, \mathbf{s}) = 1$ *for all* $j \in X$ *and for all* v. *From the goal formulas it also follows that there exists* $\dot{s'}_1 \neq \dot{s}_1$ *such that* $\min \mathbf{u}_1(v, (\dot{s'}_1, \dot{s}_j^{\forall}, \mathbf{s}_{N-\{1,j\}})) = 1$ *for all* $j \in X$ *and for all* v.

In Table 1 we list all such possibilities. The first column in Table 1 lists the formula in A, B, C, D *that is satisfied by a globally uniform strategy* $\dot{s}_1$. *The second column lists the players* $j \in \{2, \dots, 7\}$ *who can ensure an outcome 1 with* $\dot{s}_j^{\forall}$ *given* $\dot{s}_1$. *The third column gives the corresponding formulas in* A, B, C, D *that player 1 should satisfy to achieve an outcome 1. For example suppose* $\dot{s}_1$ *satisfies* A *(first row), players 2, 5 and 6 can reveal their proposition to player 8 to ensure an outcome of 1 given* $\dot{s}_1$. *Player 1 can then choose to satisfy* D *(corresponding to* α_1*),* C *(corresponding to* α_4*),* B *(corresponding to* α_5*), respectively to achieve an outcome of 1. Using Table 1 it can be verified that any* $\mathbf{s} \in \mathbf{S}$ *where* $\mathbf{s}_1$ *is a globally uniform strategy is not a pessimist Nash equilibrium.*

Arbitrary Uniform Strategies of Player 1. *Next, note that since player 1 controls a single proposition* p_1, *any uniform strategy (of player 1) can satisfy at most two of* A, B, C, D. *For example, consider the uniform strategy* $\mathbf{s}_1$*: when* p_1 *is true reveal* p_1 *to 3 but not to 4 (B), when* p_1 *is false do not reveal* p_1 *to 3 and do not reveal* p_1 *to 4 (D). Using an argument similar to the one above we can show that if* $\mathbf{s} \in NE_{\mathsf{pess}}(G)$ *then* $\min \mathbf{u}_1(v, \mathbf{s}) = 1$ *for all* $v \in V$.

Now consider the uniform strategy $\mathbf{s}_1$ *which is mentioned above for player 1 that satisfies* B *or* D*. We can argue that there is no uniform strategy* $\mathbf{s}^*$ *with* $\mathbf{s}_1^* = \mathbf{s}_1$ *such that* $\mathbf{s}^* \in NE_{\mathsf{pess}}(G)$*. Given the uniform strategy* $\mathbf{s}_1$ *of player 1, we have the following.*

- *Player 2 can ensure an outcome of 1 (for all* $v \in V$*) by not revealing* p_2 *to player 8.*
- *Player 3 can ensure an outcome of 1 (for all* $v \in V$*) by revealing* p_3 *to player 8. But this would in turn imply that player 1 can satisfy* α_2 *by deviating to a uniform strategy that satisfies* A*.*
- *If player 4 reveals* p_4 *to player 8 then player 1 can satisfy* α_3 *by deviating to a uniform strategy that satisfies* B*.*
- *If player 5 reveals* p_5 *to player 8 then player 1 can satisfy* α_4 *by deviating to a uniform strategy that satisfies* C*.*
- *If player 6 reveals* p_6 *to player 8 then player 1 can satisfy* α_5 *by deviating to a uniform strategy that satisfies* B*.*
- *If player 7 reveals* p_7 *to player 8 then player 1 can satisfy* α_6 *by deviating to a uniform strategy that satisfies* A*.*

Recall that the goal of player 1 is $\gamma_1 := \bigvee_{j=1}^{6} \alpha_j$*. Now consider any uniform strategy profile* $\mathbf{s}^*$ *where* $\mathbf{s}_1^* = \mathbf{s}_1$*, players 2 and 3 are playing their best responses and players 4,5,6 and 7 do not reveal the proposition that they control to player 8. Then* α_3*,* α_4*,* α_5 *and* α_6 *are not satisfied by* $\mathbf{s}^*$*. From items 1 and 2 above, we have that in* s_3^**, player 3 reveals* p_3 *to player 8 and subsequently player 1 has a profitable deviation to a uniform strategy that satisfies* A*. Thus* $s^* \notin NE_{\mathsf{pess}}(G)$*. If at least one of the players* $4, \dots, 7$ *reveal the proposition that they control to player 8, then for player 1 to ensure the outcome 1 for all* v*, it need not necessarily have to satisfy* α_2 *but can choose to satisfy the corresponding formula* $\alpha_3, \dots, \alpha_6$*. But this would also imply deviating from* $\mathbf{s}_1$ *as listed in items 3-6 above. Thus we can conclude that for all uniform strategy* $\mathbf{s}^*$ *with* $\mathbf{s}_1^* = \mathbf{s}_1$ *we have that* $\mathbf{s}^* \notin NE_{\mathsf{pess}}(G)$*.*

In Table 2 we enumerate all possibilities. In the first column we list all the possible combinations of the formulas in A, B, C, D *that player 1 can possibly satisfy in any uniform strategy. Corresponding to each row which denotes a uniform strategy* $\mathbf{s}_1$ *of player 1, in the second column in Table 2, we list the minimal set of players* X *which satisfy the following conditions. Given the uniform strategy* $\mathbf{s}_1$ *of player 1,*

- *For all* $j \in X$*, player* j *cannot ensure the outcome 1 (for all* $v \in V$*) by not revealing its proposition to player 8 (assuming player 1 chooses* $\mathbf{s}_1$*).*
- *If for all* $j \in X$*,* $\mathbf{s}_j$ *is a uniform strategy of player* j *that reveals* p_j *to player 8 then in the resulting uniform strategy profile* $\mathbf{s}$*, we have* $\mathbf{u}_1(v, \mathbf{s}) = 1$ *for all* $v \in V$*.*

In other words, if all the players in X *reveal their proposition to player 8 then the outcome for player 1 under the strategy* $\mathbf{s}_1$ *is 1 for all* v*. For example, consider the uniform strategy* $\mathbf{s}_1'$ *of player 1 defined as follows: "when* p_1 *is true reveal* p_1 *to 3 and 4 (A), when* p_1 *is false reveal* p_1 *to 4 but not to 3 (C)". Given* $\mathbf{s}_1'$*, player 3 violates the first condition above as*

$C \vee A$	$\{5, 7\}$
$B \vee D$	$\emptyset$
$D \vee C$	$\emptyset$
$A \vee B$	$\{3, 6\}, \{6, 7\}$
$A \vee D$	$\{2, 3\}$
$B \vee C$	$\{4, 5\}$

Table 2: Uniform strategies for player 1. Explanations are given in the text.

player 3 can ensure the outcome 1 by not revealing p_3 to player 8. If both the players 5 and 7 reveal p_5 and p_7, respectively, to player 8, then in the resulting uniform strategy profile $\mathbf{s}'$ we have $\mathbf{u}_1(v, \mathbf{s}') = 1$ for all $v \in V$. However, note that then $\mathbf{s}' \notin NE_{\mathsf{pess}}(G)$. In $\mathbf{s}'$ since player 7 reveals p_7 to player 8 and $\mathbf{s}'_1$ satisfies $C \vee A$, player 5 can deviate to not reveal p_5 and ensure an outcome of 1 (for all $v \in V$). Thus player 5 has a profitable deviation from $\mathbf{s}'$ and therefore, $\mathbf{s}' \notin NE_{\mathsf{pess}}(G)$. A similar reasoning applies to the other rows in Table 2. From the goal formulas $\gamma_1, \dots, \gamma_7$ we can verify that for every such set X, there is a player $k \in X$ who can ensure an outcome of 1 by not revealing p_k to player 8. Therefore $NE_{\mathsf{pess}}(G) = \emptyset$.

5.2 Existence of Nash Equilibrium in the General Case

Following up on Example 26 we now determine more generally for which fragments of observation games the existence of a Nash equilibrium is guaranteed. An initial step would be to consider observation games where the goal formulas for all players are restricted to the positive fragment of L^K. For this fragment, the following result is straightforward.

Proposition 27. *Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ be an observation game where $\gamma_i \in L^+$ for all $i \in N$. Then $NE(G) \neq \emptyset$ for all outcome relations.*

Proof. Observe that when $\gamma_i \in L^+$ for all $i \in N$, the globally uniform strategy $\dot{s}_i^{\forall}$ (public announcement by player i of P_i) is dominant for all $i \in N$. Thus $\dot{s}^{\forall} \in NE(G)$ for any outcome relation. $\square$

In this section, we present a more general structural result that identifies a class of observation games in which a Nash equilibrium is guaranteed to exist. Our results show that the existence of equilibrium crucially depends on the combination of positive/negative epistemic assertions made by players in their goal formulas. Observation games where the goal formulas are restricted to the positive fragment of L^K can be viewed as a particular simple case in this setting.

We assume that the goal formulas are in negation normal form. For $i, j \in N$ (where j may be i) and γ_i in L^K_{nnf}, we first define $x_i^j(\gamma_i)$ for $x \in \{+, -\}$. Intuitively, $+_i^j(\gamma_i)$ and $-_i^j(\gamma_i)$ encode the fact that player i makes a positive and negative (respectively) epistemic assertion about a variable assigned to player j in the goal formula γ_i. Formally, $x_i^j(\gamma_i)$ is defined as follows.

- For $\gamma_i = p_j$ and $\gamma_i = \neg p_j$ we have $+_i^j(p_j)$ and $+_i^j(\neg p_j)$.

- $\gamma_i = K_k\varphi$ (where $k \in N$): $+_i^j(K_k\varphi)$ iff $+_i^j(\varphi)$ and $-_i^j(K_k\varphi)$ iff $-_i^j(\varphi)$.
- $\gamma_i = \hat{K}_k\varphi$ (where $k \in N$): $+_i^j(\hat{K}_k\varphi)$ iff $-_i^j(\varphi)$ and $-_i^j(\hat{K}_k\varphi)$ iff $+_i^j(\varphi)$.
- $\gamma_i = \varphi \wedge \psi$:
  - $+_i^j(\varphi \wedge \psi)$ iff $+_i^j(\varphi)$ or $+_i^j(\psi)$ and $-_i^j(\varphi \wedge \psi)$ iff $-_i^j(\varphi)$ or $-_i^j(\psi)$.
- $\gamma_i = \varphi \vee \psi$:
  - $+_i^j(\varphi \vee \psi)$ iff $+_i^j(\varphi)$ or $+_i^j(\psi)$ and $-_i^j(\varphi \vee \psi)$ iff $-_i^j(\varphi)$ or $-_i^j(\psi)$.

Note that the definition of $+_i^j(\gamma_i)$ is intended to encode the fact that player i makes a positive epistemic assertion about a variable assigned to player j in the goal formula γ_i. So in item 4, we have that $+_i^j(\varphi \wedge \psi)$ holds iff the same holds for at least one of the subformulas φ or ψ. A similar comment applies to definition of $-_i^j(\gamma_i)$.

For every player i, we define $\mathit{type}(i) \subseteq \{+, -, c+, c-\}$ as follows. For $x \in \{+, -\}$,

- $x \in \mathit{type}(i)$ if there is a player $j \neq i$ such that $x_i^j(\gamma_i)$,
- $cx \in \mathit{type}(i)$ if $x_i^i(\gamma_i)$.

In other words, $+$ and $-$ are in $\mathit{type}(i)$ if there exists some player j with $j \neq i$ such that player i makes a positive and negative (respectively) epistemic assertion about a variable assigned to player j in γ_i. Likewise, $c+$ and $c-$ is in $\mathit{type}(i)$ if player i makes a positive and negative (respectively) epistemic assertion about its own variable in γ_i.

For example, for $i \in N$, consider the goal formula γ_i given in Example 12, under its translation to negation normal form (NNF). We have the following for player 1.

- $K_1 p_2$ occurs as a subformula in the NNF of γ_1 and therefore $+ \in \mathit{type}(1)$. $\hat{K}_1 p_2$ occurs as a subformula in the NNF of γ_1 and therefore $- \in \mathit{type}(1)$.
- $K_2 p_1$ occurs as a subformula in the NNF of γ_1 and therefore $c+ \in \mathit{type}(1)$. $\hat{K}_2 p_1$ occurs as a subformula in the NNF of γ_1 and therefore $c- \in \mathit{type}(1)$.

For player 2, the reasoning is similar and therefore, we have that $\mathit{type}(i) = \{+, -, c+, c-\}$ for all $i \in N$. In fact, Theorem 32 given below shows that it is crucial that $|\mathit{type}(i)| > 3$ in this example.

Based on the notion of *type*, we define the following subsets of N. Let

- $X_+ = \{i \in N \mid c+ \in \mathit{type}(i)\}$, $X_- = \{i \in N \mid c- \in \mathit{type}(i)\}$,
- $W_l = \{i \in N \mid \mathit{type}(i) = \{c+, c-\}\}$,
- $W_+ = \{i \in N \mid \mathit{type}(i) = \{+, c+, c-\}\}$, $W_- = \{i \in N \mid \mathit{type}(i) = \{-, c+, c-\}\}$.

For the proofs in this section, we also find it useful to define an ordering $\succcurlyeq$ over the set of strategy profiles. Let $X \subseteq N$ and $s_X, t_X \in S_X$. We say that $s_X \succcurlyeq t_X$ if for all $i \in X$ and $j \in N$, $t_i(j) \subseteq s_i(j)$. We can then show the following existence result.

Algorithm 1:

Input: $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$.

Output: A uniform strategy profile $\mathbf{s} \in NE_{\mathsf{pess}}(G)$.

1 Let $W_o := \overline{X_+} \cup \overline{X_-} \cup W_l$;

2 $\forall i \in \overline{X_+}$, $\forall v \in V$, set $\mathbf{s}_i(v) := s_i^{\emptyset}$; `/* a dominant uniform strategy */`

3 $\forall i \in \overline{X_-} \setminus \overline{X_+}$, $\forall v \in V$, set $\mathbf{s}_i(v) := s_i^{\forall}$; `/* a dominant uniform strategy */`

4 $\forall i \in W_l$, $\forall v \in V$, if $\exists s \in S$ such that $\forall w : w \sim_i v$, $u_i(w, s) = 1$ then $\forall w : w \sim_i v$, set $\mathbf{s}_i(w) := s_i$ else $\mathbf{s}_i(w) := s_i^{\emptyset}$; `/* u_i does not depend on others' choice */`

5 $\forall i \in W_+$, $\forall j \in W_-$, $\forall v \in V$ set $\mathbf{s}_i(v) := s_i^{\emptyset}$; $\mathbf{s}_j(v) := s_j^{\forall}$; `/* initialisation */`

6 $\forall v \in V$, set $Y(v) := \emptyset$; $Z(v) := \emptyset$;

7 **repeat**

8 $\quad$ $\forall v \in V$, set $Y'(v) := Y(v)$; $Z'(v) := Z(v)$;

$\quad$ `/* process players who make positive assertions about variables controlled by others */`

9 $\quad$ **while** $\exists v \in V$, $\exists i \in W_+ \setminus Y(v)$, $\exists s_i$, *such that* $\forall w : w \sim_i v$, $\forall s_{W_- \setminus Z(w)}$, *we have* $(s_i, \mathbf{s}_{W_+ \setminus \{i\}}(w), \mathbf{s}_{Z(w)}(w), s_{W_- \setminus Z(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$ **do**

10 $\quad\quad$ $\forall w : w \sim_i v$, set $\mathbf{s}_i(w) := s_i$; $Y(w) := Y(w) \cup \{i\}$;

$\quad$ `/* process players who make negative assertions about variables controlled by others */`

11 $\quad$ **while** $\exists v \in V$, $\exists i \in W_- \setminus Z(v)$, $\exists s_i$ *such that* $\forall w : w \sim_i v$, $\forall s_{W_+ \setminus Y(w)}$, *we have* $(s_i, \mathbf{s}_{W_- \setminus \{i\}}(w), \mathbf{s}_{Y(w)}(w), s_{W_+ \setminus Y(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$ **do**

12 $\quad\quad$ $\forall w : w \sim_i v$, set $\mathbf{s}_i(w) := s_i$; $Z(w) := Z(w) \cup \{i\}$;

13 **until** $\forall v \in V$, $Y(v) = Y'(v)$ *and* $Z(v) = Z'(v)$;

14 $\forall i \in W_+ \setminus Y(v)$, $\forall j \in W_- \setminus Z(v)$ and $\forall v \in V$, set $\mathbf{s}_j(v)(i) := \emptyset$;

15 $\forall i \in W_- \setminus Z(v)$, $\forall j \in W_+ \setminus Y(v)$ and $\forall v \in V$, set $\mathbf{s}_j(v)(i) := P_j$;

16 **return** $\mathbf{s}$;

Theorem 28. *Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ be an observation game where all goals γ_i are guarded. If for all $i \in N$, $|type(i)| \leq 3$, then*

1. $NE_{\mathsf{pess}}(G) \neq \emptyset$.

2. $NE_{\mathsf{opt}}(G) \neq \emptyset$.

Proof. **Part 1.** $NE_{\mathsf{pess}}(G) \neq \emptyset$. Consider the procedure described as Algorithm 1. We argue that Algorithm 1 always terminates and constructs a uniform strategy profile $\mathbf{s} \in NE_{\mathsf{pess}}(G)$. First, we note that the sets $\overline{X_+}$, $\overline{X_-} \setminus \overline{X_+}$, W_l, W_+ and W_- form a partition of N.

In each iteration of the outer loop in Algorithm 1 (steps 7 - 13), the size of the set $Y(v)$ or $Z(v)$ strictly increases for some $v \in V$. We also have that for all $v \in V$, $0 \leq |Y(v)| \leq |N|$ and $0 \leq |Z(v)| \leq |N|$. It follows that Algorithm 1 always terminates. Let $\mathbf{s}$ be the strategy profile constructed by Algorithm 1. From the description of the procedure, it can also be verified that $\mathbf{s}$ is a uniform strategy profile. Thus to prove the claim, it suffices to show that $\mathbf{s} \in NE_{\mathsf{pess}}(G)$.

Note that for all $i \in \overline{X_+}$, $\mathbf{s}_i^\emptyset$ is a dominant uniform strategy and for all $i \in \overline{X_-}$, $\mathbf{s}_i^\forall$ is a dominant uniform strategy. Therefore, for all $v \in V$, for all $i \in \overline{X_+} \cup \overline{X_-}$ and for all $s'_i \in S_i$, $u_i(v, \mathbf{s}(v)) \geq u_i(v, (s'_i, \mathbf{s}_{-i}(v)))$.

For all $i \in W_l$, for all $v \in V$, we have $\mathbf{u}_i(v, (\mathbf{t}_i, \mathbf{t}_{-i})) = \mathbf{u}_i(v, (\mathbf{t}_i, \mathbf{t}'_{-i}))$ for all $v \in V$, $\mathbf{t}_i \in \mathbf{S}_i$ and for all $\mathbf{t}_{-i}, \mathbf{t}'_{-i} \in \mathbf{S}_{-i}$. Therefore, by the choice of $\mathbf{s}_i$ made in line 4 of Algorithm 1, we have for all $i \in W_l$, for all $v \in V$, for all $\mathbf{s}'_i \in \mathbf{S}_i$, $\mathbf{u}_i(v, \mathbf{s}) \geq \mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i}))$.

Now consider a player $i \in W_+$. For $v \in V$, suppose $\mathbf{s}_i(v)$ is assigned a value in the while loop (steps 9-10). Let $\mathbf{s}^k$ denote the resulting strategy profile after this assignment (step 10) and Z^k denote the value of Z in the corresponding iteration. By definition of the while loop, for all w with $v \sim_i w$, for all $s_{W_- \setminus Z^k(w)}$, $(s_i, \mathbf{s}^k_{W_+\setminus\{i\}}(w), \mathbf{s}_{Z^k}(w), s_{W_-\setminus Z^k(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$. Since $i \in W_+$, this implies that for all $t_{W_+\setminus\{i\}} \in S_{W_+\setminus\{i\}}$ such that $t_{W_+\setminus\{i\}} \succcurlyeq \mathbf{s}^k_{W_+\setminus\{i\}}(w)$, for all $s_{W_-\setminus Z^k(w)}$, $(s_i, t_{W_+\setminus\{i\}}, \mathbf{s}_{Z^k(w)}(w), s_{W_-\setminus Z^k(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$. By definition, we have $\mathbf{s}_{W_+\setminus\{i\}}(w) \succcurlyeq \mathbf{s}^k_{W_+\setminus\{i\}}(w)$ and for all $j \in Z^k(w)$, $\mathbf{s}^k_j(w) = \mathbf{s}_j(w)$. Therefore, it follows that $(s_i, \mathbf{s}_{W_+\setminus\{i\}}, \mathbf{s}_Z(w), s_{W_-\setminus Z(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$ and $\mathbf{u}_i(v, \mathbf{s}) = 1$.

Consider a player $i \in W_-$. For $v \in V$, suppose $\mathbf{s}_i(v)$ is assigned a value in the while loop (steps 11-12). Let $\mathbf{s}^k$ denote the resulting strategy profile after this assignment (step 12) and Y^k denote the value of Y in the corresponding iteration. By definition of the while loop, for all w with $v \sim_i w$, for all $s_{W_+\setminus Y^k(w)}$, $(s_i, \mathbf{s}^k_{W_-\setminus\{i\}}(w), \mathbf{s}_{Y^k(w)}(w), s_{W_+\setminus Y^k(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$. Since $i \in W_-$, this implies that for all $t_{W_-\setminus\{i\}} \in S_{W_-\setminus\{i\}}$ such that $\mathbf{s}^k_{W_-\setminus\{i\}}(w) \succcurlyeq t_{W_-\setminus\{i\}}$, for all $s_{W_+\setminus Y^k(w)}$, $(s_i, t_{W_-\setminus\{i\}}, \mathbf{s}_{Y^k(w)}(w), s_{W_+\setminus Y^k(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$. By definition, $\mathbf{s}^k_{W_-\setminus\{i\}}(w) \succcurlyeq \mathbf{s}_{W_-\setminus\{i\}}(w)$. Thus $(s_i, \mathbf{s}_{W_-\setminus\{i\}}, \mathbf{s}_Y(w), s_{W_+\setminus Y(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$. Therefore, $\mathbf{u}_i(v, \mathbf{s}) = 1$.

Now suppose there exists $v \in V$ and $i \in W_+$ such that $i \notin Y(v)$ (on termination of the repeat loop, steps 7-13). By definition, for all s_i, there exists w with $v \sim_i w$ and there exists $t_{W_-\setminus Z(w)}$ such that $(s_i, \mathbf{s}_{W_+\setminus\{i\}}(w), \mathbf{s}_{Z(w)}(w), t_{W_-\setminus Z(w)}, \mathbf{s}_{W_o}(w)), w \not\models \gamma_i$. Since $i \in W_+$ and $s_j(v)(i) = \emptyset$ for all $j \in W_- \setminus Z(v)$, it follows that for all s_i, there exists a w with $v \sim_i w$ such that $(s_i, \mathbf{s}_{W_+\setminus\{i\}}(w), \mathbf{s}_Z(w), s_{W_-\setminus Z(w)}, \mathbf{s}_{W_o}(w)), w \not\models \gamma_i$. Therefore, for all $\mathbf{s}'_i \in \mathbf{S}_i$, $\mathbf{u}_i(v, \mathbf{s}) \geq \mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i}))$.

Suppose there exists $v \in V$ and $i \in W_+$ such that $i \notin Z(v)$. Using a similar proof as above and using the fact that $s_j(v)(i) = \emptyset$ for all $j \in W_- \setminus Z(v)$ we can argue that for all s_i, there exists a w with $v \sim_i w$ such that $(s_i, \mathbf{s}_{W_-\setminus\{i\}}(w), \mathbf{s}_{Y(w)}(w), s_{W_+\setminus Y(w)}, \mathbf{s}_{W_o}(w)), w \not\models \gamma_i$. Therefore, for all $\mathbf{s}'_i \in \mathbf{S}_i$, $\mathbf{u}_i(v, \mathbf{s}) \geq \mathbf{u}_i(v, (\mathbf{s}'_i, \mathbf{s}_{-i}))$.

Part 2. To show that $NE_{\mathsf{opt}}(G) \neq \emptyset$, we modify Algorithm 1 to reflect the optimist decision rule. This is achieved by changing the conditional in both the While loops (line 9 and line 11) as described below. Note that the only change is a switch to existential quantification over the valuations w in order to capture the definition of the optimist decision rule.

*L*ine 9.
While $\exists v \in V$, $\exists i \in W_+ \setminus Y(v)$, $\exists s_i$, such that $\exists w : w \sim_i v$, $\forall s_{W_-\setminus Z(w)}$, we have $(s_i, \mathbf{s}_{W_+\setminus\{i\}}(w), \mathbf{s}_{Z(w)}(w), s_{W_-\setminus Z(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$ **do**.

*L*ine 11.
While $\exists v \in V$, $\exists i \in W_- \setminus Z(v)$, $\exists s_i$ such that $\exists w : w \sim_i v$, $\forall s_{W_+\setminus Y(w)}$, we have $(s_i, \mathbf{s}_{W_-\setminus\{i\}}(w), \mathbf{s}_{Y(w)}(w), s_{W_+\setminus Y(w)}, \mathbf{s}_{W_o}(w)), w \models \gamma_i$ **do**. □

The result in Theorem 28 is tight in the sense that there exist observation games where $|type(i)| = 4$ for $i \in N$ and $NE_{\mathsf{pess}}(G) = \emptyset$. This is illustrated in Example 26 where for players $i \in \{2, \dots, 7\}$, $type(i) = \{+, -, c+, c-\}$.

An interesting corollary of Theorem 28 is for *self-positive* goals: my objective is never to remain ignorant of others' variables even when it may be for others to remain ignorant. (We recall their definition in Section 2.2.)

Corollary 29. *Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ be an observation game where all $(\gamma_i)_{i \in N}$ are guarded and self-positive, then $NE_{\mathsf{pess}}(G) \neq \emptyset$ and $NE_{\mathsf{opt}}(G) \neq \emptyset$.*

Proof. Follows from Theorem 28, since $\forall i$, $- \notin type(i)$. □

For $NE_{\mathsf{max}}(G)$ we show a weaker result (Theorem 30) which can be strengthened for *Kw* games (Theorem 32).

Theorem 30. *Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ be an observation game where the goal formulas $(\gamma_i)_{i \in N}$ are guarded. If for all $i \in N$, $|type(i)| \leq 2$ then $NE_{\mathsf{max}}(G) \neq \emptyset$.*

Proof. Let $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ be an observation game. Let $X_+ = \{i \in N \mid c+ \in type(i)\}$ and $X_- = \{i \in N \mid c- \in type(i)\}$. Consider the uniform strategy profile $\mathbf{s}$ defined as follows.

- For $i \in X_+ \cap X_-$, we define $\mathbf{s}_i$ using the iterative procedure: for $v \in V$ where $\mathbf{s}_i(v)$ is not defined, if there exists $s \in S$ such that $u_i(v, \mathbf{s}) = 1$ then set $\mathbf{s}_i(w) = s_i$ for all $w : v \sim_i w$. Otherwise set $\mathbf{s}_i(w) = s_i^{\emptyset}$ for all $w : v \sim_i w$.
- For all $i \in \overline{X_+}$, for all $v \in V$, let $\mathbf{s}_i(v) = s_i^{\emptyset}$.
- For all $i \in N \setminus [(X_+ \cap X_-) \cup \overline{X_+}]$, for all $v \in V$, let $\mathbf{s}_i(v) = s_i^{\forall}$.

Note that for all $i \in \overline{X_+}$, $\mathbf{s}_i^{\emptyset}$ is a dominant uniform strategy. For all $i \in \overline{X_-}$, $\mathbf{s}_i^{\forall}$ is a dominant uniform strategy and for all $i \in (\overline{X_+} \cup \overline{X_-})$, both $\mathbf{s}_i^{\emptyset}$ and $\mathbf{s}_i^{\forall}$ are dominant uniform strategies.

Since the goal formulas are guarded, we have for all $i \in N$ and for all $v \in V$, $v, \mathbf{s}(v) \models \gamma_i$ iff $w, \mathbf{s}(v) \models \gamma_i$ for all $w : v \sim_i w$. Also, for all $i \in X_+ \cap X_-$, with $type(i) \leq 2$, we have that $u_i(v, \mathbf{s}(v)) = u_i(v, (\mathbf{s}_i(v), s'_{-i}))$ for all $s'_{-i} \in S_{-i}$. It then follows that $\mathbf{s} \in NE_{\mathsf{max}}(G)$. □

5.3 Existence of Maximal Nash Equilibrium in Knowing-Whether Games

For the subclass of *Kw* games, we show that Theorem 30 can be strengthened. We argue that if G is a *Kw* game where for all $i \in N$, $|type(i)| \leq 3$ then the output of Algorithm 2 is a globally uniform strategy profile $\mathbf{s}$ such that $\mathbf{s} \in NE_{\mathsf{max}}(G)$.

Lemma 31. *Algorithm 2 always terminates and it satisfies the following properties.*

- *After each iteration of the while loops, steps 9-10 and steps 11-12, the strategy profile* $\mathbf{s}$ *constructed is a globally uniform strategy profile.*
- *The strategy profile* $\mathbf{s}$ *which is the output of Algorithm 2 is a globally uniform strategy profile.*

Algorithm 2:

Input: A *Kw* game $G = (N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$.

Output: A uniform strategy profile $\mathbf{s} \in NE_{\mathsf{max}}(G)$.

1 Let $W_o := \overline{X_+} \cup \overline{X_-} \cup W_l$;

2 $\forall i \in \overline{X_+}$, $\forall v \in V$, set $\mathbf{s}_i(v) := s_i^{\emptyset}$; `/* a dominant uniform strategy */`

3 $\forall i \in \overline{X_-} \setminus \overline{X_+}$, $\forall v \in V$, set $\mathbf{s}_i(v) := s_i^{\forall}$; `/* a dominant uniform strategy */`

4 $\forall i \in W_l$, if $\exists s \in S$, and $\exists v \in V$ such that $u_i(v, s) = 1$ then $\forall w \in V$ set $\mathbf{s}_i(w) := s_i$
else set $\mathbf{s}_i(w) := s_i^{\emptyset}$; /* u_i `does not depend on others' choice */`

5 $\forall i \in W_+$, $\forall j \in W_-$, $\forall v \in V$ set $\mathbf{s}_i(v) := s_i^{\emptyset}$; $\mathbf{s}_j(v) := s_j^{\forall}$; `/* initialisation */`

6 Set $Y := \emptyset$; $Z := \emptyset$;

7 **repeat**

8   set $Y' := Y$; $Z' := Z$;

    `/* process players who make positive assertions about variables controlled by others */`

9   **while** $\exists i \in W_+ \setminus Y(v)$, $\exists s_i$ *and* $\exists v \in V$ *such that* $\forall s_{W_- \setminus Z}$, *we have* $v(s_i, \mathbf{s}_{W_+ \setminus \{i\}}(v), \mathbf{s}_Z(v), s_{W_- \setminus Z}, \mathbf{s}_{W_o}(v)) \models \gamma_i$ **do**

10         $\forall w \in V$, set $\mathbf{s}_i(w) := s_i$; $Y := Y \cup \{i\}$;

    `/* process players who make negative assertions about variables controlled by others */`

11   **while** $\exists i \in W_- \setminus Z$, $\exists s_i$, $\exists v \in V$, *such that* $\forall s_{W_+ \setminus Y}$, *we have* $v(s_i, \mathbf{s}_{W_- \setminus \{i\}}(v), \mathbf{s}_Y(v), s_{W_+ \setminus Y}, \mathbf{s}_{W_o}(v)) \models \gamma_i$ **do**

12         $\forall w \in V$, set $\mathbf{s}_i(w) := s_i$; $Z := Z \cup \{i\}$;

13 **until** $Y = Y'$ *and* $Z = Z'$;

14 $\forall i \in W_+ \setminus Y$, $\forall j \in W_- \setminus Z$, $\forall v \in V$, set $\mathbf{s}_j(v)(i) := \emptyset$;

15 $\forall i \in W_- \setminus Z$, $\forall j \in W_+ \setminus Y$, $\forall v \in V$, set $\mathbf{s}_j(v)(i) := P_j$;

16 **return** $\mathbf{s}$;

Proof. First, note that in each iteration of the outer loop in Algorithm 2 (steps 7 - 13), the size of the set Y or Z strictly increases. Therefore Algorithm 2 always terminates.

At the end of the initialization steps (2 - 6), $\mathbf{s} \in \mathbf{S}^g$ by definition. So it suffices to argue that at the end of each iteration of the two While loops (steps 9 - 10 and 11 - 12), the following invariant is maintained: $\mathbf{s} \in \mathbf{S}^g$. We can argue by induction on the number of iterations of the while loops (steps 7 - 13). The claim follows from the definition of the assignment statements: steps 10 and 12.

Thus on termination of the outer loop (steps 7 - 13) we have that $\mathbf{s} \in \mathbf{S}^g$. It follows from the definition of lines 14 and 15 that the output of Algorithm 2, $\mathbf{s} \in \mathbf{S}^g$. □

Theorem 32. *Let $G = (N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$ be a Kw game. If for all $i \in N$, $|type(i)| \leq 3$ then $NE_{\mathsf{max}}(G) \neq \emptyset$.*

Proof. We argue that the output of Algorithm 2 is a globally uniform strategy profile $\mathbf{s}$ such that $\mathbf{s} \in NE_{\mathsf{max}}(G)$. As in the case of Theorem 28, note that the sets $\overline{X_+}$, $\overline{X_-} \setminus \overline{X_+}$, W_l, W_+ and W_- form a partition of N.

By Lemma 31, Algorithm 2 always terminates. Let $\mathbf{s}$ be the profile constructed by Algorithm 2. To prove the claim, it suffices to argue that $\mathbf{s} \in NE_{\mathsf{max}}(G)$.

Note that for all $i \in \overline{X_+}$, $\mathbf{s}_i^{\emptyset}$ is a dominant uniform strategy and for all $i \in \overline{X_-}$, $\mathbf{s}_i^{\forall}$ is a uniform strategy that is dominant. Therefore, for all $v \in V$, for all $i \in \overline{X_+} \cup \overline{X_-}$, $u_i(v, \mathbf{s}(v)) \geq u_i(v, (s'_i, \mathbf{s}_{-i}(v)))$.

For all $i \in W_l$ we have that $u_i(v, \mathbf{s}(v)) = u_i(v, (\mathbf{s}_i(v), s'_{-i}))$ for all $s'_{-i} \in S_{-i}$. Since the goals are knowing whether formulas, we have if there exists $v \in V$ and there exists $s \in S$ such that $u_i(v, s) = 1$ then for all $w \in V$, for all $s'_{-i} \in S_{-i}$, $u_i(w, (s_i, s'_{-i})) = 1$. Therefore, for all $v \in V$, for all $i \in W_l$, $u_i(v, \mathbf{s}(v)) \geq u_i(v, (s'_i, \mathbf{s}_{-i}(v)))$.

Now consider a player $i \in W_+$. For $v \in V$, suppose $\mathbf{s}_i(v)$ is assigned a value in the while loop (steps 9-10). Let $\mathbf{s}^k$ denote the resulting strategy profile after this assignment (step 10). Let Z^k denote the value of Z in the corresponding iteration. By Lemma 31, we have $\mathbf{s}^k \in \mathbf{S}^g$ and by definition of the while loop, there exists v such that for all $s_{W_- \setminus Z^k}$, $v, (s_i, \mathbf{s}^k_{W_+ \setminus \{i\}}(v), \mathbf{s}_{Z^k}, s_{W_- \setminus Z^k}, \mathbf{s}_{W_o}(v)) \models \gamma_i$. By Lemma 31 and the fact that $\gamma_i \in L^{Kw}$ it follows that for all w, for all $s_{W_- \setminus Z^k}$, $v, (s_i, \mathbf{s}^k_{W_+ \setminus \{i\}}(v), \mathbf{s}_{Z^k}, s_{W_- \setminus Z^k}, \mathbf{s}_{W_o}(v)) \models \gamma_i$. Since $i \in W_+$, this implies that for all $w \in V$, for all $t_{W_+ \setminus \{i\}} \in S_{W_+ \setminus \{i\}}$ such that $t_{W_+ \setminus \{i\}} \succcurlyeq s_{W_- \setminus Z^k}$, $w, (s_i, t_{W_+ \setminus \{i\}}, \mathbf{s}_{Z^k}(w), s_{W_- \setminus Z^k(w)}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. By definition, $\mathbf{s}_{W_+ \setminus \{i\}}(w) \succcurlyeq \mathbf{s}^k_{W_+ \setminus \{i\}}(w)$ and for all $j \in Z^k$, $\mathbf{s}^k_j(w) = \mathbf{s}_j(w)$. Thus we have
$w, (s_i, \mathbf{s}_{W_+ \setminus \{i\}}, \mathbf{s}_Z(w), s_{W_- \setminus Z}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. Therefore, $u_i(w, \mathbf{s}) = 1$ for all $w \in V$.

Consider a player $i \in W_-$. For $v \in V$, suppose $\mathbf{s}_i(v)$ is assigned a value in the while loop (steps 11-12). Let $\mathbf{s}^k$ denote the resulting uniform strategy profile after this assignment (step 12) and Y^k denote the value of Y in the corresponding iteration. By Lemma 31, $\mathbf{s}^k \in \mathbf{S}^g$. By definition of the while loop, there exists v such that for all $s_{W_+ \setminus Y^k}$, $v, (s_i, \mathbf{s}^k_{W_- \setminus \{i\}}(v), \mathbf{s}_{Y^k}(v), s_{W_+ \setminus Y^k}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. By Lemma 31 and the fact that $\gamma_i \in L^{Kw}$, we have for all w such that for all $s_{W_+ \setminus Y^k}$, $w, (s_i, \mathbf{s}^k_{W_- \setminus \{i\}}(w), \mathbf{s}_{Y^k}(w), s_{W_+ \setminus Y^k}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. Since $i \in W_-$, this implies that for all $t_{W_- \setminus \{i\}} \in S_{W_- \setminus \{i\}}$ such that $\mathbf{s}^k_{W_- \setminus \{i\}}(w) \succcurlyeq t_{W_- \setminus \{i\}}$, for all $s_{W_+ \setminus Y^k(w)}$, $w, (s_i, t_{W_- \setminus \{i\}}, \mathbf{s}_{Y^k(w)}(w), s_{W_+ \setminus Y^k(w)}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. By definition, $\mathbf{s}^k_{W_- \setminus \{i\}}(w) \succcurlyeq \mathbf{s}_{W_- \setminus \{i\}}(w)$. Thus $w, (s_i, \mathbf{s}_{W_- \setminus \{i\}}, \mathbf{s}_Y(w), s_{W_+ \setminus Y(w)}, \mathbf{s}_{W_o}(w)) \models \gamma_i$. Therefore, $u_i(w, \mathbf{s}) = 1$ for all $w \in W$.

Now suppose there exists $i \in W_+$ such that $i \notin Y$ (on termination of the repeat loop, steps 7-13). By definition, for all s_i, for all $v \in V$, there exists $t_{W_- \setminus Z}$ such that $v, (s_i, \mathbf{s}_{W_+ \setminus \{i\}}(v), \mathbf{s}_Z(v), t_{W_- \setminus Z}, \mathbf{s}_{W_o}(v)) \not\models \gamma_i$. Since $i \in W_+$ and $s_j(v)(i) = \emptyset$ for all $j \in W_- \setminus Z$, it follows that for all s_i, for all $v \in V$ $v, (s_i, \mathbf{s}_{W_+ \setminus \{i\}}(v), \mathbf{s}_Z(v), s_{W_- \setminus Z}, \mathbf{s}_{W_o}(v)) \not\models \gamma_i$.

Suppose there exists $v \in V$ and $i \in W_+$ such that $i \notin Z$. Using a similar proof as above and using the fact that $s_j(v)(i) = \emptyset$ for all $j \in W_- \setminus Z(v)$ we can argue that for all s_i, for all $v \in V$, $v, (s_i, \mathbf{s}_{W_- \setminus \{i\}}(w), \mathbf{s}_Y(w), s_{W_+ \setminus Y}, \mathbf{s}_{W_o}(w)) \not\models \gamma_i$. It follows that $\mathbf{s} \in NE_{\mathsf{max}}(G)$. $\square$

Examples 33 and 34 show that Theorems 30 and 32 are tight.

Example 33. *Consider the two-player game where $N = \{1, 2\}$, $P_1 = \{p\}$ and $P_2 = \{q^1, q^2, q^3\}$. Let $\gamma_1 = (Kw_1 q^2 \wedge Kw_2 p) \vee (Kw_1 q^3 \wedge \neg Kw_2 p)$ and $\gamma_2 = (q^1 \to Kw_1 q^2) \wedge (\neg q^1 \to Kw_1 q^3) \wedge (\neg Kw_1 q^2 \vee \neg Kw_1 q^3)$. Note that in this game, $|type(1)| = 3$ and $|type(2)| = 2$. The goal of player 1 is a Kw formula. It can be verified that $NE_{\mathsf{max}}(G) = \emptyset$.*

Example 34. *Consider the two-player game where* $P_1 = \{p_1, q_1\}$ *and* $P_2 = \{p_2\}$. *Let the goal formulas be:* $\gamma_1 = (\neg Kw_1 p_2 \to (Kw_2 p_1 \wedge \neg Kw_2 q_1)) \wedge (Kw_1 p_2 \to (\neg Kw_2 p_1 \wedge Kw_2 q_1))$ *and* $\gamma_2 = (Kw_2 p_1 \wedge Kw_1 p_2) \vee (Kw_2 q_1 \wedge \neg Kw_1 p_2)$. *In this game, both goals are Kw formulas. We have* $|type(1)| = 4$ *and* $|type(2)| = 3$. *It can be verified that* $NE_{\mathsf{max}}(G) = \emptyset$.

6. Representation and Complexity

For an observation game $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$, let $|N| = n$, $|P| = k$ and $\max_{i \in N} |\gamma_i| = m$ (where $|\gamma_i|$ denotes the number of symbols in γ_i). For $i \in N$, every strategy $s_i : N \to \mathcal{P}(P_i)$, can be represented in size $\mathcal{O}(nk)$. That is, both observation games and strategies have compact representations — linear in n, k and m.

On the other hand, each uniform strategy $\mathbf{s}_i : V \to S_i$ can be encoded as a tuple of Boolean functions $(\mathbf{s}_i^j(p_i))_{j \in N, p_i \in P_i}$ where each $\mathbf{s}_i^j(p_i) : \mathcal{P}(P) \to \{\top, \bot\}$. Here $\mathbf{s}_i^j(p_i)(v) = \top$ is viewed as player i revealing the variable p_i to player j under the valuation v. We assume that the Boolean function $\mathbf{s}_i^j(p_i)$ is represented as a propositional formula $\beta_i^j(p_i)$ over the propositions P. It is well known that every such Boolean function can be represented as a propositional formula, in the worst case the size of $\mathbf{s}_i^j(p_i)$ can be exponential in k.

In this section we address the computational complexity of the following two basic algorithmic questions.

- **Verification.** Given an observation game G and a uniform strategy profile $\mathbf{s} \in \mathbf{S}$, is $\mathbf{s} \in NE_{\mathsf{x}}(G)$ for an outcome relation $\mathsf{x} \in \{\mathsf{pess}, \mathsf{opt}, \mathsf{max}\}$?
- **Emptiness.** Given an observation game G is $NE_{\mathsf{x}}(G) = \emptyset$ for an outcome relation $\mathsf{x} \in \{\mathsf{pess}, \mathsf{opt}, \mathsf{max}\}$?

We show that the verification and emptiness questions are PSPACE-complete and NEXPTIME-complete respectively for the maximal outcome relation. We also show that for the pessimist and optimist outcome relations, the verification and emptiness questions are in PSPACE and NEXPTIME respectively. To obtain these results it is crucial to establish the complexity of the model checking problem of the logic L^K. The following result shows that the model checking problem is PSPACE-complete. It follows directly from Proposition 2 in (Ågotnes et al., 2013b)[6].

Theorem 35. *Given* $\alpha \in L^K$ *along with a strategy profile* $s \in S$ *and a valuation* $v \in V$, *checking if* $v, s \models \alpha$ *is PSPACE-complete.*

It is well known that the model checking problem for epistemic logic formulas over Kripke structures (for example, formulas of multi-agent S5) can be solved in polynomial time (Fagin, Halpern, Moses, & Vardi, 1995a; Halpern & Vardi, 1991). Note that in our setting, a Kripke structure is not explicitly part of the input, rather the underlying relational structure is compactly presented in terms of the valuation v and strategy s. This is the reason for PSPACE-hardness of the model checking problem.

6. We thank Paul Harrenstein for providing us an unpublished full version of (Ågotnes et al., 2013b) which includes a proof of Proposition 2. For the sake of completeness, we give a full proof of Theorem 35 in the Appendix.

6.1 Verification

In the rest of this section, we refer to valuations over various sets of variables and therefore find it convenient to use the following notations. Let A be a finite set of variables. We use $V(A)$ to denote the set of all valuations over A.

Theorem 36. *Given an observation game $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ and a uniform strategy profile $\mathbf{s} \in \mathbf{S}$, checking if $\mathbf{s} \in NE_{\mathsf{max}}(G)$ is PSPACE-complete.*

Proof. We can argue that the complement of the problem is in PSPACE. That is, given G and $\mathbf{s} \in \mathbf{S}$, the problem is to verify if $\mathbf{s} \notin NE_{\mathsf{max}}(G)$. This can be solved with the following two steps:

1. Guess $i \in N$, $v \in V$ and $s_i \in S_i$.
2. Verify that $u_i(v, (s_i, \mathbf{s}(v)_{-i})) > u_i(v, \mathbf{s}(v))$.

For step 1 note that the size of a strategy $|s_i| = \mathcal{O}(nk)$. So the triple (i, v, s) which forms a possible witness to the fact that $\mathbf{s} \notin NE_{\mathsf{max}}(G)$ has a polynomial representation. By Theorem 35 it follows that step 2 can be solved in PSPACE. Since PSPACE is closed under complementation and NPSPACE = PSPACE due to Savich's Theorem, the membership in PSPACE follows.

To show PSPACE-hardness, we give a reduction from the model checking problem for L^K. That is, given $\alpha \in L^K$, a strategy profile $s \in S$ and a valuation $v \in V$ we construct an observation game G and a uniform strategy $\mathbf{s}$ as follows. Let $P(\alpha)$ denote the set of variables occurring in α and $p_1 \in P(\alpha)$ be an arbitrary fixed variable. Let q be a variable such that $q \notin P(\alpha)$.

The set of players $N = \{1, 2\}$. $P_1 = P(\alpha)$ and $P_2 = \{q\}$. To define the goal formulas we make use of the following notations. Let δ_v denote the Boolean formula over P_1 which uniquely characterises the valuation v. That is, $\delta_v := \bigwedge_{p \in v} p \wedge \bigwedge_{p \notin v} \neg p$. For the (fixed) variable $p_1 \in P_1$, we define the formula $\mathit{flip}(p_1)$ as follows.

$$\mathit{flip}(p_1) = \begin{cases} Kw_2 q & \text{if } p_1 \notin s_1(2), \\ \neg Kw_2 q & \text{if } p_1 \in s_1(2). \end{cases}$$

The goal formulas are then defined as:

- $\gamma_1 = \delta_v \wedge (\alpha \vee \mathit{flip}(p_1))$
- $\gamma_2 = \top$.

Let $\mathbf{s}$ be any uniform strategy profile such that for all $w \in V(P_1 \cup \{q\})$ with $w \cap P_1 = v$ we have $\mathbf{s}(w) = s$. Now consider a $w \in V(P_1 \cup \{q\})$ such that $w \cap P_1 = v$.

Suppose $w, s \not\models \alpha$. By the definition of $\mathit{flip}(p_1)$, we have that $w, \mathbf{s}(w) \not\models \mathit{flip}(p_1)$ and thus $w, \mathbf{s}(w) \not\models \gamma_1$. Again, by the definition of $\mathit{flip}(p_1)$, there exists s_1' such that $w, (s_1', \mathbf{s}_{-1}(w)) \models \mathit{flip}(p_1)$ and therefore $u_1(w, (s_1', \mathbf{s}_{-1}(w))) > u_1(w, \mathbf{s}(w))$. Thus $\mathbf{s} \notin NE_{\mathsf{max}}(G)$.

Conversely, suppose $w, s \models \alpha$. Then for player 1, $w, \mathbf{s}(w) \models \gamma_1$. For all $w' \in V(P_1 \cup \{q\})$ such that $w' \cap P_1 \neq v$, for all $s_1' \in S_1$, $w', (s_1', \mathbf{s}_{-1}(w) \not\models \delta_v$ and therefore, $w', (s_1', \mathbf{s}_{-1}(w')) \not\models \gamma_1$. For player 2, for all valuations $u \in V(P_1 \cup \{q\})$, we have $u, \mathbf{s}(u) \models \gamma_2$. Therefore $\mathbf{s} \in NE_{\mathsf{max}}(G)$.

By Theorem 35 PSPACE-hardness follows, which gives the desired result. □

In the case of pessimist and optimist outcome relations, the following computational upper bounds for the verification question are relatively straightforward. Whether matching lower bounds can be shown is an interesting question.

Theorem 37. *Given an observation game $G = (N, (P_i)_{i \in N}, (\gamma_i)_{i \in N})$ and a uniform strategy profile $\mathbf{s} \in \mathbf{S}$, checking if $\mathbf{s} \in NE_{\mathsf{x}}(G)$ is in PSPACE where $\mathsf{x} \in \{\mathsf{pess}, \mathsf{opt}\}$.*

Proof. Observe that by Theorem 35, for $i \in N$, $\mathbf{s} \in \mathbf{S}$ and $v \in V$, checking if $u_i(v, \mathbf{s}(v)) = 1$ can be done in PSPACE. It follows that checking if $\max \mathbf{u}_i(v, \mathbf{s}) = 1$ (respectively, if $\min \mathbf{u}_i(v, \mathbf{s}) = 1$) can be checked in PSPACE. Therefore, to check if $\mathbf{s} \notin NE_{\mathsf{pess}}(G)$ (respectively, if $\mathbf{s} \notin NE_{\mathsf{opt}}(G)$), it suffices to perform the following two steps.

1. Guess a player i, a valuation v and a strategy $s'_i \in S_i$.

2. Verify if $\min \mathbf{u}_i(v, \mathbf{s}) < \min \mathbf{u}_i(v, (\dot{s}'_i, \mathbf{s}_{-i}))$,
(respectively, if $\max \mathbf{u}_i(v, \mathbf{s}) < \max \mathbf{u}_i(v, (\dot{s}'_i, \mathbf{s}_{-i})))$.

For step 1 note that the size of the strategy $|s_i| = \mathcal{O}(nk)$. Thus the triple (i, v, s_i) that forms a possible witness to the fact that $\mathbf{s} \notin NE_{\mathsf{pess}}(G)$ (respectively, $\mathbf{s} \notin NE_{\mathsf{opt}}(G)$), has a polynomial representation. By the observation above, step 2 can be solved in PSPACE. Since PSPACE is closed under complementation and NPSPACE = PSPACE, the membership in PSPACE follows. □

6.2 Emptiness

Next we address the complexity of checking for emptiness of maximal Nash equilibria in observation games. We find it useful to introduce the following definitions. Let $A = \{a_1, \ldots, a_l\}$ and $B = \{b_1, \ldots, b_l\}$ be two finite sets of variables where $|A| = |B|$ and let $\zeta : A \to B$ be a bijection. For valuations $v^1 \in V(A)$ and $v^2 \in V(B)$, we say that $cons_\zeta(v^1, v^2)$ holds if for all $j : 1 \leq j \leq l$, $a_j \in v^1$ iff $\zeta(a_j) \in v^2$. We also define the formula $\mathcal{C}_\zeta(A, B) := \wedge_{j=1}^{l}(a_j \leftrightarrow \zeta(a_j))$.

Given a uniform strategy $\mathbf{s}_i$ and a set $Z \subseteq P_i$, we say that $\mathbf{s}_i$ is *globally Z-uniform* if for all $v, v' \in V$, if $v \cap Z = v' \cap Z$ then $\mathbf{s}_i(v) = \mathbf{s}_i(v')$. For $i \in N$, let $\mathbf{S}_i^Z = \{\mathbf{s}_i \in \mathbf{S}_i \mid \mathbf{s}_i \text{ is globally } Z\text{-uniform}\}$. Note that $\mathbf{S}_i^Z$ can be viewed as a natural generalisation of $\mathbf{S}_i^g$ by parameterising the uniform strategies on the set Z.

An NEXPTIME-complete Problem. We now show that given an observation game G, checking if $NE_{\mathsf{max}}(G)$ is empty is NEXPTIME-complete. To prove the hardness, we give a reduction from the Dependency quantifier Boolean formula game (Dqbfg) (Hearn & Demaine, 2009, p.87). Dqbfg involves a three player game with players 1, 2 and 3. There are four finite sets of variables which are mutually disjoint, X_2, X_3, A_2 and A_3 along with a Boolean formula φ over the variables $X_2 \cup X_3 \cup A_2 \cup A_3$. Let $X = X_2 \cup X_3$ and $A = A_2 \cup A_3$. For the rest of this section we use L^B to denote the set of Boolean formulas over the variables $X \cup A$. Players' strategies are defined as follows.

- Player 1: a strategy $t_1 \in V(X)$.

- Player 2: a strategy $t_2 : V(X_2) \to V(A_2)$.

- Player 3: a strategy $t_3 : V(X_3) \to V(A_3)$.

In other words, a strategy for player 1 is to select a valuation for variables in X. Player 2 chooses a valuation for variables in A_2 and his strategy can depend on the valuation for variables in X_2. Similarly, a strategy for player 3 is to choose a valuation for variables in A_3 which can depend on the valuation of variables in X_3.

For player $i \in \{1,2,3\}$ let T_i denote the set of strategies of player i and T the set of strategy profiles. It is easy to observe that a strategy profile $t = (t_1, t_2, t_3)$ defines a valuation over the set of variables $X \cup A$. For a formula $\alpha \in L^B$ we then have the natural interpretation for $t \models \alpha$. Given strategies $t_2 \in T_2$ and $t_3 \in T_3$, we say that the pair (t_2, t_3) is a *winning strategy* for the coalition of players 2 and 3 if for all $t_1 \in T_1$, $(t_1, t_2, t_3) \models \neg\varphi$.

An instance of DQBFG is then given by the tuple $H = ((X_i)_{i\in\{2,3\}}, (A_i)_{i\in\{2,3\}}, \varphi)$ and the associated decision problem is to check if the coalition of players 2 and 3 have a winning strategy in H.

Theorem 38 ((Hearn & Demaine, 2009))**.** DQBFG *is NEXPTIME-complete.*

The reduction. Given an instance of DQBFG $H = ((X_i)_{i\in\{2,3\}}, (A_i)_{i\in\{2,3\}}, \varphi)$, we construct an observation game $G_H = (N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$ as follows. The set of players $N = \{1,2,3\}$. For $i \in \{2,3\}$, let Y_i be a copy of the variables in X_i, so $|Y_i| = |X_i|$ and let $Y = Y_2 \cup Y_3$. Let $P_1 = X$, $P_2 = A_2 \cup Y_2 \cup \{q\}$ and $P_3 = A_3 \cup Y_3 \cup \{r\}$. For the rest of this section, we use V and L^K to denote the set of all valuations and the set of all formulas over the variables in the observation game G_H respectively (so $V = V(X \cup Y \cup A \cup \{q, r\})$).

We also define the bijection $\zeta : X \to Y$ as the function that maps each variable in X_i to its corresponding copy in Y_i. Formally, let $X_1 = \{x_1^1, \dots, x_1^l\}$, $Y_1 = \{y_1^1, \dots, y_1^l\}$, $X_2 = \{x_2^1, \dots, x_2^h\}$ and $Y_2 = \{y_2^1, \dots, y_2^h\}$. Then $\zeta(x_1^j) = y_1^j$ for all $j \in \{1, \dots, l\}$ and $\zeta(x_2^j) = y_2^j$ for all $j \in \{1, \dots, h\}$. To simplify notation, we denote $cons_\zeta$ by $cons$ and $\mathcal{C}_\zeta$ by $\mathcal{C}$ for this fixed bijection ζ.

In order to define the goal formulas, we first inductively define a function $\lambda : L^B \to L^K$ that transforms φ to a formula in L_K as follows.

- For $p \in X$, $\lambda(p) := p$.
- For $p \in A_2$, $\lambda(p) := Kw_3 p$.
- For $p \in A_3$, $\lambda(p) := Kw_2 p$.
- $\lambda(\neg\alpha) := \neg\lambda(\alpha)$.
- $\lambda(\alpha_1 \vee \alpha_2) := \lambda(\alpha_1) \vee \lambda(\alpha_2)$.

Let $\psi_2 = (Kw_2 r \leftrightarrow \neg Kw_3 q)$ and $\psi_3 = (Kw_2 r \leftrightarrow Kw_3 q)$. Recall that Example 12 shows that already for the Kw fragment of observation games, NE_{max} need not always exist. Observe that the formulas ψ_2 and ψ_3 precisely correspond to γ_1 and γ_2 respectively as used in Example 12. We define the players' goal formulas as follows.

- $\gamma_1 = \top$.
- For $i \in \{2,3\}$, $\gamma_i = (\lambda(\neg\varphi) \vee \psi_i) \wedge \mathcal{C}(X_2, Y_2) \wedge \mathcal{C}(X_3, Y_3)$.

Properties of G_H. It is easy to see that the resulting observation game G_H is polynomial in the size of H. We first make the following observations about G_H.

Lemma 39. *Let G_H be the observation game corresponding to H and let $\mathbf{s} \in \mathbf{S}$. If there exists $v \in V$ such that $cons(v \cap X_1, v \cap Y_1)$, $cons(v \cap X_2, v \cap Y_2)$ and $v, \mathbf{s}(v) \models \lambda(\varphi)$ then $\mathbf{s} \notin NE_{\mathsf{max}}(G_H)$.*

Proof. Suppose there exists $v \in V$ such that $cons(v \cap X_1, v \cap Y_1)$, $cons(v \cap X_2, v \cap Y_2)$ and $v, \mathbf{s}(v) \models \lambda(\varphi)$. Then $v, \mathbf{s}(v) \models \mathcal{C}(X_2, Y_2) \wedge \mathcal{C}(X_3, Y_3)$. By Example 12, we have that there exists $i \in \{2, 3\}$ such that $v, \mathbf{s}(v) \not\models \psi_i$ and there exists $s_i \in S_i$ such that $v, (s_i, \mathbf{s}_{-i}(v)) \models \psi_i$. Therefore we have $u_i(v, \mathbf{s}(v)) < u_i(v, (s_i, \mathbf{s}_{-i}(v)))$. Thus $\mathbf{s} \notin NE_{\mathsf{max}}(G_H)$. □

Lemma 40. *For $i \in \{2, 3\}$, for all $s \in S$, for all $v, v' \in V$ such that $v \cap (X \cup Y) = v' \cap (X \cup Y)$ we have $v, s \models \gamma_i$ iff $v', s \models \gamma_i$.*

Proof. For $i \in \{2, 3\}$, the claim clearly holds for formulas ψ_i and $\mathcal{C}(X_i, Y_i)$. Thus for γ_i, the claim follows by a simple induction on φ. □

Next, we show that if the set of maximal Nash equilibria in G_H is non-empty then this set contains certain restricted types of maximal Nash equilibria.

Let $\mathbf{R}$ denote the set of uniform strategy profiles $\mathbf{s} \in \mathbf{S}$ that satisfy the following conditions:

- $\mathbf{s}_1 \in \mathbf{S}_1^g$,
- for $i \in \{2, 3\}$, $\mathbf{s}_i \in \mathbf{S}_i^{Y_i}$.

In other words, $\mathbf{R}$ consists of the set of all uniform strategy profiles $\mathbf{s}$ such that $\mathbf{s}_1$ is globally uniform and for $i \in \{2, 3\}$, s_i is globally Y_i-uniform.

Lemma 41. *If $NE_{\mathsf{max}}(G_H) \neq \emptyset$ then there exists $\mathbf{s}^* \in NE_{\mathsf{max}}(G_H)$ such that $\mathbf{s}^* \in \mathbf{R}$.*

Proof. For players $i \in \{2, 3\}$ we define an equivalence relation $\cong_i \subseteq V \times V$ as follows. For $v, v' \in V$, $v \cong_i v'$ if $v \cap Y_i = v' \cap Y_i$. For $v \in V$, let $[v]_i$ denote the equivalence class containing the valuation v and $c_v^i \in [v]_i$ denote a fixed valuation which is interpreted as the canonical representative element in the equivalence class $[v]_i$.

Suppose $\mathbf{s} \in NE_{\mathsf{max}}(G_H)$. Consider the uniform strategy profile $\mathbf{s}^* \in \mathbf{R}$ defined as follow.

- For player 1, fix a valuation $w \in V$ and let $\mathbf{s}_1^*(v) = \mathbf{s}_1(w)$ for all $v \in V$.
- For players $i \in \{2, 3\}$, for all $v \in V$, $\mathbf{s}_i^*(v) = \mathbf{s}_i(c_v^i)$.

We claim that $\mathbf{s}^* \in NE_{\mathsf{max}}(G_H)$. Suppose not, then there exists $i \in \{2, 3\}$, there exists $w \in V$, there exists $s_i \in S_i$ such that $u_i(w, (s_i, \mathbf{s}_{-i}^*(w))) > u_i(w, \mathbf{s}^*(w))$. Then $w, (s_i, \mathbf{s}_{-i}^*(w)) \models \gamma_i$ and $w, \mathbf{s}^*(w) \not\models \gamma_i$.

Now consider the valuation u defined as follows: $u \cap P_1 = w \cap P_1$ and for $i \in \{2, 3\}$, $u \cap P_i = c_w^i \cap P_i$. By definition of u we have that $u \cap (X \cup Y) = w \cap (X \cup Y)$ and therefore, $w \cong_i u$. From the definition of $\mathbf{s}^*$ it follows that $\mathbf{s}^*(w) = \mathbf{s}(u)$.

Since $w, (a_i, \mathbf{s}^*_{-i}(w)) \models \gamma_i$ we have that $w, (a_i, \mathbf{s}^*_{-i}(u)) \models \gamma_i$. By Lemma 40 we have that $u, (a_i, \mathbf{s}^*_{-i}(u)) \models \gamma_i$. Since $w, \mathbf{s}^*(w) \not\models \gamma_i$ we have that $w, \mathbf{s}^*(u) \not\models \gamma_i$. By Lemma 40 we have that $u, \mathbf{s}^*(u) \not\models \gamma_i$. However, this implies that $\mathbf{s}^* \notin NE_{\mathsf{max}}(G_H)$ which is a contradiction. $\square$

Strategy Translation. Note that by the construction of G_H, the strategies of player 1 are irrelevant in terms of existence of maximal Nash equilibria. Player 1 can ensure a utility of 1 by choosing any strategy. We now define two functions which translate strategies of players 2 and 3 between H and G_H. For the rest of the section we make use of the following concise notation. For $i = 2$, let $i^+ = 3$ and for $i = 3$, let $i^+ = 2$.

For $i \in \{2, 3\}$, let $\chi_i : T_i \to \mathbf{S}_i^{Y_i}$ be the function that translates every strategy t_i of player i in H to a globally Y_i-uniform strategy $\mathbf{s}_i = \chi_i(t_i)$ in G_H as defined below.

- For all $v \in V$, if $cons(v \cap X_i, v \cap Y_i)$ then $\mathbf{s}_i(v)(i^+) = t_i(v \cap X_i)$ and $\mathbf{s}_i(v)(i^+) = \emptyset$ otherwise. For all $v \in V$, $\mathbf{s}_i(v)(1) = \emptyset$ and $\mathbf{s}_i(v)(i) = P_i$.

For $i \in \{2, 3\}$, let $\mu_i : \mathbf{S}_i^{Y_i} \to T_i$ be the function that translates every globally Y_i-uniform strategy $\mathbf{s}_i$ of player i in H to a strategy $t_i = \mu_i(\mathbf{s}_i)$ in G_H as defined below.

- For all $v \in V$, such that $cons(v \cap X_i, v \cap Y_i)$, $t_i(v \cap X_i) = \mathbf{s}_i(v)(i^+)$.

Note that since $\mathbf{s}_i \in \mathbf{S}_i^{Y_i}$, μ_i is well defined.

Lemma 42. *For all $i \in \{2, 3\}$ and for all $\mathbf{s}_i \in \mathbf{S}^{Y_i}$, let $\mathbf{s}'_i = \chi_i(\mu_i(\mathbf{s}_i))$. For all $\mathbf{s}_1, \mathbf{s}'_1 \in \mathbf{S}_1$, for all $i \in \{2, 3\}$ and for all $v \in V$ such that $cons(v \cap X_i, v \cap Y_i)$ we have $v, (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)(v)) \models \gamma_i$ iff $v, (\mathbf{s}'_1, \mathbf{s}'_2, \mathbf{s}'_3)(v)) \models \gamma_i$.*

Proof. For $i \in \{2, 3\}$, the claim clearly holds for the formulas ψ_i and $\mathcal{C}(X_i, Y_i)$. Thus for γ_i, the claim follows by induction on φ. $\square$

Lemma 43. *For all $\alpha \in L^B$, for all $t \in T$, for all $i \in \{2, 3\}$ and for all $v \in V$ such that $t \cap X = v \cap X$ and $cons(v \cap X_i, v \cap Y_i)$ we have $t \models \alpha$ iff $v, (\mathbf{s}_1, \chi_2(t_2), \chi_3(t_3))(v)) \models \lambda(\alpha)$ for all $\mathbf{s}_1 \in \mathbf{S}_1$.*

Proof. For $i \in \{2, 3\}$, let $\mathbf{s}_i = \chi_i(t_i)$. The proof is by induction on the structure of α where the interesting cases involve the three base cases.

- $\alpha = p \in X$. Then we have $\lambda(p) = p$ and the following sequence of equivalences. $t \models p$ iff $p \in t_1$ iff $p \in v$ (since $t_1 \cap X = v \cap X$) iff $v, (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)(v)) \models p$ for all $\mathbf{s}_1 \in \mathbf{S}_1$.
- $\alpha = p \in A_2$. Then we have $\lambda(p) = Kw_3 p$ and the following sequence of equivalences. $t \models p$ iff $p \in t_2(t_1 \cap X_2)$ iff $p \in \mathbf{s}_2(v)(3)$ (since $cons(v \cap X_2, v \cap Y_2)$) iff $v, (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)(v)) \models Kw_3 p$ for all $\mathbf{s}_1 \in \mathbf{S}_1$.
- $\alpha = p \in A_3$. Then we have $\lambda(p) = Kw_2 p$ and the following sequence of equivalences. $t \models p$ iff $p \in t_3(t_1 \cap X_3)$ iff $p \in \mathbf{s}_3(v)(2)$ (since $cons(v \cap X_3, v \cap Y_3)$) iff $v, (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)(v)) \models Kw_2 p$ for all $\mathbf{s}_1 \in \mathbf{S}_1$.
- For $\alpha = \neg\alpha_1$ and $\alpha = \alpha_1 \vee \alpha_2$ the claim follows by a direct application of the induction hypothesis.

□

Lemma 44. *Let $H = ((X_i)_{i\in\{2,3\}}, (A_i)_{i\in\{2,3\}}, \varphi)$ be an instance of* DQBFG *and G_H the associated observation game. The coalition of players 2 and 3 have a winning strategy in H iff $NE_{\mathsf{max}}(G_H) \neq \emptyset$.*

Proof. Let (t_2, t_3) be a winning strategy for the coalition of players 2 and 3 in H. By definition of a winning strategy, for all $t_1 \in T_1$, we have $(t_1, t_2, t_3) \models \neg\varphi$. Let $t = (t_1, t_2, t_3)$ and consider the observation game G_H.

Note that in G_H, by the definition of player 1's goal γ_1, we have for all $v \in V$ and for all $\mathbf{s} \in \mathbf{S}$, $u_1(v, \mathbf{s}(v)) = 1$. Now consider an arbitrary valuation $v \in V$. There are two cases to consider.

Case 1. Suppose there exists $i \in \{2,3\}$ such that $cons(v \cap X_i, v \cap Y_i)$ does not hold. By semantics, for all $\mathbf{s} \in \mathbf{S}$ and for all $i \in \{2,3\}$ we have $v, \mathbf{s}(v) \not\models \mathcal{C}(X_2, Y_2) \wedge \mathcal{C}(X_3, Y_3)$ and thus $v, \mathbf{s}(v) \not\models \gamma_i$. Therefore, $u_i(v, \mathbf{s}(v)) = 0$.

Case 2. Suppose for all $i \in \{2,3\}$, $cons(v\cap X_i, v\cap Y_i)$ holds. By semantics we have for all $\mathbf{s} \in \mathbf{S}$, for all $i \in \{2,3\}$, $v, \mathbf{s}(v) \models \mathcal{C}(X_2, Y_2)\wedge\mathcal{C}(X_3, Y_3)$. Let $t'_1 = v\cap X$ and $t' = (t'_1, t_2, t_3)$. By definition of t', we have $t'\cap X = v\cap X$. Since (t_2, t_3) is a winning strategy for players 2 and 3 in H, we have $(t'_1, t_2, t_3) \models \neg\varphi$. By Lemma 43 $v, (\mathbf{s}_1, \chi_2(t_2), \chi_3(t_3))(v) \models \lambda(\neg\varphi)$.

Since the choice of v was arbitrary, we can conclude that $(\mathbf{s}_1, \chi_2(t_2), \chi_3(t_3)) \in NE_{\mathsf{max}}(G_H)$. In fact, note that the argument shows a stronger claim - for all $\mathbf{s}'_1 \in \mathbf{S}_1$, the uniform strategy profile $(\mathbf{s}'_1, \chi_2(t_2), \chi_3(t_3)) \in NE_{\mathsf{max}}(G_H)$.
$(\Leftarrow)$ Suppose $NE_{\mathsf{max}}(G_H) \neq \emptyset$. By Lemma 41, there exists a $\mathbf{s} \in \mathbf{R}$ such that $\mathbf{s} \in NE_{\mathsf{max}}(G_H)$. Let $(t_2, t_3) = (\mu_2(\mathbf{s}_2), \mu_3(\mathbf{s}_3))$. We argue that (t_2, t_3) is a winning strategy for the coalition of players 2 and 3 in H.

Suppose not, then there exists $t'_1 \in T_1$ such that for the strategy profile $t' = (t'_1, t_2, t_3)$, we have $t' \models \varphi$. Consider the pair of strategies $(\mathbf{s}'_2, \mathbf{s}'_3) = (\chi_2(t_2), \chi_3(t_3))$ and a valuation $v \in V$ such that $v \cap X = t' \cap X$ and for all $i \in \{2,3\}$, $cons(v \cap X_i, v \cap Y_i)$. By Lemma 43 we have that $v, (\mathbf{s}'_1, \mathbf{s}'_2, \mathbf{s}'_3) \models \lambda(\varphi)$ for all $\mathbf{s}'_1 \in \mathbf{S}_1$. In particular, $v, (\mathbf{s}_1, \mathbf{s}'_2, \mathbf{s}'_3) \models \lambda(\varphi)$. Since $\mathbf{s}'_i = \chi_i(\mu_i(\mathbf{s}_i))$ for $i \in \{2,3\}$, by Lemma 42 we have that $v, (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) \models \lambda(\varphi)$. By Lemma 39, $\mathbf{s} \notin NE_{\mathsf{max}}(G_H)$ which is a contradiction. □

Theorem 45. *Given an observation game G, checking if $NE_{\mathsf{max}}(G) \neq \emptyset$ is NEXPTIME-complete.*

Proof. Recall that for each player, a uniform strategy $\mathbf{s}_i$ can be encoded as a tuple of Boolean functions $(\mathbf{s}^j_i(p_i))_{j\in N, p_i\in P_i}$ each of which can be represented by a propositional formula $\beta^j_i(p_i)$ whose size is at most exponential in k. To show that the problem is in NEXPTIME, we first guess a uniform strategy profile $\mathbf{s}$. This involves guessing n^2k formulas each of which can be exponential in k. Membership in NEXPTIME then follows from Theorem 36.

By Lemma 44, it follows that checking if $NE_{\mathsf{max}}(G) \neq \emptyset$ is NEXPTIME-hard. Thus the claim follows. □

In the case of pessimist and optimist outcome relations, an argument similar to that given in the proof of Theorem 45 along with Theorem 37 immediately gives us an upper bound on the complexity of emptiness problem.

Theorem 46. *Given an observation game G, checking if $NE_{\mathsf{x}}(G) \neq \emptyset$ is in NEXPTIME where* $\mathsf{x} \in \{\mathsf{pess}, \mathsf{opt}\}$.

6.3 Knowing-Whether Observation Games

In the case of Kw games, we show that both the verification problem and the emptiness problem have "better" complexity bounds which match the known complexity results for the corresponding questions in Boolean games. We first recall the relevant results for Boolean games.

Theorem 47 ((Harrenstein, Turrini, & Wooldridge, 2017))**.** *(Verification) Given a Boolean game B along with a strategy profile v checking if $v \in NE(B)$ is co-NP-complete.*

Theorem 48 ((Bonzon et al., 2006))**.** *(Emptiness) Given a Boolean game B, checking if $NE(B) \neq \emptyset$ is Σ_2^p-complete.*

In the context of Kw games, as an immediate consequence of Proposition 5 we get that the model checking question for the fragment L^{Kw} is in polynomial time.

Corollary 49. *Given $\alpha \in L^{Kw}$ along with a strategy profile $s \in S$ and a valuation $v \in V$, checking if $v, s \models \alpha$ is in PTIME.*

We then have the following results for the complexity of verification and emptiness in Kw games.

Theorem 50. *Given a Kw game $G = (N, (P_i)_{i\in N}, (\gamma_i)_{i\in N})$ and a uniform strategy profile $\mathbf{s} \in \mathbf{S}$, checking if $\mathbf{s} \in NE_{\mathsf{max}}(G)$ is co-NP-complete.*

Proof. Membership in co-NP follows immediately from Corollary 49. For hardness, we show a reduction from the corresponding verification problem in Boolean games which is: given a Boolean game B and a strategy profile v in B, to check if $v \in NE(B)$. By Theorem 47 this problem is known to be co-NP-complete.

Given a Boolean game B and a strategy profile w in B, let G_B and $\dot{s}^w$ be the corresponding observation game and the globally uniform strategy profile in G_B as defined in Section 4.2. We argue that $w \in NE(B)$ iff $\dot{s}^w \in NE_{\mathsf{max}}(G_B)$.

($\Rightarrow$) This direction is exactly the same as the first part of the proof of Theorem 19. Suppose $w \in NE(B)$ and $\dot{s}^w \notin NE_{\mathsf{max}}(G_B)$. Then there exists $i \in N$, $v \in V$ and $t_i \in S_i$ such that $u_i(v, (t_i, \dot{s}^w{}_{-i}(v))) > u_i(v, \dot{s}^w(v))$. Let $w' = \chi^{-1}([\dot{t}_i, \mathbf{s}_{-i}])$ From Lemmas 14, 17 and 18 it follows that $u_i^B(w') = u_i(v, (\dot{t}_i, \mathbf{s}_{-i})(v)) > u_i(v, \dot{s}^w(v)) = u_i^B(w)$ for all $v \in V$. Therefore $w \notin NE(B)$ which is a contradiction.

($\Leftarrow$) Suppose $\dot{s}^w \in NE_{\mathsf{max}}(G_B)$ and $w \notin NE(B)$. Then there exists $i \in N$ and w_i' such that $u_i^B((w_i', w_{-i}) > u_i^B(w)$. Let $w' = (w_i', w_{-i})$. From Lemma 14 we have that $u_i(v, \dot{s}^{w'}(v)) = u_i^B(w') > u_i^B(w) = u_i(v, \dot{s}^w(v))$. This implies that $\dot{s}^w \notin NE_{\mathsf{max}}(G_B)$ which is a contradiction. $\square$

Theorem 51. *Given a Kw game G, checking if $NE_{\mathsf{max}}(G) \neq \emptyset$ is Σ_2^P-complete.*

Proof. Membership in Σ_2^P follows immediately from Corollary 16 and Theorem 50. For Σ_2^P-hardness, notice that the translation from observation games to Boolean games that we provide in Section 4.3 is polynomial time computable. Thus given an instance of an observation game G, we can construct a Boolean game B_G in polynomial time. By Theorem 24, $NE_{\mathsf{max}}(G) \neq \emptyset$ iff $NE(B_G) \neq \emptyset$. From Theorem 48 it follows that checking if $NE_{\mathsf{max}}(G) \neq \emptyset$ is Σ_2^P-complete. $\square$

7. Discussion and Conclusion

Summary. We introduced Boolean observation games as a qualitative model which combines aspects of imperfect and incomplete information games. For these games we studied Nash equilibria based on different ways to compare sets of outcomes, that result in different expectations of outcomes. Our main technical contributions are for the existence of Nash equilibria, for the computational analysis of Nash equilibria, as well as for identifying knowing-whether games, a fragment of observation games that precisely corresponds to Boolean games in terms of existence of Nash equilibria. A summary of our results are listed in Table 3.

		Existence			Complexity	
		$\|type(i)\| \leq 2$	$\|type(i)\| \leq 3$	$\|type(i)\| > 3$	Verification	Emptiness
Observation games	NE_{pess}	Yes (Theorem 28)	Yes (Theorem 28)	No (Example 26)	PSPACE (Theorem 37)	NEXPTIME (Theorem 46)
	NE_{opt}	Yes (Theorem 28)	Yes (Theorem 28)	—	PSPACE (Theorem 37)	NEXPTIME (Theorem 46)
	NE_{max}	Yes (Theorem 30)	No (Example 33)	No (Example 33)	PSPACE-complete (Theorem 36)	NEXPTIME-complete (Theorem 45)
Kw games	NE_{pess}	Yes (Theorem 28)	Yes (Theorem 28)	—	—	—
	NE_{opt}	Yes (Theorem 28)	Yes (Theorem 28)	—	—	—
	NE_{max}	Yes (Theorem 32)	Yes (Theorem 32)	No (Example 34)	Co-NP-complete (Theorem 50)	Σ_2^p-complete (Theorem 51)

Table 3: Summary of results.

Complexity and Existence. Note that in Boolean observation games, the underlying relational structure (the Kripke model) is not explicit. It is implicitly presented in terms of a valuation v and strategy profile s. Therefore, even the basic model checking problem is PSPACE-complete given the compact presentation. This in turn is one of the main reasons for the "high" complexity bounds that we obtain for the computational analysis of this model.

An alternative would be to explicitly have a Kripke model as part of the input. Suppose the Kripke model is defined over a set of worlds W. Then a uniform strategy can be thought of as a uniform function from W to the set S_i of strategies for player i, which would have a polynomial representation in terms of the number of worlds $|W|$, the number of agents n, and the number of variables (atoms) $|P|$. Computing $u_i(v, s(v))$ can then also be done in polynomial time. As a consequence it can be shown that the verification problem is in co-NP and the emptiness problem is in Σ_2^P. However, the size of the Kripke structure can in the worst case be exponential in $|P|$.

Clearly, the computational complexity of the model requires further analysis. There are two approaches which are interesting. The first is to try and identify fragments of the model which provide better complexity bounds. In the subclass of knowing-whether games we obtain bounds which match the known bounds for the corresponding questions in Boolean games. It is also known that in two player Boolean games where the goal formulas are restricted to Horn-renamable DNF, 2CNF or monotone CNF, the emptiness of Nash equilibrium can be checked in polynomial time (Bonzon et al., 2006). By modifying the arguments appropriately, we can identify subclasses of knowing-whether games in which the corresponding emptiness question can be solved in polynomial time. In general, any natural restriction of the logical specification language which results in the corresponding model checking question to have "better" complexity is a promising fragment.

The second approach would be to identify the specific parameters within the model which contribute to the exponential complexity bounds. Some of the natural candidates are the number of players and the number of variables used in the goal formulas. Since the hardness results given in Theorem 36 and Theorem 44 are for two and three players respectively, bounding the number of players alone is not sufficient. Analysis of fragments where the number of variables in the goal formulas are bounded appears to be a promising research direction which require more careful study.

Analysing the lower bounds in the case of pessimist and optimist outcome relations is another question which is relevant.

In Section 5.1, we analyse existence of Nash equilibria in knowing-whether games, and in Section 5.2 we identify conditions based on positive/negative epistemic assertions which ensure existence of Nash equilibria. Identifying other fragments where Nash equilibria are guaranteed to exist is an obvious direction of future research. It would be particularly interesting if the existence result can be related to structural properties of the underlying game.

Extensions of the Model. There are many extensions of the model which are interesting for further research. One could imagine a whole and ever widening range of qualitative incomplete information games of imperfect information. For the strategies, instead of merely revealing the value of propositional variables, we could consider revealing the value of any epistemic proposition, as already considered in (Ågotnes & van Ditmarsch, 2011; Ågotnes et al., 2011) for more complex, arbitrary, Kripke models. Instead of having merely partitions (exhaustive and exclusive) of all variables, one could consider overlapping sets of variables (exhaustive but not exclusive, so more than one player may observe the same variables)[7] as for example employed in (Belardinelli, Grandi, Herzig, Longin, Lorini, Novaro, & Perrussel, 2017). Doing the same for Boolean games would create the possibility of conflict, as not more than one player can control the value of a variable. But as many agents as you wish can make the same observation.

Another interesting extension to explore would be to consider iterated Boolean observation games, wherein players can gradually reveal more and more of their variables. This would be a generalization similar to that already studied for Boolean games in (Gutierrez

7. Kindly suggested by Paul Harrenstein.

et al., 2015, 2016). It would involve epistemic temporal goals or dynamic epistemic goals. Different from iterated Boolean games, in iterated Boolean observation games one can only reveal more and more variables in every round, until all have been revealed. This should therefore considerably reduce the complexity of iterated Boolean observation games with respect to otherwise comparable iterated Boolean games.

Yet another relevant direction is (epistemic) incentive engineering in Boolean observation games, similar to what is studied in Boolean games (Wooldridge, Endriss, Kraus, & Lang, 2013; Turrini, 2013; Harrenstein et al., 2017).

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Appendix A. Dynamic Epistemic Logic

A.1 Proof in Section 2.2

Proposition 5. *For all $\varphi \in L^{Kw}$, valuations v, and strategy profiles s: $v, s \models \varphi$ iff $s \models \varphi$.*

Proof. The proof is by induction on the structure of Kw formulas in negation normal form (L^{Kw}_{nnf}). The direction from right to left is by definition. For the direction from left to right we proceed as follows.

Case atom: $v, s \models Kw_i p_j$, iff (for all $w \sim^s_i v$, $w, s \models p_j$ iff $v, s \models p_j$), iff (for all w with $w \cap P_i(s) = v \cap P_i(s)$, $w, s \models p_j$ iff $v, s \models p_j$), iff $p_j \in P_i(s)$. As v no longer appears in the final statement, v is arbitrary. Therefore, the initial statement $v, s \models Kw_i p_j$ is equivalent to "for all $w \in V$, $w, s \models Kw_i p_j$," in other words, to $s \models Kw_i p_j$.

Case negated atom: $v, s \models \neg Kw_i p_j$, iff (there are $w, x \in V$ with $w \sim^s v$ and $x \sim^s v$ and such that $w, s \models p_j$ and $x, s \models \neg p_j$), iff (there are $w, x \in V$ with $w \cap P_i(s) = x \cap P_i(s) = v \cap P_i(s)$ and such that $w, s \models p_j$ and $x, s \models \neg p_j$), iff $p_j \notin P_i(s)$. As in the previous case, the final statement is independent from v and therefore the initial statement is equivalent to $s \models \neg Kw_i p_j$.

Case conjunction: $v, s \models \alpha \wedge \beta$, iff $v, s \models \alpha$ and $v, s \models \beta$, iff (IH) $s \models \alpha$ and $s \models \beta$, iff $s \models \alpha \wedge \beta$.

Case disjunction: $v, s \models \alpha \vee \beta$, iff $v, s \models \alpha$ or $v, s \models \beta$, iff (IH) $s \models \alpha$ or $s \models \beta$, iff $s \models \alpha \vee \beta$. □

A.2 Strategies as Epistemic Actions

In this section we compare our modelling and our results with related work in epistemic logic. We model strategy profiles as epistemic actions in a dynamic epistemic logic, where we also discuss an alternative semantics of strategies resulting in far larger models. The alternatives can be compared on their game theoretical implications, which may help to motivate our preference.

The situation wherein each player only observes the value of its own variables, corresponds to a Kripke model where the accessibility relation is the initial observation relation, and a strategy profile corresponds to an action model that, when executed in this Kripke model, results in an updated model wherein the accessibility relation is the observation relation (for that strategy profile). In this section we make precise how. It may serve to illustrate that our setting is very simple. This was why we were able to obtain modelling and computational results for Boolean observation games that are close or analogous to those for Boolean games.

An *epistemic model* (Kripke model) M is a triple $(W, \sim, \pi)$ where W is an (abstract) domain of *worlds* or *states*, where $\sim$ is a collection of equivalence relations on W, one for each agent, denoted $\sim_a$ (also known as *indistinguishability relations*), and where π is a *valuation* (function) mapping each state $w \in W$ to the subset of the propositional variables P that are true in that state. A pointed epistemic model (M, w) is a pair consisting of an epistemic model and a state $w \in W$.

Now consider the situation in our observation games where each of n players $1, \dots, n$ only observes the value of its own variables P_i, but before they enact/play a strategy s_i. We have implicitly modelled this as the strategy profile $s^\emptyset$ wherein no player reveals any variable. We can identify this situation with the following epistemic model.

> The *initial observation model* (IM, v), where $IM = (V, \sim, \pi)$, is such that:
>
> - domain V is the set of valuations of P ($V = \mathcal{P}(P)$);
> - for each player $i \in N$ and valuations $v, w \in V$, $v \sim_i w$ iff $v \cap P_i = w \cap P_i$;
> - for each $v \in V$, $\pi(v) = v$.

Note that the relations are exactly as in interpreted systems (Fagin, Halpern, Moses, & Vardi, 1995b).

Similarly, the result of playing strategy profile $s \in S$ given valuation $v \in V$ of observed variables, corresponds to an updated epistemic model.

> The *observation model* (IM^s, v), where $IM^s = (V, \sim^s, \pi)$, is such that V and π are as for IM, whereas in this case $v \sim_i^s w$ iff $v \cap P_i(s) = w \cap P_i(s)$.

We recall that $P_i(s) = \{p \in P \mid \text{there is a } j \in N \text{ with } p \in s_j(i)\}$, the variables revealed to i in s, where by definition $P_i(i) = P_i$ so that always $P_i \subseteq P_i(s)$.

Surely more interestingly, we can model a strategy profile as an independent semantic primitive namely as an *action model* U such that

$$v, s \models \varphi \text{ iff } IM \otimes U, (v, s) \models \varphi$$

where the former is the satisfaction relation in our logical semantics for L^K and the latter is the satisfaction relation in action model semantics. In order to establish that we first need to define action models and their execution (following details as in (Baltag, Moss, & Solecki, 1998; van Ditmarsch, van der Hoek, & Kooi, 2008; Moss, 2015)).

An *action model* U is a triple $(E, \approx, \mathsf{pre})$ where E is a domain of *actions*, for each player $i = 1, \dots, n$, $\approx_i$ is an equivalence relation on E, and pre is a *precondition* function mapping each action $e \in E$ to an executability precondition $\mathsf{pre}(e)$ that is a formula in some logical language L. The execution of an action model in an epistemic model $M = (W, \sim, \pi)$ is then defined as the restricted modal product $M \otimes U = (W', \sim', \pi')$ where $W' = \{(w, e) \mid w \in W, e \in E, M, w \models \mathsf{pre}(e)\}$, where $(w, e) \sim'_i (w', e')$ iff $w \sim_i w'$ and $e \approx_i e'$, and where $\pi'(w, e) = \pi(w)$.

In the case of strategy profiles for observation games, the logical language of action model preconditions can be restricted to L^B, the Booleans (the language required to describe preconditions is therefore simpler than the language L^K to describe epistemic goals), and a rather simple action model corresponds to a strategy profile s. A strategy profile can be identified with the following action model. In the definition, $\delta_v \in L^B$ is the description of the valuation v, defined as $\delta_v := \bigwedge_{p \in v} p \wedge \bigwedge_{p \notin v} \neg p$.

> A *strategy profile action model* U^s is a triple $(V, \sim^s, \mathsf{pre})$ where the set of actions is the set of valuations V, where for each $i = 1, \dots, n$, $v \sim^s_i w$ iff $v \cap P_i(s) = w \cap P_i(s)$, and where for each action $v \in V$, $\mathsf{pre}(v) = \delta_v$.

The domain of the strategy profile action model is therefore the same as the domain of an observation model, namely the set of all valuations.

In can be verified that

> $IM \otimes U^s$ is isomorphic to IM^s.

This is fairly elementary. We note that each action can only be executed in a single world — $IM, v \models \delta_v$, so that the size of IM^s is the same as the size of IM. Then, $(v, v) \sim_i (w, w)$ iff, by definition of action model execution, $v \sim_i v$ (in IM) and $v \sim^s_i w$ (in U^s), iff, by definition of these relations, $v \cap P_i = w \cap P_i$ and $v \cap P_i(s) = w \cap P_i(s)$. As the latter is a refinement of the former, the desired result that $v \cap P_i(s) = w \cap P_i(s)$ follows. Finally, $\pi'(v, v) = \pi(v) = v$. And the valuations π do not change.

In fact, already U^s is isomorphic to IM^s (slightly abusing the notion, but when we identify valuations with their description). It should be noted that it is common that action models are isomorphic to updated models when executed in initial models consisting of all valuations (and representing some sort of initial maximal ignorance over those valuations).

As a word of warning: the 'actions' that are the points in our action model U^s do **not** correspond to the strategies, that are sometimes also called actions. The action model 'action' combines the strategies of all players simultaneously, so they rather correspond to strategy profiles.

More Succinct Action Models. A slightly more succinct modelling of strategy profiles as action models is conceivable, that is a quotient of the action model U^s defined above with respect to variables that are not revealed by any player. Let us call this set $\overline{P^s}$, that is therefore defined as the complement of the set $P^s := \{p \in P \mid \exists i, j \in [1..n], i \neq j, p \in s_i(j)\}$. We can now redefine U^s_{small} as $(\mathcal{P}(P^s), \sim^s, \mathsf{pre})$ where in this case for any $v, w \subseteq P^s$ (so for *partial* valuations of atoms revealed by some agent only), $v \sim_i w$ iff $v \cap P_i(s) = w \cap P_i(s)$. This looks the same as before, but note that $P_i(s)$ may involve far more variables, namely in $\overline{P^s}$, than v and w, that are both restricted to P^s. Also, still $\mathsf{pre}(v) = v$ for all $v \in P^s$ (and where $\mathsf{pre}(\emptyset) = \top$ in case $P^s = \emptyset$).

Again, it is elementary to show that $IM \otimes U^s_{\mathsf{small}}$ is isomorphic to IM^s. We now have that $IM, w \models \mathsf{pre}(v)$ iff $v \subseteq w$. But in this case U^s_{small} is typically smaller than the resulting updated model IM^s. The resulting IM^s, as before, has the same domain as the initial model IM.

We now have, for example, that the action model corresponding to the 'reveal nothing' strategy profile $s^\emptyset$ is the trivial singleton action model $U^{s^\emptyset}_{\mathsf{small}}$ with precondition $\top$ (as $P^{s^\emptyset} = \emptyset$), and in this case $IM \otimes U^{s^\emptyset}_{\mathsf{small}}$ is isomorphic to the initial observation model IM again: the relations $\sim_i$ have not changed.

A.3 Strategies for Weaker Observations give Bigger Models

In our modelling, it is common knowledge to all players what variables have been revealed by who and to whom: the strategy profile s is common knowledge 'after the fact'. But, although I therefore know what variables are revealed by other players to yet other players, I still have not learnt the *values* of these variables.

> For example: After player 1 reveals atom p_1 to player 2 and atom q_1 to player 3, player 2 knows whether p_1 and player 3 knows whether q_1. Also, player 2 knows that player 3 knows whether q_1, and player 3 knows that player 2 knows whether p_1.

In a different modelling, each player *only* learns what variables have been revealed by other players to herself, and what variables she reveals to others.

> For example: After player 1 reveals atom p_1 to player 2 and atom q_1 to player 3, player 2 knows whether p_1 and player 3 knows whether q_1. However, player 2 does not know that player 3 knows whether q_1, and player 3 does not know that player 2 knows whether p_1. Player 2 also considers it possible that no variable has been revealed to 3, in which case 3 does not know whether q_1. And similarly for player 3.

So, clearly, depending on which modelling one prefers, different goal formulas γ of observation games may be satisfied, and it will therefore affect the existence of Nash equilibria and what the optimal strategies are.

Let us first formalize this as an action model, and let us be explicit about the (rather different) updated model as well. The strategies s_i and profiles $s = (s_1, \dots, s_n)$ remain the same, and thus also the $P_i(s)$, the set of atoms revealed to agent i. However, we can no longer define an updated observation model as one wherein only the indistinguishability relations have been changed, namely as $v \sim^s_i w$ iff $v \cap P_i(s) = w \cap P_i(s)$, while keeping the domain (and the valuation).

Instead of models consisting of valuations (domain V) we now need much larger models consisting of pairs (v, t) for valuations v and profiles t (domain $V \times S$) and define:

> For all $v, v' \in V$ and for all $s, t, t' \in S$ and for all players $i \in N$: $(v, t) \sim^s_i (v', t')$ if $v \cap P_i(s) = v' \cap P_i(s)$ [same valuation inasfar observed], $t_i = t'_i = s_i$ [same variables revealed to others], and $P_i(s) = P_i(t) = P_i(t')$ [same variables revealed by others to you].

As a consequence, we cannot describe the initial observation model as the one wherein $s^\emptyset$ is executed, because that would still blow up the model and introduce maximal uncertainty about what is revealed by who. So the initial observation model IM needs to be given separately (namely as the model already defined in Appendix A.2). However, once this is done, that is all. An action model can also be given for this modelling.

In this alternative modelling the players would remain far more ignorant about other players: optimist expected outcome would be more optimist, pessimist expected outcome would be more pessimist, realist expected outcome would quantify over a far larger set of possible outcomes. Basically, any epistemic feature is diluted. It therefore appeared to us that our preferred modelling provides more interesting results and variations.

Beyond that, the envisaged iterated Boolean observation games would become less meaningful for such strategies encoding weaker observations, as a player remains unaware of other players' increasing knowledge over such iterations, unless as a consequence of that player informing those other players.

Appendix B. Representation and Complexity

Theorem 35. *Given $\alpha \in L^K$ along with a strategy profile $s \in S$ and a valuation $v \in V$, checking if $v, s \models \alpha$ is PSPACE-complete.*

Proof. The membership in PSPACE is straightforward. For PSPACE-hardness, we give a reduction from Quantified Boolean Formula (QBF) which is a canonical PSPACE-complete problem (Papadimitriou, 1994). A QBF instance consists of a formula of the form $Q_1x_1Q_2x_2 \dots Q_nx_n\ \psi(x_1, x_2, \dots, x_n)$ where every Q_i is either a $\exists$ or $\forall$ quantifier, every x_i is a propositional variable and $\psi(x_1, x_2, \dots, x_n)$ is a Boolean formula over the variables $x_1, \dots, x_n$. From the definition, it follows that every QBF instance is either true or false (irrespective of the valuation under which it is evaluated).

Given an instance $\varphi = Q_1x_1Q_2x_2 \dots Q_nx_n\ \psi(x_1, x_2, \dots, x_n)$ of QBF, we associate with each variable x_i, a player i (thus $N = \{1, \dots, n\}$) and let $P = \{x_1, \dots, x_n\}$. We use the following notation introduced in Section 4.2: for $i = 1, \dots, n-1$ let $i^+ := i+1$ and $n^+ := 1$. For all $i \in N$, let $P_i = \{x_{i^+}\}$ and let s_i^* denote the strategy where player i reveals x_{i^+} to all players except player i^+. That is, $s_i^*(i^+) = \emptyset$ and $s_i^*(j) = P_i$ for all $j \neq i^+$.

Let $\alpha_\varphi \in L^K$ be the formula obtained from φ by replacing all occurrence of $\forall x_i$ by K_i and all occurrence of $\exists x_i$ by $\neg K_i \neg$. Let $v_\perp = \emptyset$ denote the valuation that assigns all variables the value false. We show that the QBF instance φ is true iff $v_\perp, s^* \models \alpha_\varphi$.

We first argue that for all QBF instances φ and for all valuations v over P, $v \models \varphi$ iff $v, s^* \models \alpha_\varphi$. The proof is by induction on the structure of φ and the non-trivial cases involve quantifiers. Suppose $\varphi = \forall x_i \psi$ so that $\alpha_\varphi = K_i \alpha_\psi$, then

$$\begin{array}{lll} v \models \forall x_i \psi & \text{iff} & \text{for all valuations } u \text{ where } u \cap (P \setminus \{x_i\}) = v \cap (P \setminus \{x_i\}),\ u \models \psi \\ & \text{iff} & \text{for all valuations } u \text{ where } u \cap (P \setminus \{x_i\}) = v \cap (P \setminus \{x_i\}),\ u, s^* \models \alpha_\psi \\ & \text{iff} & \text{for all } u \text{ where } u \sim_i^{s^*} v \text{ we have } u, s^* \models \alpha_\psi \\ & \text{iff} & v, s^* \models K_i \alpha_\psi. \end{array}$$

Since all variables in the QBF instance φ are bound, we have the following. φ is true iff $v_\perp \models \varphi$ iff $v_\perp, s^* \models \alpha_\varphi$. The claim then follows from the PSPACE-completeness of QBF. $\square$

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