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Best of Both Worlds: Agents with Entitlements

Martin Hoefer MHOEFER@CS.RWTH-AACHEN.DE
RWTH Aachen University, Germany
Marco Schmalhofer SCHMALHOFER@EM.UNI-FRANKFURT.DE
Goethe University Frankfurt, Germany
Giovanna Varricchio GIOVANNA.VARRICCHIO@UNICAL.IT
University of Calabria, Italy

Abstract

Fair division of indivisible goods is a central challenge in artificial intelligence. For many prominent fairness criteria including *envy-freeness* (EF) or *proportionality* (PROP), no allocations satisfying these criteria might exist. Two popular remedies to this problem are randomization or relaxation of fairness concepts. A timely research direction is to combine the advantages of both, commonly referred to as *Best of Both Worlds* (BoBW).

We consider fair division *with entitlements*, which allows to adjust notions of fairness to heterogeneous priorities among agents. This is an important generalization to standard fair division models and is not well-understood in terms of BoBW results. Our main result is a lottery for additive valuations and different entitlements that is ex-ante *weighted envy-free* (WEF), as well as ex-post *weighted proportional up to one good* (WPROP1) and *weighted transfer envy-free up to one good* (WEF$(1,1)$). We show that this result is tight – ex-ante WEF is incompatible with any stronger ex-post WEF relaxation.

In addition, we extend BoBW results on group fairness to entitlements and explore generalizations of our results to instances with more expressive valuation functions.

1. Introduction

Fair division of a set of indivisible goods is a prominent challenge at the intersection of economics and computer science. It has attracted a lot of attention over the last decades due to many applications in both simple and complex real-world scenarios. Formally, we face an allocation problem with finite sets $\mathcal{N}$ of n agents and $\mathcal{G}$ of m goods. Each agent $i \in \mathcal{N}$ has a valuation function $v_i : 2^{\mathcal{G}} \to \mathbb{R}_{\geq 0}$. The goal is to compute a "fair" and "efficient" allocation $\mathcal{A} = (A_1, \ldots, A_n)$, i.e., a partition of the goods among the agents, where agent i receives the bundle $A_i \subseteq \mathcal{G}$.

What is fair can certainly be a matter of debate. For this reason, several fairness criteria have been introduced and studied. *Envy-freeness* (EF) is probably one of the most intuitive concepts – it postulates that once goods are allocated no agent strictly prefers goods received by any other agent, i.e., $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in \mathcal{N}$. EF is a comparison-based notion. In contrast, there are also threshold-based ones such as *proportionality* (PROP). Allocation $\mathcal{A}$ is proportional if every agent receives a bundle whose value is at least her proportional share, i.e., $v_i(A_i) \geq v_i(\mathcal{G})/n$ for every $i \in \mathcal{N}$.

Unfortunately, for indivisible goods, neither PROP- nor EF-allocations may exist. Two natural conceptual remedies to this non-existence problem are (1) randomization or (2) relaxation of fairness concepts. Towards (1), a random allocation that is EF in expectation always exists (for every set of valuation functions): Select an agent uniformly at random and give the entire set of goods $\mathcal{G}$ to her. Then, however, every realization in the support is highly unfair – there is always an agent who receives everything, while all others get nothing. Moreover, it is easy to see that such an allocation might not even be Pareto-optimal. Towards (2), a well-known relaxation of EF is *envy-freeness up to one good* ($\mathsf{EF}1$) (Lipton, Markakis, Mossel, & Saberi, 2004; Budish, 2011): Every agent shall value her own bundle at least as much as any other agent's bundle after removing some good from the latter, i.e., for every $i, j \in \mathcal{N}$ there is $g \in \mathcal{G}$ such that $v_i(A_i) \geq v_i(A_j \setminus \{g\})$. Whenever the valuations of the agents are monotone, an $\mathsf{EF}1$ allocation always exists and can be computed in polynomial time (Lipton et al., 2004). However, different $\mathsf{EF}1$ allocations may advantage different agents. Similarly to $\mathsf{EF}1$, *proportionality up to one good* ($\mathsf{PROP}1$) has also been studied (Conitzer, Freeman, & Shah, 2017).

A timely research direction is to combine the advantages of both randomization and relaxation, commonly referred to as *Best of Both Worlds* (BoBW) results. An important result was obtained by both Aziz (2020) and Freeman, Shah, and Vaish (2020) for additive valuations – a lottery that is EF in expectation (ex-ante) and $\mathsf{EF}1$ for every allocation in the support (ex-post). Moreover, it can be implemented as a lottery over deterministic allocations with polynomial-sized support. Both papers generalize the Probabilistic Serial (PS) rule by Bogomolnaia and Moulin (2001) for the matching case, when there are n agents and $m = n$ goods. PS is ex-ante EF. By the Birkhoff-von Neumann decomposition, it can be represented as a lottery over polynomially many deterministic allocations. Furthermore, any allocation in the support assigns to each agent exactly one good. This implies ex-post $\mathsf{EF}1$. Both (Aziz, 2020; Freeman et al., 2020) generalize the application of the Birkhoff-von Neumann decomposition to instances with arbitrarily many goods.

In our work, we consider a more general framework to allow more flexibility in the definition of fairness. Concepts like EF or PROP imply that all agents are symmetric, i.e., they are ideally treated as equals. In many scenarios, however, there is an inherent asymmetry in the agent population. Alternatively, it can be beneficial for an allocation mechanism to have the option to reward certain agents. We follow the formal framework of *entitlements* (Chakraborty, Igarashi, Suksompong, & Zick, 2021a; Aziz, Moulin, & Sandomirskiy, 2020) that enables increased expressiveness. Formally, each agent $i \in \mathcal{N}$ now has a weight (or priority) $w_i > 0$. Fairness notions like EF or PROP are then refined based on the weights (see Section 2 for formal definitions). Generally, we will use a prefix "W" to refer to a fairness concept in the context of entitlements.

1.1 Our Contribution

We study lotteries satisfying both ex-ante and ex-post fairness guarantees for additive valuations and different entitlements. As main result, we provide a lottery that is ex-ante *weighted stochastic-dominance envy-free* ($\mathsf{WSD\text{-}EF}$) and consequently ex-ante WEF. Differently from (Aziz, 2020; Freeman et al., 2020), we make use of a stronger decomposition theorem by Budish, Che, Kojima, and Milgrom (2013) and show it is possible to achieve ex-post $\mathsf{WPROP}1$ and $\mathsf{WEF}(1,1)$. The latter means that in every allocation $\mathcal{A}$ in the support, weighted envy from agent i towards j can be eliminated by moving entirely one good from A_j to A_i. Perhaps surprisingly, our result is tight – we show that ex-ante WEF is

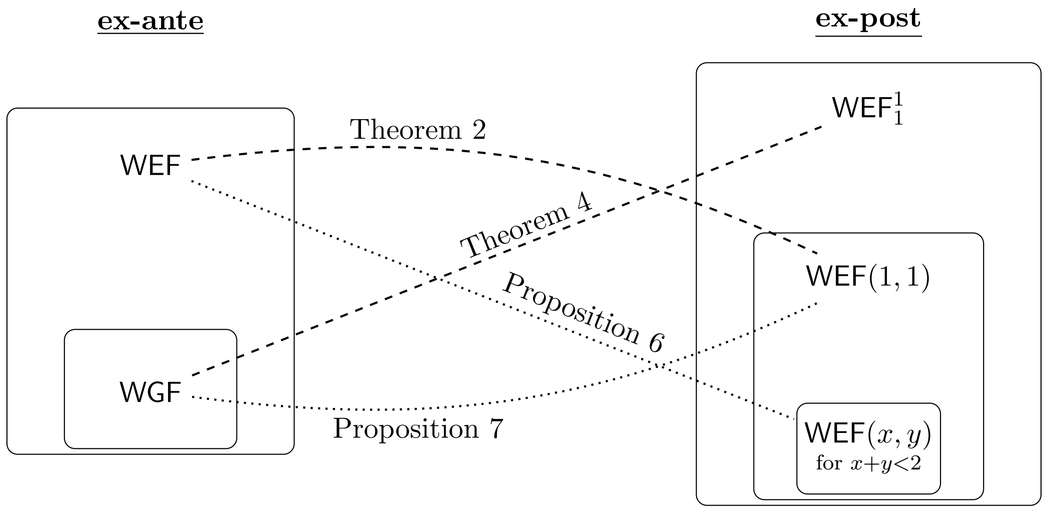

Figure 1: Summary of our results on additive valuations. Positive results are shown as dashed lines, negative results as dotted. Rectangles represent set inclusion. All our algorithms also guarantee WPROP1 ex-post.

incompatible with any ex-post fairness notion stronger than $\mathsf{WEF}(1,1)$. In particular, a direct extension of (Aziz, 2020; Freeman et al., 2020) to a lottery with ex-ante WEF and ex-post WEF1 is impossible.

Freeman et al. (2020) investigate further combinations of ex-ante and ex-post properties; namely, they provide a lottery that is ex-ante *group fair* (GF) as well as ex-post PROP1 and EF_1^1. In an EF_1^1 allocation $\mathcal{A}$, we can eliminate envy from i to j when we remove one good from A_j and add one good to A_i; differently from $\mathsf{EF}(1,1)$, the good added to A_i is *not required* to come from A_j. We prove that this result can be adapted to hold also for instances with entitlements, i.e., we show that ex-ante WGF together with ex-post WEF_1^1 is possible. We again show tightness in the sense that the ex-ante fairness guarantee is incompatible with any stronger ex-post fairness notion. Figure 1 gives an overview of our results for additive valuations.

Finally, we expand the scope of BoBW towards more general valuations. We circumvent the need to evaluate fractional allocations and the usage of fractional extensions of valuation functions. Note that for non-additive valuations the expected utility of an agent in a given lottery is usually not uniquely determined by the marginal probabilities (i.e., the fractional allocation) of individual goods – it depends on entire bundles in the support of the lottery.

For equal entitlements, ex-ante EF and ex-post EF1 is possible in more general cases. For different entitlements, ex-ante WEF and ex-post $\mathsf{WEF}(1,1)$ or WPROP1 are no longer compatible (even for two agents, one additive and one unit-demand). For this reason, we focus on threshold-based guarantees – we show that it is possible to construct a lottery that is ex-ante WPROP and ex-post WPROP1, even for XOS valuations.

1.2 Related Work

Fair division attracted an enormous amount of attention, and there is a large number of surveys. We refer to a rather recent one by Amanatidis, Birmpas, Filos-Ratsikas, and Voudouris (2022) and restrict attention to more directly related works.

An orthogonal direction is pursued by Caragiannis, Kanellopoulos, and Kyropoulou (2021) who introduce interim EF, a trade-off between ex-ante and ex-post EF. Interim EF requires that agent i, knowing the realization of her bundle A_i, obtains more value from A_i than the (conditional) expected value of the bundle of any other agent j. Formally, $v_i(S) \geq \mathbb{E}\left[v_i(A_j) \mid A_i = S\right]$ for each $S \subseteq \mathcal{G}$ such that $\mathbb{P}(A_i = S) > 0$. Interim-EF implies ex-ante EF, and ex-post EF implies interim EF. Furthermore, any interim EF allocation is ex-post PROP, which implies that interim EF allocations may not exist. In the matching case, a cleverly crafted separation oracle for a linear program and an extension of the Birkhoff-von Neumann decomposition can be combined to design a polynomial-time algorithm that either provides an interim EF allocation or determines it does not exist.

When agents are endowed with ordinal preferences rather than cardinal valuation functions, stochastic-dominance envy-freeness is the most prominent fairness notion for lotteries. It was first considered by Bogomolnaia and Moulin (2001) and later systematically studied by Aziz, Gaspers, Mackenzie, and Walsh (2014).

Feldman, Mauras, Narayan, and Ponitka (2023) consider BoBW lotteries in instances where agents have subadditive valuations, a generalization of additive. The authors provide a lottery guaranteeing $\frac{1}{2}$-EF ex-ante and a $\frac{1}{2}$-approximation of envy-freeness up to any good (EFX) as well as $\mathsf{EF1}$ ex-post.

For the Maximin-share (MMS), Babaioff, Ezra, and Feige (2022) design a lottery simultaneously achieving ex-ante PROP and ex-post $\mathsf{PROP1} + \frac{1}{2}$-$\mathsf{MMS}$.

For the model with entitlements, there is a focus on characterizing picking sequences to guarantee fairness properties (Chakraborty et al., 2021a; Chakraborty, Schmidt-Kraepelin, & Suksompong, 2021b; Chakraborty, Segal-Halevi, & Suksompong, 2022), on the problem of maximizing Nash social welfare (Garg, Kulkarni, & Kulkarni, 2020; Suksompong & Teh, 2022; Garg, Husic, Li, Végh, & Vondrák, 2023), and on the definition of appropriate shares (Farhadi, Ghodsi, Hajiaghayi, Lahaie, Pennock, Seddighin, Seddighin, & Yami, 2019; Babaioff, Ezra, & Feige, 2021).

Another recent line of research in fair division focuses on chores and mixed manna. There, items either have negative utility for all the agents (in case of chores), or can have positive and negative utility for different agents (mixed manna). In the chores setting, there has been research on picking sequences (Feige & Huang, 2023) and weighted $\mathsf{EF1}$ (Wu, Zhang, & Zhou, 2023).

Parts of this paper appeared in the proceedings of AAMAS 2023 (Hoefer, Schmalhofer, & Varricchio, 2023). Our existence results for additive valuation functions in Theorems 2 and 4 were discovered simultaneously by Aziz, Ganguly, and Micha (2023).

1.3 Outline

In Section 2 we formally describe the model and our notation as well as known results from the literature that are useful for our analysis. The above mentioned generalization of the Birkhoff-von Neumann Theorem by Budish et al. (2013) is discussed in Section 3.

In particular, while the result per se only provides an *algorithm to efficiently sample* from the distribution, Carathéodory's theorem can be used to compute a *representation of the entire* underlying lottery in strongly polynomial time. Our approach is a simplification of arguments in (Freeman et al., 2020, Section 3.2). In Section 4 we present our main results for additive valuations, depicted in Figure 1. Finally, in Section 5 we investigate to which extent our and previous BoBW results can be achieved for more general valuation functions, namely, XOS, multi-demand, and cancelable valuations.

2. Preliminaries

A fair division instance $\mathcal{I}$ is given by a triple $(\mathcal{N}, \mathcal{G}, \{v_i\}_{i\in\mathcal{N}})$, where $\mathcal{N}$ is a set of n agents and $\mathcal{G}$ is a set of m indivisible goods. Every agent $i \in \mathcal{N}$ has a valuation function $v_i : 2^{\mathcal{G}} \to \mathbb{R}^+$, where $v_i(A)$ represents the value, or utility, of i for the bundle $A \subseteq \mathcal{G}$. We assume that valuations are monotone ($v(A) \le v(B)$ for $A \subseteq B$) and normalized ($v(\emptyset) = 0$).

Classes of Valuations. For each $i \in \mathcal{N}$ and $g \in \mathcal{G}$, $v_i(g) \ge 0$ represents the value i assigns to the good g. A valuation v_i is *additive* if $v_i(A) = \sum_{g\in A} v_i(g)$ for each bundle $A \subseteq \mathcal{G}$.

A valuation function v_i is *multi-demand* if there exists $k \in \mathbb{N}$ such that, for every $A \subseteq \mathcal{G}$, $v_i(A)$ is given by the sum of the k most valuable goods in A for i, formally $v_i(A) = \max\limits_{M\subseteq A, |M|\le k} \sum_{g\in M} v_i(g)$. If $k = 1$ we talk about *unit-demand* valuations.

A valuation function v_i is *cancelable* if $v_i(S\cup\{g\}) > v_i(T\cup\{g\}) \Rightarrow v_i(S) > v_i(T)$, for all $S, T \subseteq \mathcal{G}$ and $\forall g \in \mathcal{G} \setminus (S \cup T)$. Cancelable valuations generalize several classes studied in the literature, e.g., additive, weakly-additive, budget-additive, product, and unit-demand (see, e.g., Berger, Cohen, Feldman, & Fiat, 2022).

A valuation function v_i is *XOS* if there is a family of additive set functions $\mathcal{F}_i$ such that $v_i(A) = \max_{f\in\mathcal{F}_i} f(A)$. XOS valuations generalize additive and submodular ones (while the latter also include all multi-demand ones). Note that there is no direct inclusive relation between XOS and cancelable valuations.

Remark: In what follows, whenever we sort the goods in $\mathcal{G}$ according to the valuation function of a specific agent, we assume ties are broken according to a fixed ordering of $\mathcal{G}$. This serves to avoid technical and tedious tie-breaking issues.

Entitlements. We study fair division with entitlements. Each agent $i \in \mathcal{N}$ is endowed with an *entitlement*, also called *weight*, $w_i > 0$. For convenience, we assume w.l.o.g. $\sum_{i\in\mathcal{N}} w_i = 1$. We say that agents have *equal entitlements* if $w_i = \frac{1}{n}$, for all $i \in \mathcal{N}$, and refer to this as the *unweighted setting.*

We next give an example of a fair division instance with entitled and additive agents.

Example 1 (A fair division instance $\mathcal{I}^*$ with entitlements)**.** We outline an instance $\mathcal{I}^*$ given by $(\mathcal{N}, \mathcal{G}, \{v_i\}_{i\in\mathcal{N}})$ and entitlements $w_i > 0$ for each $i \in \mathcal{N}$. The agents are $\mathcal{N} = \{1, 2, 3\}$, the goods $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$, and $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{3}$ and $w_3 = \frac{1}{6}$ are the entitlements of agent $1, 2$ and 3, respectively. The valuation functions are additive with values of the agents for single goods shown in Table 1.

Throughout the paper, we use $\mathcal{I}^*$ to refer to this particular example instance. ■

g	g_1	g_2	g_3	g_4
$v_1(g)$	8	8	5	2
$v_2(g)$	3	5	4	1
$v_3(g)$	4	7	6	2

Table 1: Agents' valuations in $\mathcal{I}^*$

2.1 Weighted Fairness Notions

An allocation $\mathcal{A} = (A_1, \dots, A_n)$ is a partition of the set of goods $\mathcal{G}$ among the agents, i.e., A_i is the bundle given to agent i, $A_i \cap A_j = \emptyset$, for each $i \neq j$, and $\cup_{i \in \mathcal{N}} A_i = \mathcal{G}$. An allocation $\mathcal{A}$ is *weighted proportional* (WPROP) if $v_i(A_i) \geq w_i \cdot v_i(\mathcal{G})$ for each $i \in \mathcal{N}$. An allocation $\mathcal{A}$ is *weighted envy-free* (WEF) if for each $i, j \in \mathcal{N}$,

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j)}{w_j} \ .$$

Since goods are indivisible such allocations may not always exist, and relaxed versions have been defined. An allocation $\mathcal{A}$ is *weighted proportional up to one good* (WPROP1) if for each $i \in \mathcal{N}$ there exists $g \in \mathcal{G}$ such that $v_i(A_i \cup \{g\}) \geq w_i \cdot v_i(\mathcal{G})$. For additive valuations, WEF $\Rightarrow$ WPROP, but – differently from equal entitlements – WEF1 $\not\Rightarrow$ WPROP1.

Concerning envy-freeness, we have already discussed EF and EF1 in the introduction. We here work with a broader definition that generalizes these notions.

Definition 1 (WEF(x, y))**.** For $x, y \in [0, 1]$, an allocation $\mathcal{A}$ is called WEF(x, y) if for each $i, j \in \mathcal{N}$ either $A_j = \emptyset$ or there exists $g \in A_j$ such that

$$\frac{v_i(A_i) + y \cdot v_i(g)}{w_i} \geq \frac{v_i(A_j) - x \cdot v_i(g)}{w_j} \ .$$

The definition of WEF(x, y) was introduced by Chakraborty et al. (2022). It is meaningful mostly for additive valuations. For general valuations, the idea of WEF$(1, 1)$ can be expressed by $w_j \cdot v_i(A_i \cup \{g\}) \geq w_i \cdot v_i(A_j \setminus \{g\})$, and analogously for WEF$(0, 1)$ and WEF$(1, 0)$. Conceptually, WEF$(1, 0)$ coincides with a notion of *weighted envy-freeness up to one good* (WEF1) while WEF$(0, 1)$ with a notion termed *weak weighted envy-freeness up to one good* (WWEF1). WEF$(1, 1)$ has also been called *transfer envy-freeness up to one good* in the unweighted setting (Chakraborty et al., 2021a). Note that in WEF$(1, 1)$ the good g added to A_i must come from A_j. Assuming that g may come from any other bundle leads to the following (weaker) notion introduced in (Barman & Krishnamurthy, 2019).

Definition 2 (WEF$^1_1$)**.** An allocation $\mathcal{A}$ is called *weighted envy-free up to one good more and less* (WEF$^1_1$) if for each $i, j \in \mathcal{N}$ there exist $g_i, g_j \in \mathcal{G}$ such that $w_j \cdot v_i(A_i \cup \{g_i\}) \geq w_i \cdot v_i(A_j \setminus \{g_j\})$.

We move on to fairness concepts for fractional allocations. A *fractional allocation* $X = (x_{ig})_{i \in \mathcal{N}, g \in \mathcal{G}} \in [0, 1]^{n \times m}$ specifies the fraction of good g that agent i receives. We assume fractional allocations are complete, i.e., $\sum_{i \in \mathcal{N}} x_{ig} = 1$ for every $g \in \mathcal{G}$.

Group fairness was first introduced by Conitzer, Freeman, Shah, and Vaughan (2019) and extended to fractional allocations by Freeman et al. (2020). Towards extending group fairness to weighted agents, consider a subset of agents $S \subseteq \mathcal{N}$. We define $w_S = \sum_{i \in S} w_i$ as the weight of the set, and $\cup_{j \in S} X_j = \left(\sum_{j \in S} x_{jg}\right)_{g \in \mathcal{G}}$ as the total fraction of each good $g \in \mathcal{G}$ assigned to the agents of S.

Definition 3 (WGF)**.** A fractional allocation X is *weighted group fair* (WGF) if for all non-empty subsets of agents $S, T \subseteq \mathcal{N}$, there is no fractional allocation X' of $\cup_{j \in T} X_j$ to the agents in S such that $w_S \cdot v_i(X'_i) \geq w_T \cdot v_i(X_i)$, for all $i \in S$ and at least one inequality is strict.

Similarly to the unweighted setting, weighted group fairness implies other (weighted) envy and efficiency notions, for example, WEF (if $|S| = |T| = 1$), WPROP (if $|S| = 1, T = \mathcal{N}$), and Pareto-optimality (if $S = T = \mathcal{N}$).

We finally focus on stochastic dominance, a standard fairness notion for random allocations. For convenience, we first define it using fractional allocations. Given any $i \in \mathcal{N}$, let us denote by X_i and X'_i the fractional bundles of agent i in the allocations X and X', respectively. Agent i SD prefers X_i to X'_i, written $X_i \succeq_i^{\mathsf{SD}} X'_i$, if for any $g^* \in \mathcal{G}$,

$$\sum_{g \,:\, v_i(g) \geq v_i(g^*)} x_{ig} \geq \sum_{g \,:\, v_i(g) \geq v_i(g^*)} x'_{ig} \ ,$$

where x_{ig} and x'_{ig} represents the fraction of g assigned to i in the two fractional allocations.

We say $X_i \succ_i^{\mathsf{SD}} X'_i$, if $X_i \succeq_i^{\mathsf{SD}} X'_i$ and not $X'_i \succeq_i^{\mathsf{SD}} X_i$. Notice that $\{g \mid v_i(g) \geq v_i(g^*)\}$ is the set of goods that i likes at least as much as i likes g^*. We defined it by means of cardinal values of v_i, but the set is determined already by the ordinal valuation of single goods for i.

Definition 4 (SD-EF and WSD-EF)**.** A fractional allocation X is SD-*envy-free* (SD-EF) if for all $i, j \in \mathcal{N}$, $X_i \succeq_i^{\mathsf{SD}} X_j$. Similarly, we say X is WSD-*envy-free* (WSD-EF) if for all $i, j \in \mathcal{N}$, $w_j \cdot X_i \succeq_i^{\mathsf{SD}} w_i \cdot X_j$.

Put differently, X is WSD-EF if for all $i, j \in \mathcal{N}$, and all $k \in [m]$,

$$w_j \cdot \sum_{g \in G_k} x_{ig} \geq w_i \cdot \sum_{g \in G_k} x_{jg} \ ,$$

where G_k is the set of the k most valued goods of agent i.

Now we extend this definition towards random allocations. Consider a probability distribution D over a set of deterministic allocations. If an allocation $\mathcal{A}$ is chosen randomly according to D, then the matrix of marginal assignment probabilities $X = (x_{ig})_{i \in \mathcal{N}, g \in \mathcal{G}}$, where x_{ig} is the probability that i gets assigned good g, forms a fractional allocation. We say that random allocation $\mathcal{A}$ is WSD-EF if the corresponding X is WSD-EF. In Proposition 4 we show that WSD-EF implies WEF for additive valuations.

2.2 Picking Sequences

For additive valuations and equal entitlements, a straightforward round-robin algorithm yields an $\mathsf{EF1}$ allocation. Clearly, when agents have different entitlements, the round-robin algorithm might no longer provide a $\mathsf{WEF1}$ allocation. Different entitlements impose different priorities among agents, which has resulted in the consideration of picking sequences.

A *picking sequence* for n agents and m goods is a sequence $\pi = (i_1, \ldots, i_m)$, where $i_h \in \mathcal{N}$, for $h = 1, \ldots, m$. An allocation $\mathcal{A}$ is the result of the picking sequence π if it is the output of the following procedure: Initially every bundle is empty; then, at time step h, i_h inserts in her bundle the most preferred good among the available ones. Once a good is selected, it is removed from the set of available goods.

In (Chakraborty et al., 2021b, 2022), the authors characterize picking sequences for several fairness criteria, including $\mathsf{WEF1}$, $\mathsf{WWEF1}$ as well as $\mathsf{WEF}(x, 1-x)$. For our purposes, we will rely on the following characterization for $\mathsf{WEF}(x, y)$ (in the context of additive valuations).

Proposition 1. *Let t_i, t_j be the number of picks of agents i, j, respectively, in a prefix of π. A picking sequence π is $\mathsf{WEF}(x, y)$ if and only if for every prefix of π and every pair of agents i, j, we have $\frac{t_i + y}{w_i} \geq \frac{t_j - x}{w_j}$.*

Chakraborty et al. (2022) prove this proposition using the assumption $x + y = 1$, since $\mathsf{WEF}(x, y)$ allocations might not exist for $x + y < 1$. The proof can be easily extended beyond that assumption to show the statement for all $x, y \in [0, 1]$. For completeness, we provide a full proof below.

Round-robin is not the only picking sequence achieving $\mathsf{EF1}$ for equal entitlements. Any picking sequence that is recursively balanced (RB), i.e., $|t_i - t_j| \leq 1$ in any prefix of π, results in an $\mathsf{EF1}$ allocation (Aziz, 2020; Freeman et al., 2020).

Proof of Proposition 1. Our proof very closely follows the arguments for Theorem 3.2 in Appendix A.2 of the full version of (Chakraborty et al., 2022). We point out that our adapted proof requires only a minor modification in the inductive proof of Inequality (3).

($\Rightarrow$) Assume that π fulfills $\mathsf{WEF}(x, y)$. Since the $\mathsf{WEF}(x, y)$ condition must be satisfied for every instance, we can choose a special one that forces the utility of every agent to equal her number of picks up to a certain point. Consider a prefix of π. Every agent values each good which has been picked so far with 1, and the remaining ones with 0. If $t_j = 0$, the claim trivially holds. Otherwise, $\mathsf{WEF}(x, y)$ gives us existence of $g \in A_j$ such that

$$\frac{v_i(A_i) + y \cdot v_i(g)}{w_i} \geq \frac{v_i(A_j) - x \cdot v_i(g)}{w_j} \ .$$

Plugging in $v_i(g) = 1$ and $v_i(A_i) = t_i$ as well as $v_i(A_j) = t_j$ yields the claim.

($\Leftarrow$) Consider any two agents i, j. We show that the $\mathsf{WEF}(x, y)$ condition for agent i towards agent j is fulfilled after every pick of j. Consider the t_j-th pick of agent j. We divide the sequence of picks up to this point into *phases*, where each phase $\ell \in \{1, \ldots, t_j\}$ consists of the picks after agent j's $(\ell - 1)$-th pick up to (and including) the agent's ℓ-th pick. We use the following notation:

- $\tau_\ell :=$ the number of times agent i picks in phase ℓ (i.e., between agent j's $(\ell-1)$-th and ℓ-th picks),
- $\alpha_\ell :=$ the total utility gained by agent i in phase ℓ,
- $\beta_\ell :=$ agent i's utility for the good that agent j picks at the end of phase ℓ.

Let $\rho := w_i/w_j$. For any integer $s \in [t_j]$, applying the condition in the theorem statement to the picking sequence up to and including phase s, we have

$$y + \sum_{\ell=1}^{s} \tau_\ell \;\geq\; \rho(s-x) \;. \tag{1}$$

Every time agent i picks, she picks a good with the highest value for her. In particular, in each phase ℓ, she picks τ_ℓ goods each of which gives at least as high value to her as each good picked by agent j after (and including) phase ℓ. Hence for all phases $\ell \in \{1, \ldots, t_j\}$,

$$\alpha_\ell \geq \tau_\ell \cdot \max_{\ell \leq r \leq t_j} \beta_r \;. \tag{2}$$

To show the claim, we prove the following inequality for all $s \in [t_j]$:

$$\begin{aligned} y \cdot \max_{1 \leq r \leq t_j} \beta_r + \sum_{\ell=1}^{s} \alpha_\ell \geq \rho \left(\sum_{\ell=1}^{s} \beta_\ell - x\beta_1 \right) \\ + \left(y + \sum_{\ell=1}^{s} \tau_\ell - \rho(s-x) \right) \max_{s \leq r \leq t_j} \beta_r \;. \end{aligned} \tag{3}$$

We prove (3) by induction on s. The base case $s = 1$ can be obtained by setting $\ell = 1$ in (2) and adding the term $y \cdot \max_{1 \leq r \leq t_j} \beta_r$ on both sides:

$$\begin{aligned} y \cdot \max_{1 \leq r \leq t_j} \beta_r + \alpha_1 &\geq (y + \tau_1) \cdot \max_{1 \leq r \leq t_j} \beta_r \\ &\geq \rho(1-x)\beta_1 + (y + \tau_1 - \rho(1-x)) \cdot \max_{1 \leq r \leq t_j} \beta_r \end{aligned}$$

For the inductive step, assume that (3) holds for some $s-1$. Using the inductive hypothesis (i.h.), we have

$$\begin{aligned} y \cdot \max_{1 \leq r \leq t_j} \beta_r + \sum_{\ell=1}^{s} \alpha_\ell &= y \cdot \max_{1 \leq r \leq t_j} \beta_r + \sum_{\ell=1}^{s-1} \alpha_\ell + \alpha_s \\ &\overset{\text{(i.h.)}}{\geq} \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \left(y + \sum_{\ell=1}^{s-1} \tau_\ell - \rho(s-1-x) \right) \cdot \max_{s-1 \leq r \leq t_j} \beta_r + \alpha_s \\ &\overset{(1)}{\geq} \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \left(y + \sum_{\ell=1}^{s-1} \tau_\ell - \rho(s-1-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r + \alpha_s \end{aligned}$$

$$
\begin{aligned}
&\overset{(2)}{\geq} \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \left(y + \sum_{\ell=1}^{s-1} \tau_\ell - \rho(s-1-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r + \tau_s \max_{s \leq r \leq t_j} \beta_r \\
&= \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \left(y + \sum_{\ell=1}^{s} \tau_\ell - \rho(s-1-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r \\
&= \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \rho \max_{s \leq r \leq t_j} \beta_r + \left(y + \sum_{\ell=1}^{s} \tau_\ell - \rho(s-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r \\
&\geq \rho \left(\sum_{\ell=1}^{s-1} \beta_\ell - x\beta_1 \right) + \rho\beta_s + \left(y + \sum_{\ell=1}^{s} \tau_\ell - \rho(s-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r \\
&= \rho \left(\sum_{\ell=1}^{s} \beta_\ell - x\beta_1 \right) + \left(y + \sum_{\ell=1}^{s} \tau_\ell - \rho(s-x) \right) \cdot \max_{s \leq r \leq t_j} \beta_r \ .
\end{aligned}
$$

This completes the induction and shows (3).

Using (3) with $s = t_j$ together with (1) yields

$$
y \cdot \max_{1 \leq r \leq t_j} \beta_r + \sum_{\ell=1}^{t_j} \alpha_\ell \ \geq \ \rho \left(\sum_{\ell=1}^{t_j} \beta_\ell - x\beta_1 \right) \ .
$$

Now letting A_i and A_j be the bundles of agents i and j after agent j's t_j-th pick, and g be a good in A_j for which agent i has highest utility, we obtain from the last inequality that

$$
\begin{aligned}
y \cdot v_i(g) + v_i(A_i) &\geq \frac{w_i}{w_j} \cdot (v_i(A_j) - x\beta_1) \\
&\geq \frac{w_i}{w_j} \cdot (v_i(A_j) - x \cdot v_i(g)) \ .
\end{aligned}
$$

Therefore the $\mathsf{WEF}(x, y)$ condition for agent i towards agent j is fulfilled, completing the proof. □

3. Fractional Allocations and Lotteries

A random allocation is a probability distribution $\mathcal{L}$ over deterministic allocations. We mostly focus on additive valuations, so we conveniently use a representation as matrix X of marginal assignment probabilities for each good to each agent (i.e., a complete fractional allocation as defined above). We denote by $X^{\mathcal{L}}$ the fractional allocation corresponding to a lottery $\mathcal{L}$. Notice that different lotteries might produce the very same fractional allocation.

Throughout the paper, we denote by X (resp., Y) fractional (resp., integral) allocations in matrix form. Further, X_i (resp., Y_i) denotes the fractional (resp., integral) bundle of i.

Decomposing Fractional Matrices. As mentioned in the introduction, to obtain our best of both worlds result, we make use of a strong matrix decomposition theorem by Budish et al. (2013). In this section, we describe the ingredients we need to apply it in our context. Roughly speaking a decomposition theorem guarantees that a given fractional matrix can be expressed as a convex combination of integral ones; moreover, if the fractional matrix satisfies certain properties (sometimes termed constraints), then every allocation in the decomposition will satisfy the same properties/constraints as well.

A *decomposition* of a fractional matrix (allocation) $X \in [0,1]^{\mathcal{N}\times\mathcal{G}}$ is a convex combination of deterministic matrices (allocations), i.e., $X = \lambda_1 Y^1 + \cdots + \lambda_k Y^k$, where $\sum_{h=1}^{k} \lambda_h = 1$, $\lambda_h \geq 0$ and $Y^h \in \{0,1\}^{\mathcal{N}\times\mathcal{G}}$ for each $h \in [k] = \{1,\ldots,k\}$.

A *constraint structure* $\mathcal{H}$ consists of a collection of subsets $S \subseteq \mathcal{N} \times \mathcal{G}$. Every $S \in \mathcal{H}$ comes with a lower and upper *quota* denoted by $\underline{q}_S$ and $\overline{q}_S$, respectively. Quotas are integer numbers stored in $\mathbf{q} = (\underline{q}_S, \overline{q}_S)_{S\in\mathcal{H}}$.

A matrix Y is feasible under quotas $\mathbf{q}$ if for each $S \in \mathcal{H}$,

$$\underline{q}_S \leq \sum_{(i,g)\in S} y_{ig} \leq \overline{q}_S \ .$$

A constraint structure $\mathcal{H}$ is a *hierarchy* if, for every $S, S' \in \mathcal{H}$, either $S \cap S' = \emptyset$ or one is contained in the other. $\mathcal{H}$ is a *bihierarchy* if it can be partitioned into $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, such that $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ and both $\mathcal{H}_1$ and $\mathcal{H}_2$ are hierarchies.

Budish et al. (2013) generalize the well-known decomposition theorem by Birkhoff and von Neumann:

Theorem 1 (Budish et al., 2013). *Given any fractional matrix X, a bihierarchy $\mathcal{H}$ and corresponding quotas $\mathbf{q}$, if X is feasible under $\mathbf{q}$, then there exists a decomposition into integral matrices $Y^1,\ldots,Y^k$, i.e.,*

$$X = \lambda_1 Y^1 + \cdots + \lambda_k Y^k, \quad \lambda_h \geq 0, \ \sum\nolimits_h \lambda_h = 1,$$

such that every matrix in the decomposition is feasible under $\mathbf{q}$.

In the rest of this paper, given a fractional allocation X and a bihierarchy $\mathcal{H}$, we define the quotas in $\mathbf{q}$ as follows: for every $S \in \mathcal{H}$ we set $\underline{q}_S = \lfloor x_S \rfloor$ and $\overline{q}_S = \lceil x_S \rceil$, where $x_S = \sum_{(i,g)\in S} x_{ig}$. The decomposition obtained with these quotas and bihierarchy $\mathcal{H}$ will be called the $\mathcal{H}$-decomposition.

Efficiently Computing the Decomposition. Unfortunately, the constructive proof of Theorem 1 in (Budish et al., 2013) does not guarantee that the support size k is polynomial in the number of rows n and columns m of X, although existence of an $(nm+1)$-sized decomposition of X is guaranteed by Carathéodory's theorem (since X is in the convex hull of points in an nm-dimensional space). However, the authors provide an algorithm to sample a matrix Y from the probability distribution $\left(\lambda_h, Y^h\right)_{h=1\ldots k}$ in strongly polynomial time. We next show that their sampling algorithm can be extended to also return a decomposition of size $nm+1$ as guaranteed to exist by Carathéodory's theorem.

Let us first summarize the sampling algorithm (for a detailed description, see the full version of Budish et al., 2013). The algorithm starts with the fractional matrix X feasible under the given quota constraints $\mathbf{q}$. It then computes the *degree of integrality* of X, which is defined to be the number of integral constraints in X, i.e., $\deg(X) = |\{S \in \mathcal{H} : \sum_{(i,g)\in S} x_{ig} \in \mathbb{Z}\}|$. If $\deg(X) = |\mathcal{H}|$, the algorithm terminates. Otherwise, if $\deg(X) < |\mathcal{H}|$, the algorithm computes a decomposition of X into $X = \gamma X^1 + (1-\gamma)X^2$, where $\gamma \in [0,1]$ and X^1 and

X^2 both have a strictly higher degree of integrality than X. With this decomposition at hand, the algorithm randomly picks one of X^1 or X^2, that means, with probability γ it selects X^1 and with probability $1-\gamma$ it selects X^2. The selected matrix is set to be the new X, and the process is repeated if $\deg(X) < |\mathcal{H}|$, i.e., not all constraints are integral. It is easy to see that the number of iterations is bounded by $|\mathcal{H}|$. Moreover, in (Budish et al., 2013) it is argued that computing the decomposition in each iteration can be done in strongly polynomial time using a flow network, which overall yields strongly polynomial running time of this sampling procedure.

Now assume we want to compute the explicit lottery instead of only sampling from it, i.e., we want to compute the values $\lambda_1, \dots, \lambda_k$ and matrices $Y^1, \dots, Y^k$. An easy approach would be to not select one of X^1 or X^2 in every iteration, but to continue with both branches and construct the complete execution tree and thereby also the explicit lottery. Unfortunately, the running time of this procedure as well as the support size of the resulting lottery might be exponential in $|\mathcal{H}|$. However, it is possible to reduce the support size during the construction of the execution tree using a simple algorithmic version of Carathéodory's theorem. If after some iteration, the support size grows from $s \leq nm+1$ to some $s' > nm+1$, we know that $s' \leq 2s$. Hence the support size is still polynomial in n and m (in fact, it is linear in nm). This allows us to compute in strongly polynomial time a new support of size at most $nm+1$ using an algorithmic version of Carathéodory's theorem (see, e.g., Mulzer & Stein, 2018, Lemma 2.1). Since we only remove matrices from the support, in every iteration the degree of integrality continues to strictly increase in every matrix in the support. Hence the algorithm still terminates after at most $|\mathcal{H}|$ iterations.

Note that Freeman et al. (2020) discuss a similar approach using a complicated LP framework. The main difference is that we apply an (arguably) much simpler algorithmic version of Carathéodory's theorem to reduce the support size.

Our discussion can be summarized in the following Proposition, which is crucial to guarantee strongly polynomial running time of our algorithms.

Proposition 2. *The decomposition guaranteed in Theorem 1 can be assumed to contain at most $nm+1$ matrices and can be computed in polynomial time in n, m and $|\mathcal{H}|$, where n and m is the number of rows and columns of X, respectively.*

Utility Guarantee Bihierarchy. We next define an extremely useful bihierarchy. For a deeper understanding, we refer the reader to (Budish et al., 2013; Freeman et al., 2020).

We set $\mathcal{H}_1 = \{C_g \mid g \in \mathcal{G}\}$, where $C_g = \{(i,g) \mid i \in \mathcal{N}\}$ represent columns. Roughly speaking, the hierarchy $\mathcal{H}_1$ ensures that, in any allocation of the decomposition, every good is integrally assigned (and therefore the allocation is complete).

For agent $i \in \mathcal{N}$, we consider the goods in non-increasing order of i's valuation, i.e., $v_i(g_1) \geq \ldots \geq v_i(g_m)$. Recall that ties are broken according to a predefined ordering of $\mathcal{G}$. We set $\mathcal{S}_i = \{\{g_1\}, \{g_1, g_2\}, \dots, \{g_1, \dots, g_m\}\}$. In other words, for every $h \in [m]$, $\mathcal{S}_i$ contains a set of the h most preferred goods of i. Define $\mathcal{H}_2 = \{(i,S) \mid i \in \mathcal{N}, S \in \mathcal{S}_i\} \cup \{(i,g) | i \in \mathcal{N}, g \in \mathcal{G}\}$. The second set of constraints implies that if $x_{ig} = 0$ (resp., $x_{ig} = 1$) then $y_{ig} = 0$ (resp., $y_{ig} = 1$), for any Y in the decomposition. Note that (for convenience later on) we slightly abuse notation for $\mathcal{H}_2$ as it is not a set of (row, column)-pairs.

Finally, the *utility guarantee bihierarchy* is given by $\mathcal{H}^{\mathsf{UG}} = \mathcal{H}_1 \cup \mathcal{H}_2$. Clearly, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are hierarchies.

This bihierarchy was fundamental in (Budish et al., 2013). It allows to state the following theorem (see Freeman et al. (2020) for the proof).

Corollary 1 (Utility Guarantee up to one Good More or Less)**.** *Suppose we are given a fractional allocation X, and additive valuation functions v_i. Then for any matrix Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X the following hold:*

1. *if $v_i(Y_i) < v_i(X_i)$, then $\exists\ g \notin Y_i$ with $x_{ig} > 0$ such that $v_i(Y_i) + v_i(g) > v_i(X_i)$;*
2. *if $v_i(Y_i) > v_i(X_i)$, then $\exists\ g \in Y_i$ with $x_{ig} < 1$ such that $v_i(Y_i) - v_i(g) < v_i(X_i)$.*

In other words, Corollary 1 ensures that, in any deterministic allocation in the $\mathcal{H}^{\mathsf{UG}}$-decomposition, the valuation of any agent i differs from her expected value by at most the value of one good. Moreover, such a good must have a positive probability of occurring in i's bundle. Proposition 2 implies that the $\mathcal{H}^{\mathsf{UG}}$-decomposition can be computed in strongly polynomial time since $|\mathcal{H}^{\mathsf{UG}}| = \text{poly}(n, m)$.

4. Additive Valuations with Entitlements

In this section, we present a lottery for additive valuations that simultaneously achieves ex-ante WSD-EF (and hence ex-ante WEF) and ex-post $\mathsf{WEF}(1,1) + \mathsf{WPROP1}$. In contrast to equal entitlements, we show a weaker ex-post guarantee. However, we prove that this is necessary since no stronger envy notion is compatible with ex-ante WEF. We also generalize a result of Freeman et al. (2020) to entitlements: Similarly to the unweighted setting, we design a lottery that is ex-ante WGF and ex-post $\mathsf{WEF}_1^1 + \mathsf{WPROP1}$.

4.1 Ex-ante WSD-EF, Ex-post $\mathsf{WEF}(1,1)$ + WPROP1

The main contribution of this subsection is a proof of the following result.

Theorem 2. *For entitlements and additive valuations, we can compute in strongly polynomial time a lottery that is ex-ante* WSD-EF *and ex-post* $\mathsf{WPROP1} + \mathsf{WEF}(1,1)$.

Let us start by introducing our main algorithm DifferentSpeedsEating (DSE), which is inspired by Eating for equal entitlements in (Aziz, 2020). Agents continuously eat their most preferred available good at speed equal to their entitlement. Every agent starts eating her most preferred good; as soon as a good has been completely eaten it is removed from the set of available goods. Each agent that was eating this good continues eating her most preferred remaining one. The procedure terminates when no good remains. See Algorithm 1 for a formal description. Observe that by precomputing the times at which goods are removed, we can implement the algorithm in strongly polynomial time.

The key properties of the algorithm are summarized in the following lemma.

Lemma 1. *Let X^{DSE} be the output of* DSE*, then*

1. $\sum_{g\in\mathcal{G}} x_{ig}^{\text{DSE}} = w_i \cdot m$ *for each* $i \in \mathcal{N}$*;*
2. *the time needed for agent i to eat one unit of goods is $\frac{1}{w_i}$;*

Algorithm 1: DifferentSpeedsEating

Input: An instance $\mathcal{I} = (\mathcal{N}, \mathcal{G}, \{v_i\}_{i \in \mathcal{N}})$ and the entitlements $w_1, \ldots, w_n$
Output: A fractional allocation X

$X \leftarrow \mathbf{0}_{n \times m}$ `// current fractional allocation`
$\mathbf{z} \leftarrow \mathbf{1}_m$ `// remaining supply of each good`
while $\mathcal{G} \neq \emptyset$ **do**
  $\mathbf{s} \leftarrow \mathbf{0}_m$ `// eating speed on each good`
  for $i \in \mathcal{N}$ **do**
    $g^i \leftarrow \arg\max_{g \in \mathcal{G}} v_i(g)$ `// most favored good`
    $\mathbf{s}(g^i) \leftarrow \mathbf{s}(g^i) + w_i$ `// sum speeds of each agent`
  $\mathbf{t} \leftarrow \mathbf{1}_m$ `// eating time of each good`
  for $g \in \mathcal{G}$ **do**
    $\mathbf{t}(g) \leftarrow \frac{\mathbf{z}(g)}{\mathbf{s}(g)}$ `// compute finishing times`
  $t \leftarrow \min_{g \in \mathcal{G}} \mathbf{t}(g)$ `// time when first good is finished`
  for $i \in \mathcal{N}$ **do**
    $x \leftarrow t \cdot w_i$ `// amount of goods eaten by` i
    $x_{ig^i} \leftarrow x_{ig^i} + x$ `// eat fraction of` g^i
    $\mathbf{z}(g^i) \leftarrow \mathbf{z}(g^i) - x$ `// reduce supply of` g^i
  $\mathcal{G} \leftarrow \mathcal{G} \setminus \{g \in \mathcal{G} \mid \mathbf{t}(g) \leq \mathbf{t}(g') \text{ for all } g' \in \mathcal{G}\}$ `// remove finished goods`
return X

3. *overall, one unit of goods is consumed in one unit of time and, therefore,* DSE *runs for m time units.*

We define the *eating time* of a good g as the point in time when it has been entirely consumed (during a run of DSE). Whenever an agent starts eating a good g, she can start eating another good only after the eating time of g.

Before proceeding to the proof of Theorem 2, let us provide an example describing a run of DSE on the instance $\mathcal{I}^*$ introduced above in Example 1.

Example 2 (DSE at work)**.** Let us observe the behavior of DSE on $\mathcal{I}^*$.

The agents' priorities for the goods are the following:

$$g_1 \succ_1 g_2 \succ_1 g_3 \succ_1 g_4 \ ,$$
$$g_2 \succ_2 g_3 \succ_2 g_1 \succ_2 g_4 \ ,$$
$$g_2 \succ_3 g_3 \succ_3 g_1 \succ_3 g_4 \ .$$

Notice that, for agent 1, goods g_1 and g_2 are identical and ties are broken in favor of the good coming first in the ordering $g_1, \ldots, g_4$. Moreover, agents 2 and 3 have the same priority order, for this reason they will always be eating the same good.

During a run of DSE, whenever a good gets entirely eaten up, the behavior of agents who were eating this good changes. In the following, we only refer to time points where these events happen. Indeed, the times are the eating times of the good(s) that have been completely consumed.

Time $\mathbf{t} = \mathbf{0}$: At the beginning, $x_{ig} = 0$, for all $i \in \mathcal{N}$ and $g \in \mathcal{G}$. Agent 1 starts eating g_1 while agents 2 and 3 good g_2. Notice that agents 2 and 3 together have the same speed as agent 1.

Time $\mathbf{t} = \mathbf{2}$: g_1 and g_2 get fully consumed and $x_{1g_1} = 1$, $x_{2g_2} = \frac{2}{3}$ and $x_{3g_2} = \frac{1}{3}$, respectively. Agent 1 will start eating g_3 as well as agents 2 and 3. All the agents together have speed equal to 1. Notice that agent 1 would prefer good g_2, however, it has been consumed entirely by agents 2 and 3.

Time $\mathbf{t} = \mathbf{3}$: g_3 is now fully consumed. We have $x_{1g_3} = \frac{1}{2}$, $x_{2g_3} = \frac{1}{3}$ and $x_{3g_3} = \frac{1}{6}$, respectively. The only remaining available good is g_4, all agents are now starting to eat it.

Time $\mathbf{t} = \mathbf{4}$: All goods are fully consumed and $x_{1g_4} = \frac{1}{2}$, $x_{2g_4} = \frac{1}{3}$ and $x_{3g_4} = \frac{1}{6}$. DSE returns the fractional allocation

$$X^{\mathrm{DSE}} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} .$$

■

Our first result is that the output of DSE is WSD-EF.

Proposition 3. *X^{DSE} is* WSD-EF*.*

Proof. For convenience, we use $X = X^{\mathrm{DSE}}$. Let us consider an agent $i \in \mathcal{N}$. Note that the goods $g_1, \ldots, g_m$ are ordered in the same manner as in DSE for agent i, since we always break ties according to a predefined ordering of $\mathcal{G}$. Now consider another agent $j \in \mathcal{N}$. Using the notation $G_k = \{g_1, \ldots, g_k\}$ for the first k goods in i's ordering, we show

$$w_j \cdot \sum_{g \in G_k} x_{ig} \geq w_i \cdot \sum_{g \in G_k} x_{jg} \ , \tag{4}$$

for every $k \in [m]$, and WSD-EF follows for agent i.

Let t_k be the time when i stops eating g_k during the run of DSE. We set $t_k = t_{k-1}$ if good g_k has been completely consumed before time t_{k-1} by others. This means that, by the time t_k, no good in G_k remains available. On the one hand, until time t_k, agent i could only consume goods in G_k, implying $w_i \cdot t_k = \sum_{g \in G_k} x_{ig}$. On the other hand, every good in G_k has been fully consumed by that time, i.e., $w_j \cdot t_k \geq \sum_{g \in G_k} x_{jg}$, for every $j \in \mathcal{N}$. Combining these two properties proves Equation (4) and hence the theorem. □

It is known that SD-EF implies ex-ante EF for additive valuations (Aziz, 2020); it remains true for different entitlements. For completeness, we provide a formal proof of this statement.

Proposition 4. *Given a fractional allocation X, if X is* WSD-EF*, then X is also* WEF *under additive valuations.*

Proof. Let X be a fractional allocation satisfying WSD-EF, and let i and j be two distinct agents. Denote by $g_1, \ldots, g_m$ the goods sorted according to i's valuation, i.e., $v_i(g_1) \geq \cdots \geq$

$v_i(g_m)$. From the definition of $\mathsf{WSD\text{-}EF}$, we know that $\sum_{\ell=1}^{k} w_j x_{ig_\ell} \geq \sum_{\ell=1}^{k} w_i x_{jg_\ell}$ for each $k \in [m]$, which can be rewritten as

$$\sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) \geq 0 \ . \tag{5}$$

To show that X is WEF, we first show by induction that for each $k \in [m]$,

$$\sum_{\ell=1}^{k-1}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) v_i(g_\ell) \geq v_i(g_k) \sum_{\ell=1}^{k-1}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) \ . \tag{6}$$

The base case $k = 1$ is trivial. For the inductive step, assume (6) holds for some k. Adding $(w_j x_{ig_k} - w_i x_{jg_k}) v_i(g_k)$ on both sides yields

$$\sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) v_i(g_\ell) \geq v_i(g_k) \sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) \geq v_i(g_{k+1}) \sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) \ ,$$

where the last inequality follows from (5) and the chosen ordering of goods. This completes the induction.

Now using (6) and (5) together, we obtain

$$\sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) v_i(g_\ell) \overset{(6)}{\geq} v_i(g_k) \sum_{\ell=1}^{k}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) \overset{(5)}{\geq} 0 \ .$$

Finally, setting $k = m$ yields $\sum_{\ell=1}^{m}(w_j x_{ig_\ell} - w_i x_{jg_\ell}) v_i(g_\ell) \geq 0$, which shows X to be WEF. □

Proposition 3 shows that the outcome of DSE satisfies the ex-ante properties stated in Theorem 2. Proposition 4 ensures that this implies also WEF ex-ante.

So far we have shown ex-ante properties of lotteries having the output of the DSE as fractional matrix representation. In the remaining part of this subsection, we show how to get good properties ex-post. In particular, we start from the output of the DSE, namely, X^{DSE}. We apply the Budish's decomposition with the bihierarchy $\mathcal{H}^{\mathsf{UG}}$.

Before proceeding, we give an example of such a decomposition as well as some insights on the guarantees obtained thanks to the $\mathcal{H}^{\mathsf{UG}}$ bihierarchy. We again make use of instance $\mathcal{I}^*$; the allocation X^{DSE} was computed in Example 2.

Example 3 (The $\mathcal{H}^{\mathsf{UG}}$-decomposition)**.** The $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is a convex combination $\lambda_1 Y^1 + \cdots + \lambda_k Y^k$, for some integer k. Every allocation Y^h is deterministic and its properties are determined by the bihierarchy $\mathcal{H}^{\mathsf{UG}}$. In the following, we use Y to refer to a generic deterministic allocation in the decomposition.

Recall that $\mathcal{H}^{\mathsf{UG}} = \mathcal{H}_1 \cup \mathcal{H}_2$. Interpreting an allocation as a matrix, the hierarchy $\mathcal{H}_1$ represents columns and only ensures that any Y is complete.

Let us now consider $\mathcal{H}_2$. Recall, $\mathcal{H}_2 = \{(i, S) \mid i \in \mathcal{N}, S \in \mathcal{S}_i\} \cup \{(i, g) | i \in \mathcal{N}, g \in \mathcal{G}\}$.

By Theorem 1, every matrix of an allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition is feasible under the quotas $\underline{q}_A = \lfloor x_A^{\text{DSE}} \rfloor$ and $\overline{q}_A = \lceil x_A^{\text{DSE}} \rceil$, for every $A \in \mathcal{H}_2$, where $x_A^{\text{DSE}} = \sum_{(i,g)\in A} x_{ig}^{\text{DSE}}$. Note that only one agent appears in any pair of $\mathcal{H}_2$. Hence, we discuss the implications of Theorem 1 agent by agent.

Agent 1: The pair $(1, S)$ belongs to $\mathcal{H}_2$ if and only if $S \in \mathcal{S}_1 \cup \{\{g_2\}, \{g_3\}, \{g_4\}\}$, where $\mathcal{S}_1 = \{\{g_1\}, \{g_1, g_2\}, \{g_1, g_2, g_3\}, \{g_1, g_2, g_3, g_4\}\}$. The feasibility conditions imply:

$$\begin{aligned}
&y_{1g_1} = 1 \ , \\
&y_{1g_1} + y_{1g_2} = 1 \ , \\
&1 \le y_{1g_1} + y_{1g_2} + y_{1g_3} \le 2 \ , \\
&y_{1g_1} + y_{1g_2} + y_{1g_3} + y_{1g_4} = 2 \ ,
\end{aligned}$$

and

$$y_{1g_2} = 0 \ , \qquad\qquad 0 \le y_{1g_3} \le 1 \ , \qquad\qquad 0 \le y_{1g_4} \le 1 \ .$$

In other words, in any deterministic allocation Y, agent 1 always receives 2 goods. In particular, she always gets g_1 but never g_2. Moreover, she gets either g_3 or g_4, but not both of them.

Agent 2: The pair $(2, S)$ belongs to $\mathcal{H}_2$ if and only if $S \in \mathcal{S}_2 \cup \{\{g_1\}, \{g_3\}, \{g_4\}\}$, where $\mathcal{S}_2 = \{\{g_2\}, \{g_2, g_3\}, \{g_2, g_3, g_1\}, \{g_2, g_3, g_1, g_4\}\}$. In this case, the feasibility conditions imply

$$\begin{aligned}
&0 \le y_{2g_2} \le 1 \ , \\
&y_{2g_2} + y_{2g_3} = 1 \ , \\
&y_{2g_2} + y_{2g_3} + y_{2g_1} = 1 \ , \\
&1 \le y_{2g_2} + y_{2g_3} + y_{2g_1} + y_{2g_4} \le 2 \ ,
\end{aligned}$$

and

$$y_{2g_1} = 0 \ , \qquad\qquad 0 \le y_{2g_3} \le 1 \ , \qquad\qquad 0 \le y_{2g_4} \le 1 \ .$$

Therefore, the bundle of agent 2 is of size either 1 or 2. It never contains g_1, but must contain one good between g_2 and g_3, and possibly contains g_4.

Agent 3: The pair $(3, S)$ belongs to $\mathcal{H}_2$ if and only if $S \in \mathcal{S}_3 \cup \{\{g_1\}, \{g_3\}, \{g_4\}\}$, where $\mathcal{S}_3 = \{\{g_2\}, \{g_2, g_3\}, \{g_2, g_3, g_1\}, \{g_2, g_3, g_1, g_4\}\}$. In this case, $0 \le y_{3g_2} + y_{3g_3} + y_{3g_1} + y_{3g_4} \le 1$ and $y_{3g_1} = 0$, hence, agent 3 can receive at most one of g_2, g_3, g_4 and never receives g_1.

Finally, we provide a concrete $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} for $\mathcal{I}^*$. Considering that rows represent agents and columns represent goods, it is easy to verify that every deterministic allocation satisfies the aforementioned properties.

$$X^{\text{DSE}} = \frac{1}{6}\underbrace{\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}}_{Y^1} + \frac{1}{6}\underbrace{\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}}_{Y^2} + \frac{1}{3}\underbrace{\begin{pmatrix}1&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}}_{Y^3} + \frac{1}{3}\underbrace{\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\end{pmatrix}}_{Y^4} \ \blacksquare$$

We notice that Y^4 in Example 3 is not $\mathsf{WEF1}$. Indeed, in Y^4 agents 1 and 2 receive two goods each while agent 3 has an empty bundle, thus agent 3 $\mathsf{WEF1}$-envies any other agent. On the other hand, every allocation Y^h, for $h = 1, \dots, 4$, is $\mathsf{WEF}(1,1)$ and $\mathsf{WPROP1}$. We next show this is always the case for the $\mathcal{H}^{\mathsf{UG}}$-decomposition of any X^{DSE}.

Theorem 3. *Every deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is* $\mathsf{WEF}(1,1)$.

To show the theorem we need some preliminary notions.

Goods Eaten by i at Time t. Recall that DSE runs for m units of time. Every agent $i \in \mathcal{N}$ exactly eats a total mass of $w_i \cdot m$ of $\mathcal{G}$ during DSE. Let $g_1, \dots, g_m$ be the ordering of goods according to v_i. For $t \geq 0$, we define $\mathsf{Eaten}(i,t) = \{g_1, \dots, g_\ell\} = G_\ell$, where g_ℓ is either a good that agent i just finished to consume (i.e., t is the eating time of g_ℓ and agent i was consuming it) or agent i at time t is eating the good $g_{\ell+1}$, which has not been finished yet. Consequently, by time t, agent i may have contributed only to the consumption of goods in $G_{\ell+1}$. In particular, all goods in G_ℓ have been entirely consumed (by i or others), since otherwise i would not start eating $g_{\ell+1}$.

Recall that w_i is the speed of i. At time $t = \frac{k}{w_i}$ agent i ate a total mass k of goods. With the next lemma, we show that the $\mathcal{H}^{\mathsf{UG}}$-decomposition guarantees agent i deterministically receives at most k goods from the ones eaten by time $\frac{k}{w_i}$.

Lemma 2. *Given any deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}, for every $i \in \mathcal{N}$ and $k = 1, \dots, \lfloor w_i \cdot m \rfloor$, $|Y_i \cap \mathsf{Eaten}(i, \frac{k}{w_i})| \leq k$. Furthermore, $\lfloor w_i \cdot m \rfloor \leq |Y_i| \leq \lceil w_i \cdot m \rceil$.*

Proof. By definition, $\mathsf{Eaten}(i, \frac{k}{w_i}) = G_\ell$, the ℓ most preferred goods of i, for some ℓ. Thus, by the time $\frac{k}{w_i}$, agent i only ate goods in G_ℓ and possibly is currently eating the next less preferred good. Moreover, goods are eaten by i in the same ordering we used to build the collection $\mathcal{S}_i$ in the definition of $\mathcal{H}^{\mathsf{UG}}$ implying $(i, G_\ell) \in \mathcal{H}^{\mathsf{UG}}$. Since $|Y_i \cap \mathsf{Eaten}(i, \frac{k}{w_i})| = \sum_{g \in G_\ell} y_{ig}$, the $\mathcal{H}^{\mathsf{UG}}$-decomposition properties imply $\sum_{g \in G_\ell} y_{ig} \leq \left\lceil \sum_{g \in G_\ell} x_{ig} \right\rceil$. This last is upper-bounded by k because of these two simple observations: g_ℓ is fully consumed by the time $\frac{k}{w_i}$, and at that time agent i ate k units of goods. The first claim follows.

The second claim immediately follows by the $\mathcal{H}^{\mathsf{UG}}$-decomposition properties, since $(i, \mathcal{G}) \in \mathcal{H}^{\mathsf{UG}}$. $\square$

Given any deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition, consider agent i and sort the goods in Y_i in a non-increasing manner with respect to v_i: $Y_i = \{g_1^i, \dots, g_{h_i}^i\}$ and $v_i(g_1^i) \geq \dots \geq v_i(g_{h_i}^i)$. By Lemma 2, we see $h_i = \lfloor w_i \cdot m \rfloor$ or $h_i = \lceil w_i \cdot m \rceil$.

Stopping vs. Eating Time. Given any deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X, for each $i \in \mathcal{N}$ and $k \in [h_i]$, we define the *stopping time* by $s(g_k^i) = \min\{t(g_k^i), \frac{k}{w_i}\}$. Here $t(g_k^i)$ is the time when g_k^i has been entirely consumed during the DSE, i.e., the eating time of g_k^i. Note that $s(g_k^i)$, differently from $t(g_k^i)$, depends on Y: Indeed, in Y_i good g_k^i is the k-th most preferred good. However, if the eating time is greater than $\frac{k}{w_i}$, this good might appear as $(k+1)$-th most preferred good in another deterministic allocation of the decomposition. For convenience, we omit Y in the notation since we only discuss stopping times of single allocations in the support. Note that in the decomposition of X^{DSE} computed in Example 3 for instance $\mathcal{I}^*$, stopping and eating times always coincide, which is due to the small size of the example.

Next, let us show a couple of useful properties of stopping times.

Lemma 3. *Given any deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}, let g_k^i be the k-th most preferred good in Y_i, it holds $s(g_k^i) \in \left(\frac{k-1}{w_i}, \frac{k}{w_i}\right]$.*

Proof. By definition, $s(g_k^i) = \min\{t(g_k^i), \frac{k}{w_i}\} \leq \frac{k}{w_i}$. For contradiction, suppose $t(g_k^i) \leq \frac{k-1}{w_i}$. Then, $g_k^i \in Y_i \cap \mathsf{Eaten}\left(i, \frac{k-1}{w_i}\right)$. Notice that $t(g_1^i) \leq \cdots \leq t(g_k^i)$, by definition of DSE, and therefore $g_h^i \in Y_i \cap \mathsf{Eaten}\left(i, \frac{k-1}{w_i}\right)$, for each $h = 1, \ldots, k$. In conclusion, $\left|Y_i \cap \mathsf{Eaten}\left(i, \frac{k-1}{w_i}\right)\right| \geq k$ which is a contradiction by Lemma 2, and hence $t(g_k^i) > \frac{k-1}{w_i}$. $\square$

For the eating time $t(g_k^i)$ the same lower bound holds, but we can only upper bound it by $\frac{k+1}{w_i}$. This difference will be crucial in the proof of Theorem 3 and requires the definition of stopping times.

Lemma 4. *Given any deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}, let g_k^i be the k-th most preferred good in Y_i. For every good g coming earlier in i's ordering of goods, it holds that $s(g) < s(g_k^i)$.*

Proof. The claim follows by the definition of stopping time and the properties of DSE. Indeed, by the definition of stopping time $s(g) \leq t(g)$, and $t(g) < \min\{t(g_k^i), \frac{k}{w_i}\} = s(g_k^i)$. The second inequality holds because at time $s(g_k^i)$ agent i is eating or finishes to eat g_k^i, and g must have been eaten before i starts eating g_k^i. Further, the inequality is strict since agent i ate a positive fraction of g_k^i (i.e., $x_{ig_k^i} > 0$); otherwise, since $(i, g_k^i) \in \mathcal{H}^{\mathsf{UG}}$, $x_{ig_k^i} = 0$ would imply $y_{ig_k^i} = 0$ and, hence, $g_k^i \notin Y_i$. $\square$

We are now ready to show Theorem 3.

Proof of Theorem 3. Let Y be any deterministic allocation in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}. The proof proceeds as follows: We first generate a picking sequence π, then show that Y is the output of such a picking sequence, and finally prove that π satisfies Proposition 1, for $x = y = 1$. This shows that Y is $\mathsf{WEF}(1, 1)$.

Defining π. We sort the goods $\mathcal{G}$ in a non-decreasing order of stopping times $s_1, \ldots, s_m$ (defined according to Y). If $g \in Y_i$ is the h-th good in this ordering, then $\pi(h) = i$.

Y is the result of π. Assume i is the h-th agent in π. Assume that $\pi(h) = i$ is the k-th occurrence of i in π. We show that for each $h \in [m]$, the most preferred available good for i is exactly g_k^i. Let us proceed by induction on h.

For $h = 1$, clearly, $k = 1$. By Lemma 4, g_1^i must be the most preferred good of i, otherwise we contradict the fact that $s_1 = s(g_1^i)$ is the minimum stopping time. At this point no good has been assigned, so i selects g_1^i.

Assume the statement is true until the h-th component of π. We show it is true for $h + 1 \leq m$. Let $\pi(h + 1) = i$ and let this be the k-th occurrence of i in π. Assume for contradiction that a good g coming before g_k^i in i's ordering is still available in round $h + 1$. By Lemma 4, there exists h' s.t. $s_{h'} = s(g) < s(g_k^i)$ with $h' \leq h$. Hence by the inductive hypothesis, g must have been assigned to $\pi(h')$. Hence no good coming before g_k^i is available in round $h + 1$. On the other hand, g_k^i is still available in round $h + 1$, otherwise, there exists $h'' \leq h$, such that $\pi(h'')$ picked g_k^i during the h''-th round – a contradiction with the inductive hypothesis.

π satisfies Proposition 1. We now show that π satisfies $\mathsf{WEF}(1, 1)$. Consider any prefix of π and any pair of agents i, j. Let us denote by t_i (resp., t_j) the number of picks of

agent i (resp., j) in the considered prefix. Let $stop_j$ and $stop_i$ be the stopping times of the good selected by j at her t_j-th pick and the stopping time of the good selected by i at her (t_i+1)-th pick, respectively. If i has no (t_i+1)-th pick, we set $stop_i = m < \frac{t_i+1}{w_i}$. Within the considered prefix of π, agent j already made its t_j-th pick but i didn't make its (t_i+1)-th pick. Now by definition of π, $stop_j \leq stop_i$. By Lemma 3, $stop_j > \frac{t_j-1}{w_j}$ and $stop_i \leq \frac{t_i+1}{w_i}$. We finally get $\frac{t_j-1}{w_j} < \frac{t_i+1}{w_i}$. This shows that the hypothesis of Proposition 1 is fulfilled for $x = y = 1$. □

Note that if we had chosen eating rather than stopping times for the picking sequence, we could only deduce $\frac{t_j-1}{w_j} < \frac{t_i+2}{w_i}$ which is not sufficient to show $\mathsf{WEF}(1,1)$.

As X^{DSE} is (ex-ante) WEF, it is also WPROP. By ex-ante WPROP and Corollary 1, the following holds.

Proposition 5. *Every deterministic allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is* $\mathsf{WPROP1}$.

Proof. The fractional allocation X^{DSE} is WEF, and hence WPROP. Therefore, $v_i(X_i) \geq w_i \cdot v_i(\mathcal{G})$. By Corollary 1, for any Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition, $v_i(Y_i) \geq v_i(X_i) - v_i(g)$, for some $g \in \mathcal{G} \setminus Y_i$. This implies $v_i(Y_i \cup \{g\}) \geq w_i \cdot v_i(\mathcal{G})$, and $\mathsf{WPROP1}$ follows. □

In conclusion, we proved that the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is a lottery achieving ex-ante $\mathsf{WSD\text{-}EF}$, and therefore ex-ante WEF, and ex-post $\mathsf{WEF}(1,1) + \mathsf{WPROP1}$. As a consequence of Theorem 1, our lottery has polynomial support and the computation requires strongly polynomial time.

While our guarantee is weaker than the ex-post $\mathsf{EF1}$ for equal entitlements, we show that our lottery is, in a sense, best possible in terms of ex-post guarantees. Indeed, we prove that no stronger ex-post envy notion is compatible with ex-ante WEF.

Proposition 6. *For every pair $x, y \in [0,1]$ such that $x+y < 2$, ex-ante* WEF *is incompatible with ex-post* $\mathsf{WEF}(x,y)$.

Proof. Consider a fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{G}, \{v_i\}_{i\in\mathcal{N}})$, with $\mathcal{N} = \{1,2\}$ and $\mathcal{G} = \{g_1, g_2\}$. Moreover, $v_i(g_1) = v_i(g_2) = 1$, for $i = 1,2$. Let us set $w_1 \in \left(\frac{y}{2+y-x}, \frac{1}{2}\right)$ and $w_2 = 1 - w_1$. Observe that $\frac{y}{2+y-x} < \frac{1}{2}$, since $x+y < 2$. In any ex-ante WEF allocation agent 1 receives in expectation less than one good. This means, the allocation $Y = (Y_1, Y_2) = (\emptyset, \mathcal{G})$ is in the support of any ex-ante WEF lottery. Now assume towards contradiction that Y is $\mathsf{WEF}(x,y)$, i.e., for some good $g \in Y_2$, it holds $w_2 \cdot (v_1(Y_1) + y \cdot v_1(g)) \geq w_1 \cdot (v_1(Y_2) - x \cdot v_1(g))$. Since $v_i(g_1) = v_i(g_2) = 1$ for $i = 1,2$, this is equivalent to $w_2 \cdot y \geq w_1 \cdot (2-x)$. Using $w_2 = 1 - w_1$ finally yields $w_1 \geq \frac{y}{2+y-x}$, a contradiction to our choice of w_1. This proves Y is not $\mathsf{WEF}(x,y)$. □

Remark: Equal Entitlements. Let us notice that for equal entitlements our approach also provides ex-ante EF and ex-post $\mathsf{EF1}$. The ex-ante property follows directly since $w_i = 1/n$. For ex-post $\mathsf{EF1}$, similarly to (Aziz, 2020), it is possible to show that any allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of the X^{DSE} is the result of an RB picking sequence. Specifically, at the end of the proof of Theorem 3, we have shown that for the defined picking sequence $\frac{t_j-1}{w_j} < \frac{t_i+1}{w_i}$ is satisfied. Being $w_i = w_j = 1/n$ and t_i, t_j are integers we can conclude

$t_j - t_i \leq 1$. This condition holds for any pair of i and j, and hence $|t_i - t_j| \leq 1$, implying the picking sequence is RB.

4.2 Ex-ante WGF and Ex-post WEF_1^1 + $\mathsf{WPROP1}$

In this subsection, we generalize a result on group fairness of Freeman et al. (2020) to entitlements. We follow the general argument and incorporate some technical extensions to allow for different agent weights.

For our purposes, we recall the weighted version of the well known Nash Welfare.

Definition 5 (Weighted Nash Welfare)**.** Given a fair division instance $\mathcal{I}$, with entitlements $w_1, \ldots, w_n$, and an allocation $\mathcal{A} = (A_1, \ldots, A_n)$, the *weighted Nash welfare* of $\mathcal{A}$ is given by $\prod_{i=1}^{n} (v_i(A_i))^{w_i}$.

Theorem 4. *For entitlements and additive valuations, there exists a lottery which is ex-ante* WGF *and ex-post* $\mathsf{WPROP1} + \mathsf{WEF}_1^1$*. The lottery can be computed in strongly polynomial time.*

Recall that ex-ante weighted group fairness (WGF) is much stronger than ex-ante weighted envy-freeness (WEF). For example, it implies ex-ante envy-freeness as well as Pareto optimality, which in turn also implies ex-post Pareto optimality.

Proof (of Theorem 4). We follow the proof steps of (Freeman et al., 2020), where the same result was shown in the unweighted setting. In a first step, we show that fractional maximum weighted Nash welfare allocations (MWN allocations) form competitive equilibria (CE) (Lemma 6), and in a second step, we show that CE allocations are WGF (Lemma 7). Finally, we show that the $\mathcal{H}^{\mathsf{UG}}$-decomposition of any fractional MWN allocation yields the desired ex-post properties (Lemma 8).

The next lemmas hold even in the more general cake cutting model introduced by Steinhaus (1948). In this model, instead of a finite number of goods, a single continuous good C (a "cake") needs to be split among the n agents. An allocation is a partition of C into n subsets, and the valuation of an agent $i \in \mathcal{N}$ is given by a measure v_i on C.

First, we use a slight generalization of Lemma 4.8 in (Segal-Halevi & Sziklai, 2019) to derive a useful inequality for MWN allocations.

Lemma 5. *Let $f_i : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $i \in \mathcal{N}$, be differentiable functions. If an allocation X of cake maximizes the welfare function $S(X) = \sum_{i \in \mathcal{N}} f_i(v_i(X_i))$, then for any two agents $i, j \in \mathcal{N}$ and any slice $Z_j \subseteq X_j$,*

$$f_j'(v_j(X_j)) \cdot v_j(Z_j) \geq f_i'(v_i(X_i)) \cdot v_i(Z_j) \ .$$

Proof. Similarly as Segal-Halevi and Sziklai (2019), we use a result of Stromquist and Woodall (1985), which states that for any part Z of the cake and $\alpha \in [0, 1]$, there is $Z_\alpha \subseteq Z$ such that $v_i(Z_\alpha) = \alpha \cdot v_i(Z)$ and $v_j(Z_\alpha) = \alpha \cdot v_j(Z)$. Let X' be the allocation obtained by giving Z_α from agent j to i. The welfare difference $D = S(X') - S(X)$ is now a function of α, i.e.,

$$D(\alpha) = f_i\big(v_i(X_i')\big) - f_i\big(v_i(X_i)\big) + f_j\big(v_j(X_j')\big) - f_j\big(v_j(X_j)\big)$$

$$= f_i\big(v_i(X_i) + \alpha \cdot v_i(Z_j)\big) - f_i\big(v_i(X_i)\big) + f_j\big(v_j(X_j) - \alpha \cdot v_j(Z_j)\big) - f_j\big(v_j(X_j)\big) \ .$$

The derivative of D is

$$D'(\alpha) = v_i(Z_j) \cdot f_i'\big(v_i(X_i) + \alpha \cdot v_i(Z_j)\big) - v_j(Z_j) \cdot f_j'\big(v_j(X_j) - \alpha \cdot v_j(Z_j)\big) \ .$$

Since X maximizes the welfare $S(X)$, we must have $D'(0) \leq 0$, and hence $v_i(Z_j) \cdot f_i'(v_i(X_i)) \leq v_j(Z_j) \cdot f_j'(v_j(X_j))$. ☐

If we choose the functions $f_i(t) = w_i \cdot \ln t$ in Lemma 5, we obtain the following useful inequality.

Corollary 2. *If X is an MWN allocation, then for any two agents $i, j \in \mathcal{N}$ and $Z_j \subseteq X_j$,*

$$w_j \cdot \frac{v_j(Z_j)}{v_j(X_j)} \geq w_i \cdot \frac{v_i(Z_j)}{v_i(X_i)} \ .$$

For the subsequent analysis we rely on the notion of competitive equilibrium.

Definition 6. A pair (X, P) of allocation and prices is a competitive equilibrium (CE), if

1) $P(Z) > 0$ iff Z is a positive slice[1],
2) for all agents $i \in \mathcal{N}$, $Z_i \subseteq X_i$, and slice Z, $\frac{v_i(Z_i)}{P(Z_i)} \geq \frac{v_i(Z)}{P(Z)}$ (MBB),
3) for all $i \in \mathcal{N}$, $P(X_i) = w_i$.

This equilibrium notion originates in (Segal-Halevi & Sziklai, 2019), where the authors define it (only) for equally entitled agents under the name *strong competitive equilibrium from equal incomes* (sCEEI).

Lemma 6. *For every MWN allocation X there exists a price measure P such that (X, P) is a competitive equilibrium.*

Proof. Let X be an MWN allocation. For an agent $i \in \mathcal{N}$, we define the price of a slice $Z_i \subseteq X_i$ as

$$P(Z_i) = w_i \cdot \frac{v_i(Z_i)}{v_i(X_i)} \ .$$

The price of an arbitrary slice Z is then given by adding the agent-specific parts, i.e.,

$$P(Z) = \sum_{i \in \mathcal{N}} P(Z \cap X_i) \ .$$

We need to show that conditions 1) to 3) are fulfilled by (X, P). Condition 3) follows simply from the definition of prices. Condition 1) can be derived as follows: Suppose $P(Z) > 0$ for some slice Z. Then, again by the definition of prices, there must be at least one agent $i \in \mathcal{N}$ such that $P(Z \cap X_i) > 0$. Hence Z is a positive slice. For the other direction, suppose Z is a positive slice, i.e., $v_i(Z) > 0$ for some agent $i \in \mathcal{N}$. Then either $P(Z \cap X_i) > 0$ or $P(Z \cap X_i) = 0$. In the first case we are done. In the second case there needs to be an agent

1. A part Z of the cake is called positive slice if $v_i(Z) > 0$ for at least one agent $i \in \mathcal{N}$.

$j \in \mathcal{N}$ with $P(Z \cap X_j) > 0$ as otherwise Z would be liked by i but given only to agents j not liking it, a contradiction to X being an MWN allocation.

Now it remains to show condition 2). Consider any agent i, any $Z_i \subseteq X_i$, and any slice Z. We partition Z into agent specific parts $Z_j = Z \cap X_j$ (note that $Z = \bigcup_j Z_j$). By plugging in prices into Corollary 2 we obtain for each part Z_j,

$$v_i(X_i)P(Z_j) \geq v_i(Z_j)w_i \ .$$

Summing over all agent-specific parts Z_j and using additivity of prices yields

$$v_i(X_i)P(Z) \geq v_i(Z)w_i \ .$$

Finally, by definition of prices, $v_i(X_i) = w_i \frac{v_i(Z_i)}{P(Z_i)}$, and therefore $\frac{v_i(Z_i)}{P(Z_i)} \geq \frac{v_i(Z)}{P(Z)}$. □

Lemma 7. *For every competitive equilibrium (X, P), the allocation X is* WGF.

Proof. Let (X, P) be a CE. Assume towards a contradiction that X is not WGF, i.e., for some $S, T \subseteq \mathcal{N}$ there is a reallocation X' of $\cup_{j \in T} X_j$ among the agents in S, such that for all $i \in S$,

$$\frac{v_i(X'_i)}{v_i(X_i)} \geq \frac{w_T}{w_S}, \tag{7}$$

and at least one inequality is strict. Consider an agent $i \in S$. By choosing $Z_i = X_i$ and $Z = X'_i$ in the second condition of CE we obtain

$$P(X'_i) \geq \frac{v_i(X'_i)}{v_i(X_i)} P(X_i) = \frac{v_i(X'_i)}{v_i(X_i)} w_i,$$

where the last equation comes from the third condition of CE. Now summing over $i \in S$ and using (7) yields

$$\sum_{i \in S} P(X'_i) > \frac{w_T}{w_S} \sum_{i \in S} w_i = w_T \ ,$$

which is a contradiction since

$$\sum_{i \in S} P(X'_i) = P(\cup_{i \in S} X'_i) = P(\cup_{j \in T} X_j) = \sum_{j \in T} P(X_j) = \sum_{j \in T} w_j = w_T \ . \qquad \square$$

It remains to show the ex-post guarantee. At this point, we drop the consideration of the cake cutting setting and shift to the indivisible domain.

Lemma 8. *The $\mathcal{H}^{\mathsf{UG}}$-decomposition of a fractional MWN allocation is* WPROP1 *and* WEF_1^1.

Proof. We proceed like in (Freeman et al., 2020). Let X be a fractional MWN allocation and let Y be any deterministic allocation in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X.

We first show that Y is WPROP1. Consider an agent $i \in \mathcal{N}$. Observe that X as an MWN allocation is WGF and hence WPROP, i.e., it holds $v_i(X_i) \geq w_i \cdot v_i(\mathcal{G})$. Hence if $v_i(Y_i) \geq v_i(X_i)$, we are done. Otherwise, it holds $v_i(Y_i) < v_i(X_i)$. From Corollary 1 we know that in this case there exists $g^* \notin Y_i$ such that $v_i(Y_i) + v_i(g^*) > v_i(X_i) \geq w_i \cdot v_i(\mathcal{G})$. Hence Y is WPROP1.

It remains to show that Y is WEF_1^1. Consider two agents $i, j \in \mathcal{N}$. From Corollary 1 it follows that if $v_i(Y_i) < v_i(X_i)$, then there exists $g_i \notin Y_i$ such that

$$v_i(Y_i) + v_i(g_i) > v_i(X_i) \ . \tag{8}$$

Analogously, if $v_j(Y_j) > v_j(X_j)$, then there exists $g_j \in Y_j$ such that

$$v_j(Y_j) - v_j(g_j) < v_j(X_j) \ . \tag{9}$$

Next, we use Corollary 2 again, this time in the setting with discrete goods. It implies for any two agents $i, j \in \mathcal{N}$ and any $g \in \mathcal{G}$ with $x_{jg} > 0$,

$$w_j \frac{v_j(g)}{v_j(X_j)} \geq w_i \frac{v_i(g)}{v_i(X_i)} \ .$$

Summing over all $g \in Y_j \setminus \{g_j\}$ yields

$$w_j \frac{v_j(Y_j \setminus \{g_j\})}{v_j(X_j)} \geq w_i \frac{v_i(Y_j \setminus \{g_j\})}{v_i(X_i)} \ .$$

Note that the left-hand side of this inequality is strictly less than w_j due to (9). Hence, by inequality (8) it follows $w_j \cdot (v_i(Y_i) + v_i(g_i)) > w_j \cdot v_i(X_i) > w_i \cdot v_i(Y_j \setminus \{g_j\})$. This shows Y is WEF_1^1. $\square$

Theorem 4 now follows from Lemmas 6, 7, 8 and the fact that we can compute in strongly polynomial time an MWN allocation (Orlin, 2010) as well as its $\mathcal{H}^{\mathsf{UG}}$-decomposition (Proposition 2). $\square$

One might wonder whether the ex-post guarantee in Theorem 4 could be replaced by $\mathsf{WEF}(1,1)$, which in combination with $\mathsf{WPROP1}$ ex-post would strengthen Theorem 2, our main result. We next show that there are instances in which this combination is impossible even in the unweighted setting. This implies that out result is tight.

Proposition 7. *Ex-ante* WGF *is incompatible with ex-post* $\mathsf{WEF}(1,1)$*, even in the unweighted setting.*

Proof. Consider a fair division instance $\mathcal{I} = (\mathcal{N}, \mathcal{G}, \{v_i\}_{i \in \mathcal{N}})$ with 6 agents $\mathcal{N} = \{1, \ldots, 6\}$ and 10 goods $\mathcal{G} = \{\bar{g}\} \cup \{g_1, \ldots, g_9\}$. We call agents 1, 2, and 3 *left agents* and the remaining ones *right agents*. Also, we say good $\bar{g}$ is *big*, and all other goods are *small*. All left agents have the same valuation function v_{L} where $v_L(\bar{g}) = 9$ and $v_L(g) = 1$ for all $g \in \mathcal{G} \setminus \{\bar{g}\}$. All right agents have the same valuation function v_R where $v_R(\bar{g}) = 0$ and $v_R(g) = 1$ for all $g \in \mathcal{G} \setminus \{\bar{g}\}$.

Assume towards a contradiction that there exists a lottery $X = \sum_{h=1}^{k} \lambda_h Y^h$ which is GF ex-ante and $\mathsf{EF}(1,1)$ ex-post. Since group fairness implies fractional Pareto optimality, the big good $\bar{g}$ must be allocated completely to the left agents in X, i.e., $\sum_{i=1,2,3} x_{i\bar{g}} = 1$, as otherwise there would be a Pareto improvement. Moreover, all left agents must have the same utility in X, i.e., $v_i(X_i) = u_L$ for $i \in \{1, 2, 3\}$ since otherwise X would not be envy-free and thus not GF. The same holds for right agents, i.e., $v_i(X_i) = u_R$ for $i \in \{4, 5, 6\}$.

Now let $\alpha \geq 0$ be the fraction of small goods allocated to the left agents in X, i.e., $\alpha = \sum_{(i,g)\in\{1,2,3\}\times\mathcal{G}\setminus\{\bar{g}\}} x_{ig}$. By the above discussion, the utility of each left agent is $u_L = 3 + \alpha/3$, and the utility of each right agent is $u_R = 3 - \alpha/3$.

Next, we will argue that $\alpha \geq 2$. Consider an arbitrary allocation Y in the support of X. In Y, there must be at least two small goods allocated to left agents, since otherwise there would be at least one left agent with an empty bundle and a right agent with no less than three small goods, contradicting $\mathsf{EF}(1,1)$ ex-post. It immediately follows that also ex-ante there must be at least a fraction of two of small goods given to left agents, i.e., $\alpha \geq 2$.

Finally, we use the above properties of X to argue that the GF condition is violated for $S = \{1,2,3,4\}$ and $T = \{1,2,3\}$. We construct a fractional allocation X' of the goods owned by T among the agents in S such that for all $i \in S$,

$$v_i(X'_i) \geq \frac{|T|}{|S|} \cdot v_i(X_i) \tag{10}$$

where at least one inequality is strict. We construct X' as follows: Start with X and shift an $\varepsilon = \frac{9-\alpha}{4}$ fraction of small goods from the left agents to agent 4. Note that since $\alpha \geq 2$ this is possible without running out of small goods. Then, redistribute the remaining goods owned by the left agents equally among them, such that the utility of all left agents becomes $v_i(X'_i) = u_L - \varepsilon/3$ for $i \in \{1,2,3\}$. Note that agent 4 has utility $v_4(X'_4) = \varepsilon$ in X'.

It remains to show (10) for the constructed allocation X'. For agent 4 we have

$$v_4(X'_4) = \varepsilon = \frac{9-\alpha}{4} = \frac{3}{4} \cdot (3 - \alpha/3) = \frac{3}{4} \cdot u_R = \frac{|T|}{|S|} \cdot v_4(X_4) \ .$$

For any left agent $i \in \{1,2,3\}$ we have

$$\begin{aligned} v_i(X'_i) = u_L - \varepsilon/3 &= (3 + \alpha/3) - \frac{9-\alpha}{12} = \frac{9}{4} + \frac{5}{12}\alpha \\ &> \frac{3}{4} \cdot (3 + \alpha/3) = \frac{3}{4} \cdot u_L = \frac{|T|}{|S|} \cdot v_i(X_i) \ . \end{aligned}$$ □

5. Extensions to General Valuations

In this section, we explore to which extent our techniques apply to more general valuations. Since linearity of expectation no longer applies we can no longer use the marginal probabilities for a good to be in a random bundle. Instead, we shall explicitly compute the value for a random bundle by looking at the decomposition of the proposed lottery. A major challenge for non-additive valuations is that the (expected) utility of an agent for a lottery is not easily represented by the valuation of a fractional allocation. Nonetheless, both ex-ante WEF and ex-ante $\mathsf{WSD\text{-}EF}$ allocations exist for all valuations $\{v_i\}_{i\in\mathcal{N}}$. In particular, for ex-ante WEF one can simply assign $\mathcal{G}$ to agent i with probability w_i. For ex-ante $\mathsf{WSD\text{-}EF}$ it is sufficient to invoke DSE only using agents' priorities over single goods. Observe that for general valuations it is no longer true that ex-ante $\mathsf{WSD\text{-}EF}$ implies ex-ante WEF, not even in the unweighted setting. We next show that ex-ante WEF and either ex-post $\mathsf{WPROP1}$ or ex-post $\mathsf{WEF}(1,1)$ might no longer be possible.

Theorem 5. *For general valuations, ex-ante* WEF *is not compatible with* $\mathsf{WPROP1}$ *or* $\mathsf{WEF}(1,1)$*.*

Proof. Let us consider a fair division instance with two agents and four goods. Suppose agent 1 has an entitlement of $\frac{2}{3}$, and has value 1 for any bundle except the empty bundle, for which she has value 0. Agent 2 has entitlement $\frac{1}{3}$ and value k for any bundle of size k, for $k = 0, \ldots, 4$.

Let us denote by p_k the probability that agent 1 receives k goods. Note that, since the allocation must be complete, agent 2 in this case receives $4 - k$ goods. Using this notation, the ex-ante WEF condition for agent 1 is equivalent to

$$\frac{1}{3} \cdot (p_1 + p_2 + p_3 + p_4) \geq \frac{2}{3} \cdot (p_0 + p_1 + p_2 + p_3) \ .$$

On the other hand, if agent 1 receives more than two goods with positive probability, ex-post $\mathsf{WPROP1}$ as well as $\mathsf{WEF}(1,1)$ are violated for agent 2 (even WEF_1^1 is violated). Hence if either of the two holds, $p_3 = p_4 = 0$. From the ex-ante WEF condition for agent 1 it follows in this case $p_1 + p_2 \geq 2(p_0 + p_1 + p_2)$, and therefore $0 \geq 2p_0 + p_1 + p_2$, a contradiction. $\square$

Notice that both agents in the proof value 1 each good, and, hence, agent 1 is unit-demand and agent 2 is additive. This means that as soon as one agent is not additive in the weighted case our positive result no longer holds. We further observe that the proof relies on the fact that agents are asymmetric. Therefore, we next consider two questions:

1. What combination of ex-ante and ex-post properties we can guarantee in the weighted setting assuming slightly more general valuations (namely, XOS)?
2. Which valuations still guarantee ex-ante EF and $\mathsf{EF1}$ ex-post for symmetric agents?

We present answers to question 1) and 2) in Section 5.1 and 5.2, respectively.

5.1 XOS Valuations

For an agent i with XOS valuation, our algorithms only make use of the additive function f_i such that $v_i(\mathcal{G}) = \sum_{g \in \mathcal{G}} f_i(g)$. We either assume f_i to be known or have access to an XOS-oracle (using which f_i can be obtained with a single query). Given a query with a set $A \subseteq \mathcal{G}$, the XOS-oracle returns a function $f \in \mathcal{F}_i$ that maximizes $f(A)$.

Let X be the fractional allocation with $x_{ig} = w_i$, for each $i \in \mathcal{N}$ and $g \in \mathcal{G}$.

Proposition 8. *X is ex-ante* WPROP.

Proof. Let $\lambda_1 Y^1 + \cdots + \lambda_k Y^k$ be any decomposition of X. For any allocation $Y \in \{Y^\ell\}_{\ell \in [k]}$, $v_i(Y_i) = \max_{f \in \mathcal{F}_i} f(Y_i) \geq f_i(Y_i)$, since $f_i \in \mathcal{F}_i$. Hence, the expected utility of agent i in the lottery is

$$\begin{aligned}
\sum_{h=1}^{k} \lambda_h v_i(Y_i^h) \geq \sum_{h=1}^{k} \lambda_h f_i(Y_i^h) &= \sum_{h=1}^{k} \sum_{g \in Y_i^h} \lambda_h f_i(g) \\
&= \sum_{g \in \mathcal{G}} \sum_{h \,:\, g \in Y_i^h} \lambda_h f_i(g) = \sum_{g \in \mathcal{G}} f_i(g) \sum_{h \,:\, g \in Y_i^h} \lambda_h \\
&= \sum_{g \in \mathcal{G}} x_{ig} f_i(g) = w_i \sum_{g \in \mathcal{G}} f_i(g) = w_i \cdot v_i(\mathcal{G}) \ .
\end{aligned}$$

$\square$

In order to apply Theorem 1, we need to set up an appropriate additive function. For the next result, we assume that agent i has additive valuation f_i, for each $i \in \mathcal{N}$.

Proposition 9. *The* $\mathcal{H}^{\mathsf{UG}}$*-decomposition of* X *is ex-post* WPROP1*.*

Proof. Given any allocation Y of the decomposition, by definition of XOS, Corollary 1 and Proposition 8, we see that $v_i(Y_i \cup \{g\}) \geq f_i(Y_i \cup \{g\}) = f_i(Y_i) + f_i(g) > f_i(X_i) = w_i \cdot v_i(\mathcal{G})$. □

5.2 Equal Entitlements

Here we discuss to which extent we can guarantee BoBW results for equally entitled agents and general valuations. In particular, we explore valuation functions for which DSE with the $\mathcal{H}^{\mathsf{UG}}$-decomposition can be used to guarantee ex-ante EF and ex-post EF1. Recall that DSE was already introduced as the Eating procedure by Aziz (2020) for equally entitled agents.

Both DSE and the definition of the $\mathcal{H}^{\mathsf{UG}}$ bihierarchy only depend on the ranking of each agent for singleton bundles of goods. Therefore, we can determine a random allocation with DSE and compute its $\mathcal{H}^{\mathsf{UG}}$-decomposition for any class of valuation functions. Since the concept of SD-EF depends only on the ranking of single goods provided by the agents, the output of DSE is always an ex-ante SD-EF allocation, regardless of the considered valuation functions.

Unfortunately, for general valuations, it is no longer true that an SD-EF allocation X is ex-ante EF not even if X is the output of DSE. Such an impossibility holds also for agents having unit-demand valuations as the following example shows.

Example 4. Let $\mathcal{I}$ be a fair division instance with three agents and three goods. Assume agents to have identical unit-demand valuations; goods have all value 1. Having all agents the same priorities over goods, the output of DSE is given by the following:

$$X^{\text{DSE}} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} .$$

Being X^{DSE} the output of DSE it is SD-EF; we next provide a decomposition of X^{DSE} showing that X^{DSE} is not necessarily ex-ante EF. Let us consider the following decomposition:

$$X^{\text{DSE}} = \frac{1}{3} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{Y^1} + \frac{1}{3} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{Y^2} + \frac{1}{3} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{Y^3} .$$

Notice that such a decomposition is not an $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} (in any $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}, every agent receives deterministically exactly one good). We claim that agent 1 (corresponding to the first row) is not ex-ante EF. In fact, the expected utility she has for her random bundle is given by

$$\frac{1}{3} \cdot \left(v_1(Y_1^1) + v_1(Y_1^2) + v_1(Y_1^3) \right) = \frac{1}{3} \cdot (1 + 1 + 0) = \frac{2}{3} ;$$

while the expected value of agent 1 for the random bundle of agent 2 (second row) is

$$\frac{1}{3} \cdot \left(v_1(Y_2^1) + v_1(Y_2^2) + v_1(Y_2^3))\right) = \frac{1}{3} \cdot (1 + 1 + 1) = 1 \ . \qquad \blacksquare$$

Although for multi-demand valuations SD-EF does not imply ex-ante EF, we next prove that the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is indeed ex-ante EF. Moreover, such a decomposition also guarantees ex-post EF1.

Theorem 6. *For equal entitlements and multi-demand valuations, the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is ex-ante* EF *and ex-post* EF1*.*

Proof. As already observed, any allocation in the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is the result of an RB picking sequence (in case of equal entitlements our approach coincides with the one of Aziz, 2020). Hence, ex-post EF1 follows by noticing that any RB picking sequence is EF1 for multi-demand valuations.

We now prove ex-ante EF of the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE}. For convenience, we denote X^{DSE} by X and assume i to be k-unit-demand. Consider any agent $i \in \mathcal{N}$. Given any $j \in \mathcal{N} \setminus \{i\}$, we need to show that $v_i(X_i) \geq v_i(X_j)$. In what follows, we again sort goods $g_1, \dots, g_m$ in the ordering induced by agent i over $\mathcal{G}$.

Let us first show that $v_i(X_i) = \sum_{g \in \mathcal{G}} v_i(g) \cdot \overline{x}_{ig}$, where

$$\overline{x}_{ig_\ell} = \begin{cases} x_{ig_\ell} & \text{if } \sum_{h=1}^{\ell} x_{ig_h} \leq k, \\ k - \sum_{h=1}^{\ell-1} x_{ig_h} & \text{if } \sum_{h=1}^{\ell} x_{ig_h} > k \text{ and } \sum_{h=1}^{\ell-1} x_{ig_h} < k, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, only the first k fraction of goods the agent ate does count in her utility. Let us denote by g_{h^*} the least valuable good for which $\overline{x}_{ig_{h^*}} > 0$. Notice that at time $t = k$ of DSE agent i is eating or finishes eating good g_{h^*}.

Recall that any agent needs $1/w_i = n$ units of time to eat 1 unit of goods. The feasibility conditions of Theorem 1 ensure that in any allocation Y in the $\mathcal{H}^{\mathsf{UG}}$-decomposition, the h-most preferred good of agent i in her bundle Y_i is one good she ate in the time interval $(n(h-1), nh]$. Further, only one good eaten in this interval will be the h-most preferred in the bundle Y_i. Since the valuation of i is k-unit-demand, only the goods she ate by the time kn will contribute to her utility. This implies $v_i(X_i) = \sum_{g \in \mathcal{G}} v_i(g) \cdot \overline{x}_{ig}$.

We now provide an upper bound on $v_i(X_j)$, i.e., the expected utility i has for the bundle of j. Let us denote by p_h the probability that g_h is in the bundle X_j and is one of the k most preferred goods of agent i among the ones in X_j. The probability p_h is upper-bounded by the probability of having g_h in the bundle of j, and therefore $p_h \leq x_{jg_h}$. Moreover, $\sum_{h=1}^{m} p_h = \sum_{h=1}^{m} \sum_{\ell=1}^{k} p_h^\ell = \sum_{\ell=1}^{k} \sum_{h=1}^{m} p_h^\ell \leq k$, where p_h^ℓ is the probability that g_h is the ℓ-th most preferred good of i in j's bundle. Notice also that $\sum_{h=1}^{h^*-1} x_{jg_h} \leq k$ since, $\sum_{h=1}^{h^*-1} x_{jg_h} \leq \sum_{h=1}^{h^*-1} x_{ig_h}$, by the execution of DSE, and $\sum_{h=1}^{h^*-1} x_{ig_h} \leq k$ by definition of h^*. Therefore, the expected utility of i for j's bundle is given by

$$v_i(X_j) = \sum_{h=1}^{m} p_h \cdot v_i(g_h) \leq \sum_{h=1}^{h^*-1} x_{jg_h} \cdot v_i(g_h) + \left(k - \sum_{h=1}^{h^*-1} x_{jg_h}\right) \cdot v_i(g_{h^*}) \ ,$$

where the inequality holds because of the aforementioned properties of p_h and x_{jg_h}, and the fact that $g_1, \ldots, g_m$ are sorted in a decreasing manner with respect to i's valuations. Finally, by stochastic dominance,

$$\sum_{h=1}^{h^*-1} x_{jg_h} \cdot v_i(g_h) + \left(k - \sum_{h=1}^{h^*-1} x_{jg_h}\right) \cdot v_i(g_{h^*}) \leq \sum_{h=1}^{h^*-1} x_{ig_h} \cdot v_i(g_h) + \left(k - \sum_{h=1}^{h^*-1} x_{ig_h}\right) \cdot v_i(g_{h^*})$$
$$= \sum_{g \in \mathcal{G}} v_i(g) \cdot \overline{x}_{ig} \leq v_i(X_i) \ .$$

In conclusion, $v_i(X_j) \leq v_i(X_i)$ and the theorem follows. □

Notice that the proof of Theorem 6 takes into account only an agent i having multi-demand valuations for a given k, and not the valuations of the others. For this reason the theorem holds for multi-demand agents having different demands k. Moreover, when $k \geq m$ the agent is additive. These simple observations lead to the following.

Corollary 3. *For equal entitlements and any combination of additive and multi-demand valuations, the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is ex-ante* EF *and ex-post* EF1.

Cancelable Valuations Turning to more general cancelable valuations, we can show that RB picking sequences still provide an EF1 allocation, and therefore the ex-post guarantee is maintained. It remains an interesting open problem to prove that the lottery is ex-ante EF (not only ex-ante SD-EF).

As mentioned above, for additive valuations and equal entitlements the $\mathcal{H}^{\mathsf{UG}}$-decomposition of X^{DSE} is ex-ante SD-EF as well as ex-post EF1. The latter results from the fact that every allocation in the support of the lottery emerges from an RB picking sequence. We now discuss the implications of this approach for cancelable valuation functions. Given the cancelable valuation function v_i, for each $i \in \mathcal{N}$, we create a corresponding additive valuation function $\hat{v}_i$ with $\hat{v}_i(g) = v_i(g)$. Then we apply the result for additive valuations and equal entitlements. This implies the obtained lottery is ex-ante SD-EF, as the priorities over goods are the same for v_i and $\hat{v}_i$.

Let us discuss ex-post properties. We show that any RB picking sequence for $\hat{v}$ yields an allocation that is ex-post EF1 for v. To this aim, we need the following lemmas.

Lemma 9. *Let v be a cancelable valuation function. Given any $S, T \subseteq \mathcal{G}$ and any $R \subseteq \mathcal{G} \setminus (S \cup T)$, if $v(S) \geq v(T)$, then*

$$v(S \cup R) \geq v(T \cup R).$$

Proof. Let $R_i = \{g_1, \ldots, g_i\}$ and $R_k = R$. If $v(T \cup R_i) > v(S \cup R_i)$, then $v(T \cup R_{i-1}) > v(S \cup R_{i-1})$ by cancelability of v. Applying this result for all i from k down to 1, we get that if $v(T \cup R) > v(S \cup R)$, then $v(T) > v(S)$ – a contradiction. □

Lemma 10. *Let v be a cancelable valuation function. Given any $S, T \subseteq \mathcal{G}$ and any $g, g' \in \mathcal{G} \setminus (S \cup T)$, if $v(S) \geq v(T)$ and $v(g) \geq v(g')$, then*

$$v(S \cup \{g\}) \geq v(T \cup \{g'\}).$$

Proof. Applying Lemma 9 twice, $v(S \cup \{g\}) \geq v(T \cup \{g\}) \geq v(T \cup \{g'\})$ follows. $\square$

Recall that an RB picking sequence takes as input an ordering of the agents and agents' priorities over goods. Consequently, an RB picking sequence produces the very same outcome for v and $\hat{v}$ (if ties are broken in the same manner).

Proposition 10. *Given a fair division instance with cancelable valuations, any RB picking sequence yields an allocation that is* $\mathsf{EF1}$.

Proof. Let Y be an allocation obtained by an RB picking sequence.

Given any pair of agents i, j, the sizes of Y_i and Y_j differ by at most 1. We next show that agent i is $\mathsf{EF1}$ towards agent j: Without loss of generality, we can assume $|Y_j| = |Y_i| + 1$. Let us denote by g^* the most preferred good of i in Y_j. Because of the RB property, there exists a matching $M = \{(g, g')|g \in Y_i, g' \in Y_j \setminus \{g^*\}\}$ such that $v_i(g) \geq v_i(g')$ for each $(g, g') \in M$. Recursively applying Lemma 10 over the pairs in M we obtain $v_i(Y_i) \geq v_i(Y_j \setminus \{g^*\})$. Therefore, i is $\mathsf{EF1}$ towards agent j; applying the same arguments on any pair of agents the thesis follows. $\square$

Overall, our result for cancelable valuations is as follows.

Theorem 7. *For equal entitlements and cancelable valuations, the* $\mathcal{H}^{\mathsf{UG}}$*-decomposition of* X^{DSE} *is ex-ante* $\mathsf{SD\text{-}EF}$ *and ex-post* $\mathsf{EF1}$.

Unfortunately, we were not able to prove that the lottery is ex-ante EF (only ex-ante $\mathsf{SD\text{-}EF}$) for cancelable valuations, and this remains an interesting open question.

6. Conclusion

We studied Best-of-Both-World (BoBW) fairness and efficiency guarantees in the fair division setting with entitlements. In particular, we provided a lottery guaranteeing WEF ex-ante as well as $\mathsf{WEF}(1,1)$ + $\mathsf{WPROP1}$ ex-post. Our analysis makes use of a strong matrix decomposition theorem by Budish et al. (2013) and picking sequences (Chakraborty et al., 2021a, 2021b, 2022). Moreover, we generalized a result on group fairness by Freeman et al. (2020) to entitlements – we obtain the stronger ex-ante fairness notion of WGF in combination with a related ex-post guarantee of WEF_1^1 + $\mathsf{WPROP1}$. All our lotteries can be computed in strongly polynomial time. Interestingly, both results are shown to be tight.

Furthermore, we investigated to which extent the techniques can be used to obtain BoBW results for more general valuations, such as XOS, multi-demand, and cancelable ones. For XOS valuations we obtained WPROP ex-ante and $\mathsf{WPROP1}$ ex-post. Assuming oracle access, the underlying lottery can be computed in strongly polynomial time. For equal entitlements, we showed that EF ex-ante is possible together with $\mathsf{EF1}$ ex-post even for multi-demand valuations, while previously such a guarantee was only known for additive valuations (Freeman et al., 2020; Aziz, 2020). Finally, we showed that for cancelable valuations, the Budish decomposition applied to the result of a generic eating procedure is $\mathsf{SD\text{-}EF}$ ex-ante and $\mathsf{EF1}$ ex-post. Previously, this was known only for additive valuations.

There are many open problems that remain in this area. For example, it would be interesting to characterize the class of valuation functions which allow lotteries that are

EF ex-ante and EF1 ex-post. Moreover, it is an interesting open problem to obtain BoBW results with stronger ex-post guarantees, such as EFX. Recent work has started to explore the BoBW approach with approximate fairness concepts, and there are many open problems. Finally, as a more concrete problem, it remains open whether SD-EF implies EF for cancelable valuations.

Another interesting direction is to investigate the BoBW paradigm for chores and mixed manna. While the results of (Freeman et al., 2020; Aziz, 2020) easily extend to chores for unweighted agents, it remains unclear whether our results for entitled agents hold true in this setting. Besides this, exploring the BoBW paradigm for mixed manna would also be interesting both with and without entitled agents.

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