Approximating Proportional and Maximin Allocations on D-Claw-Free Graphs

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We study the problem of fair allocation of indivisible goods that form a graph and the bundles that are distributed to agents are connected subgraphs of this graph. We focus on the proportional and the maximin share fairness criteria. It is well-known that allocations satisfying these criteria may not exist for many graphs including complete graphs and cycles. Therefore, it is natural to look for approximate allocations, i.e., allocations guaranteeing each agent a certain portion of the value that is satisfactory to her. In this paper we consider the class of graphs of goods which do not contain a star with $d + 1$ edges (where $d > 1$) as an induced subgraph. This is a large class of graphs containing graphs with the maximum degree bounded by d. For this class of graphs we show a theorem which specifies what fraction of the proportional share can be guaranteed to each of n agents if the values of single goods for the agents are bounded by a given fraction of this share. Furthermore, an allocation ensuring such a guarantee can be computed in polynomial time. This theorem can be viewed as analogous to a result proved by Hill in 1987, who solved a similar problem for proportional allocations without connectivity constraints. Moreover, for the same class of graphs of goods, we prove that there is an allocation assigning each of n agents a connected bundle of value at least $\frac{n}{d(n-1)+1} > \frac{1}{d}$ of her maximin share, and this allocation can be computed in polynomial time.

JAIR Associate Editor: Piotr Skowron

JAIR Reference Format:
Zbigniew Lonc. 2026. Approximating Proportional and Maximin Allocations on D-Claw-Free Graphs. *Journal of Artificial Intelligence Research* 85, Article 23 (February 2026), 27 pages. DOI: 10.1613/jair.1.17263

1 Introduction

The problem of fair allocation of indivisible goods is a fundamental problem in social choice theory (Bouveret, Chevaleyre, et al. 2016; Brams and Taylor 1996). We assume that we have a set of items that we call goods and a set of agents, each with her own utility function, which assigns some values to all subsets of goods (called *bundles*). The utility functions are commonly assumed to be additive and so we need to specify their values on individual goods only. We make this assumption in this paper, too. The objective is to assign the goods to the agents so that some fairness criterion is satisfied. Some of these criteria like proportionality or envy-freeness originate from the problem of fair allocation of divisible goods which is known as the cake cutting problem (Brams and Taylor 1996; A. Procaccia 2016). Unlike in the case of allocation of divisible goods where proportional and envy-free allocations always exist, for indivisible goods it is very easy to construct examples when no allocations satisfying these criteria exist.

(Budish 2011) proposed a new fairness criterion for indivisible goods called the maximin share fairness criterion. It is based on the well-known cut and choose protocol in the cake cutting problem. Each of n agents first finds a partition of the set of goods into n bundles such that the least-valued bundle is maximized. The maximin share for this agent is equal to the value of this least-valued bundle. An allocation satisfies the maximin criterion if

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DOI: 10.1613/jair.1.17263

each agent receives a set of goods of value not smaller than her maximin share. We call such allocations mms-allocations. There are examples which show that an mms-allocation may not exist (Feige et al. 2022; Kurokawa et al. 2018; A. D. Procaccia and Wang 2014). Nevertheless, such examples are quite intricate. To get around the difficulty of constructing mms-allocations, several researchers proposed to relax the maximin share criterion by requiring that the value of each agent's share in an allocation be at least equal to some positive fraction of the maximin share (Akrami, Garg, et al. 2023; Amanatidis, Markakis, et al. 2017; Barman and Krishnamurthy 2020; Garg, McGlaughlin, et al. 2019; Garg and Taki 2021; Ghodsi et al. 2018; Kurokawa et al. 2018; A. D. Procaccia and Wang 2014). Currently, the strongest result of this kind shows that a $(3/4+3/3836)$-approximating mms-allocation always exists (Akrami and Garg 2024).

In the case of proportional allocations of indivisible goods, no nontrivial approximation can be guaranteed. Indeed, consider two agents and two goods, one of value 1 and one of value 0 to both agents. In this case, no allocation guarantees both agents any positive fraction of their proportional shares. However, the situation changes if we impose sufficiently small upper bounds on the values of individual goods relative to the total value of all goods for the agents. This problem was studied by (Hill 1987), who determined, for any $\alpha \in (0, 1]$, the largest fraction of the proportional share that can be guaranteed to each agent, if the values of individual goods for each agent do not exceed the fraction α of this share.

In this paper we study the problems of proportional and maximin share allocations in the setting proposed by (Bouveret, Cechlárová, Elkind, et al. 2017). In that setting the goods are vertices of an undirected graph (that we call the *graph of goods*) and the bundles to be assigned to the agents, as well as those in the definition of the maximin share, induce connected subgraphs of the graph of goods.

This setting captures important structural constraints that naturally arise in a variety of real-world applications, by excluding allocations that involve certain undesirable or impractical sets of goods. In particular, it ensures that the sets of goods allocated to agents are not just fair, but also practically usable or accessible as a whole. A typical and well-studied example is the problem of land consolidation, where individual landowners or farmers are assigned contiguous plots of land (King and Burton 1982). Without the connectivity requirement, a landowner might receive several disconnected parcels scattered across a region, which could severely reduce the utility of the allocation due to increased transportation costs, inefficiencies in land usage, or legal complications. Another representative application is the allocation of rooms within a building to university departments or research groups (Bouveret, Cechlárová, Elkind, et al. 2017). In such cases, departments often require that their assigned rooms are adjacent or at least close to one another to facilitate collaboration, improve internal communication, and maintain organizational coherence. Allocating disconnected rooms across different parts of the building would be highly undesirable and impractical.

In this setting, the original problem of fair allocation of indivisible goods becomes a special case where the graph of goods is complete. (Bouveret, Cechlárová, Elkind, et al. 2017) showed several complexity and algorithmic results on envy-free, proportional and mms-allocations for different graphs of goods. Among others, they proved that if the graph of goods is a tree then an mms-allocation always exists and can be computed in polynomial time. Moreover, they gave an example demonstrating that for some choice of utility functions of the agents an mms-allocation does not exist when the graph of goods is a cycle.

The case of cycles as graphs of goods was thoroughly studied by (Lonc and Truszczynski 2020). They proved that in this case it is possible to guarantee to each agent the fraction $\frac{\sqrt{5}-1}{2} \approx 0.62$ of her maximin share. They also found an example showing that if the graph of goods is a cycle, then for some choice of the utility functions of the agents, in any allocation, some agent receives a bundle of value at most $\frac{3}{4}$ of her maximin share.

For arbitrary graphs of goods it is not known whether a positive fraction of the maximin share can be guaranteed. In this paper we prove that for a large class of graphs (including graphs with vertex degrees bounded by a constant) such guarantee indeed exists.

We focus on a wide class of graphs of goods called *d-claw-free graphs*. These are graphs that do not contain a d-edge star as an induced subgraph. Obviously, all graphs with maximum degree at most d are $(d+1)$-claw-free. Probably the most interesting case concerns 3-claw-free graphs for which the approximation ratios that we prove are the best. The class of 3-claw-free graphs (also called just *claw-free graphs*) is a large and important class of graphs widely studied in graph theory (Chudnovsky and Seymour 2010; Faudree et al. 1997).

The problem of fair allocation for 3-claw-free graphs of goods is closely related to the following "edge variant" of fair allocation proposed by (Truszczynski 2021): Given a connected graph G, where *edges* represent goods, and a set of agents with utilities on edges, find a fair allocation of connected bundles of edges to agents. This problem is equivalent to the problem where *vertices* of a graph represent goods, considered for the graph of goods $L(G)$ being the line graph of G. (In the line graph $L(G)$ the edges of G are vertices and two vertices of $L(G)$ are adjacent if the corresponding edges in G have a common vertex.) It is well-known that line graphs are 3-claw-free, which provides additional motivation to consider fair allocations for 3-claw-free graphs of goods.

The "edge variant" described above has a natural interpretation in the context of transportation networks. Imagine a scenario in which several air carriers (agents) are competing to offer flight services between pairs of cities. In this setting, each connection (good) between two cities corresponds to an edge in a graph, where the vertices represent the cities themselves. Each carrier aims to obtain a subset of these connections that forms a connected subgraph, i.e., a network where all their serviced cities are reachable from one another using only the connections (edges) assigned to them. This connectivity is desirable as it enables carriers to optimize transfer logistics and provide efficient service to their customers without relying on routes controlled by competitors. From the perspective of a central authority (such as a government), the objective is to allocate all connections (edges) among the carriers so that every possible connection between cities is operated by some carrier and certain fairness criteria are satisfied.

1.1 Our Contribution

In this paper, we prove two main results on fair allocations for connected $(d+1)$-claw-free graphs of goods. The first concerns the approximation of proportional allocations and the second one addresses the approximation of mms-allocations.

Intuitively, for proportional allocations, the smaller the values of goods relative to the agent's proportional share, the closer to the proportional allocation we can get. Our first main contribution quantifies this intuition for connected $(d+1)$-claw-free graphs of goods, where $d \geq 2$ (Theorem 3.3). More precisely, we prove the following results.

(i) Let n be the number of agents and let S_i denote the total value of all vertices of the graph of goods to agent i . For every $\alpha > 0$, we determine a value $\beta = \beta_n(\alpha)$ such that, for any connected $(d+1)$-claw-free graph of goods and any set of n agents for whom the value of each individual vertex is bounded by $\alpha\frac{S_i}{n}$, there exists an allocation that assigns every agent i a connected bundle of value at least $\beta\frac{S_i}{n}$.

(ii) Such allocation can be computed in polynomial time.

(iii) The result in (i) is tight. For every $\alpha > 0$, we provide examples showing that the value of $\beta_n(\alpha)$ is the largest possible under the given assumptions.

These results are analogous to those obtained by (Hill 1987) and (Markakis and Psomas 2011), who solved a similar problem for proportional allocations without any connectivity constraints.

It is worth mentioning that for any class of graphs of goods that contains arbitrarily large stars, it is impossible to guarantee any non-zero fraction of the proportional share to agents whose valuations for individual vertices are bounded by an α-fraction of that share, for any $\alpha > 0$ (Proposition 3.1). This observation explains why $(d+1)$-claw-free graphs are particularly interesting in the context of proportional allocations.

Our main results on the approximation of mms-allocations (Theorem 4.1) look as follows.

(i) For any $(d+1)$-claw-free graph of goods, any set of n agents, and $d \geq 2$, there exists an allocation that guarantees each agent a bundle of value at least $\frac{n}{d(n-1)+1} > \frac{1}{d}$ of her maximin share.
(ii) Such allocation can be computed in polynomial time.

In contrast to proportional allocations, for which we provide examples demonstrating the tightness of the results, we do not have a tightness result for mms-allocations. In fact, little is known about the upper bounds on the approximation ratios achievable by allocations in this setting. To the best of our knowledge, the strongest known general result concerns cycles, which are, obviously, $(d+1)$-claw-free for every $d \geq 2$. (Lonc and Truszczynski 2020) showed that if the graph of goods is a cycle and there are at least 6 agents, then for some choice of utility functions, no allocation can guarantee more than $\frac{3}{4}$ of the maximin share to all agents. They also proved that for the same class of graphs, if the number of agents is greater than 2 but fewer than 6, then no allocation can guarantee more than $\frac{5}{6}$ of the maximin share to all agents. Finally, it is well known that for 2 agents and any connected graph of goods an mms-allocation always exists.

A crucial ingredient of the proof of Theorem 4.1 is Corollary 3.9 on approximation of proportional allocations, which follows from Theorem 3.3. In this proof, we also use another result (Lemma 4.5) that enables us to reduce the problem of the existence of an allocation assigning each agent a fraction of her maximin share value when the agents are arbitrary, to the problem of the existence of an allocation assigning each agent a fraction of her proportional share when the agents are not arbitrary, but their utility functions are bounded by a fraction of their maximin share value. Lemma 4.5, which might be of an independent interest, applies not only to the class of $(d+1)$-claw-free graphs but also to any hereditary class of graphs.

Proofs of Theorem 4.1 and Lemma 4.5 require an extension of the problem of approximation of mms-allocations to *disconnected* graphs of goods. Therefore, we formulate and prove these results in this more general setting. In this extension, we still require that bundles are sets of vertices of connected subgraphs of the graph of goods. However, we do not require that *all* vertices of the graph of goods are distributed to agents.

Even though mms-allocations for disconnected graphs of goods play an auxiliary role in this paper, we believe that studying such allocations is a natural and interesting direction for future research.

This paper is an extension of an earlier conference paper (Lonc 2023). The results presented in this paper are stronger than those in (Lonc 2023). The conference paper contains a weaker and simplified version of Theorem 3.3, which appears as Corollary 3.8 in the present version. Furthermore, the second main result of this paper (Theorem 4.1) is stronger than the corresponding result given in the conference paper in two respects. First, the approximation ratio for mms-allocation given in Theorem 4.1 is better than the one in (Lonc 2023, Theorem 2), especially for a small number of agents. Second, unlike in (Lonc 2023), we provide a polynomial construction of this approximate mms-allocation. Moreover, in this paper, we provide complete proofs of all the results.

1.2 Related Work

The decision problem of whether a proportional allocation of indivisible goods exists is NP-complete. This can be readily demonstrated through a reduction from the problem of partitioning a set of integers into two subsets of equal sums (Bouveret and Lemaître 2016).

As we mentioned earlier, approximations of proportional allocations without any connectivity constraints (i.e., when the graph of goods is complete) were studied by (Hill 1987). Consider n agents for whom the total value of all goods is 1. In his work, (Hill 1987) identified a certain function $V_n(\cdot)$ such that when the maximum value of every good is bounded by α for each agent, then there always exists an allocation where every agent receives a bundle of value at least $V_n(\alpha)$. Moreover, he showed that this result is sharp. (Markakis and Psomas 2011) strengthened the result of (Hill 1987) by presenting a polynomial-time algorithm for constructing allocations guaranteeing each agent a bundle of value at least $V_n(\alpha)$. Further results in this direction were also obtained by

(Demko and Hill 1988) and (Gourvès et al. 2015). Our Theorem 3.3 can be viewed as an analog to the result by (Hill 1987) when the graph of goods is $(d+1)$-claw-free.

The subject of fair division of a graph into connected bundles received a considerable attention in recent years (see survey papers by (Suksompong 2021) and (Biswas et al. 2023)). We have already mentioned the papers by (Bouveret, Cechlárová, Elkind, et al. 2017), which initiated research in this direction, and by (Lonc and Truszczynski 2020) who examined the case when the graph of goods is a cycle. (Bei et al. 2022) studied the concept of *price of connectivity* for a graph G which they defined as the worst-case ratio between the maximin share computed with the restriction that all bundles are connected in G and without this restriction. Their result most closely related to the problems considered in this paper (Bei et al. 2022, Theorem 3.14) implies that for every graph of goods G there exists an allocation that guarantees each agent a bundle of value at least $\frac{1}{m-n+1}$ of her maximin share, where n is the number of agents, m is the number of vertices in G and $m \geq n$. This guarantee becomes, however, very weak when the size of the graph grows. Our Theorem 4.1 guarantees that, for large classes of graphs, a fraction of the maximin share independent on the size of the graph can be assigned to each agent.

Several results on relaxations of envy-free allocations with connected bundles were proved by (Bilò et al. 2022) and (Bei et al. 2022). (Suksompong 2019) presented several approximation results concerning envy-freeness, proportionality, and equitability fairness criteria when the graph of goods is a path. (Igarashi and Peters 2019) studied the problem of allocation of connected bundles that are Pareto optimal. (Bouveret, Cechlárová, and Lesca 2019) and (Xiao et al. 2023) examined the problem of connected fair allocation of indivisible *chores* (i.e., items that yield disutility to the agents). (Greco and Scarcello 2024) studied maxileximin allocations in the context of connectivity constraints. These allocations aim to minimize agents' dissatisfaction while maximizing social welfare. (Greco and Scarcello 2020) and (Deligkas et al. 2021) examined computational complexity issues pertaining to fair allocation on graphs. Finally, (Hummel and Igarashi 2024) studied local fairness issues in graph fair division and focused on pairwise maximin share allocations.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of agents and let V be a set of goods. We assume that the set of goods V is the vertex set of some undirected (and not necessarily connected) graph G called the *graph of goods*. Every subset of V inducing in G a connected subgraph is called a G-*bundle* (or just a *bundle* if the graph G is clear from the context).

For each agent $i \in N$ there is a *utility function* u_i which assigns nonnegative real numbers to goods in V. We extend utility functions to subsets of V by assuming additivity, i.e., for a utility function u_i defined on V and for every $X \subseteq V$, we define $u_i(X) := \sum_{v \in X} u_i(v)$. We say that a good $v \in V$ (respectively, a set of goods $X \subseteq V$) is *of value x to agent i* if $u_i(v) = x$ (resp. $u_i(X) = x$). To simplify the notation, we define $S_i := u_i(V)$.

In our setting, by an *allocation* we mean an assignment of pairwise disjoint G-bundles to the agents in N. We do not assume here that *every* good of V is assigned to some agent. If it is, then we call such an allocation *complete*. We represent allocations by sequences $(A_1, \ldots, A_n)$, where each set $A_i \subseteq V$ is a G-bundle assigned to agent $i \in N$.

Any family $\{P_1, \ldots, P_n\}$ of pairwise disjoint G-bundles is called a (G,n)-*packing*. If, in addition, $\bigcup_{i=1}^{n} P_i = V$ then such a (G,n)-packing is called a (G,n)-*split* of G. For an agent with utility function u we define the *maximin share* as

$$mms^{(n)}(G,u) := \max_{\{P_1,\ldots,P_n\}} \min_{j=1,\ldots,n} u(P_j),$$

where the maximum is computed over all (G,n)-splits $\{P_1, \ldots, P_n\}$ of G. We call a (G,n)-split for which the maximum is attained an *mms-split*.

If the graph of goods G is disconnected and has more than n components then no (G,n)-split exists so the maximin share is undefined. However, if we replace in the definition of the maximin share (G,n)-splits by

(G, n)-packings, then we obtain a parameter that is defined for every graph G, whether it is connected or not. Formally, for an agent with utility function u, the *packing maximin share* is:

$$pmms^{(n)}(G, u) := \max_{\{P_1,\dots,P_n\}} \min_{j=1,\dots,n} u(P_j),$$

where the maximum is computed over all (G, n)-*packings* $\{P_1, \dots, P_n\}$ of G.[1] We call a (G, n)-packing for which the maximum is attained an *mms-packing*. Obviously, for every graph for which $mms^{(n)}(G, u)$ is defined, we have $pmms^{(n)}(G, u) \geq mms^{(n)}(G, u)$.

An allocation $(A_1, \dots, A_n)$ is an *mms-allocation* (resp. *packing mms-allocation*) if

$$u_i(A_i) \geq mms^{(n)}(G, u_i)$$

$$\big(\text{respectively, } u_i(A_i) \geq pmms^{(n)}(G, u_i)\big),$$

for every agent $i \in N$. For any $c > 0$, an allocation $(A_1, \dots, A_n)$ is a c-*mms allocation* (resp. c-*pmms allocation*) if

$$u_i(A_i) \geq c \cdot mms^{(n)}(G, u_i)$$

$$\big(\text{respectively, } u_i(A_i) \geq c \cdot pmms^{(n)}(G, u_i)\big),$$

for every agent $i \in N$ with utility function u_i.

For a *connected* graph of goods, we call an allocation $(A_1, \dots, A_n)$ *proportional* (resp. c-*proportional*) if

$$u_i(A_i) \geq \frac{u_i(V)}{n} \quad \big(\text{respectively, } u_i(A_i) \geq c \cdot \frac{u_i(V)}{n}\big),$$

for every agent $i \in N$ with utility function u_i.

Every c-proportional allocation is both a c-mms and a c-pmms allocation. Indeed, let $\{P_1, \dots, P_n\}$ be an mms-packing for a connected graph G and agent i. Then,

$$\begin{aligned} u_i(V) \geq \sum_{j=1}^{n} u_i(P_j) &\geq n \cdot pmms^{(n)}(G, u_i) \\ &\geq n \cdot mms^{(n)}(G, u_i). \end{aligned} \tag{1}$$

Let G be a *connected* graph of goods and let $c > 0$. An agent $i \in N$ with utility function u_i is c-*proportionally bounded* if

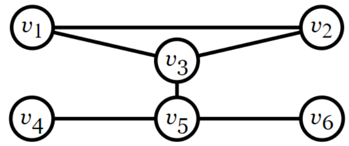
$$u_i(v) \leq c \cdot \frac{u_i(V)}{n},$$

for every vertex $v \in V(G)$.

To illustrate some of the concepts introduced above, let us consider the graph, say G, in Figure 1. The graph defines the set of goods $V = \{v_1, \dots, v_6\}$ and their adjacency relation. The table in Figure 1 shows two utility functions u_1 and u_2 of two different agents on the set of goods V. By examining all $(G, 2)$-splits one can easily show that the family $\{\{v_1, v_2\}, \{v_3, v_4, v_5, v_6\}\}$ is a $(G, 2)$-split, which is an mms-split for the utility function u_1 with its bundles having values 8 and 10, respectively. Thus, $mms^{(2)}(G, u_1) = 8$. Similarly, $mms^{(2)}(G, u_2) = 3$ because $\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$ is a $(G, 2)$-split, which is an mms-split for the utility function u_2. Hence, the sequence $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$ is an mms-allocation because $u_1(\{v_1, v_2, v_3\}) = 12 \geq 8$ and $u_2(\{v_4, v_5, v_6\}) = 3$.

Since $u_1(V) = 18$ and $u_2(V) = 10$, the sequence $(\{v_4, v_5, v_6\}, \{v_1, v_2, v_3\})$ is a $\frac{2}{3}$-proportional allocation because $u_1(\{v_4, v_5, v_6\}) = 6 \geq \frac{2}{3} \cdot \frac{18}{2} = \frac{2}{3} \cdot \frac{u_1(V)}{2}$ and $u_2(\{v_1, v_2, v_3\}) = 7 \geq \frac{2}{3} \cdot \frac{10}{2} = \frac{2}{3} \cdot \frac{u_2(V)}{2}$. The agent 1 is $\frac{1}{2}$-proportionally bounded because the largest value of a single vertex for this agent is $4 \leq \frac{1}{2} \cdot \frac{18}{2} = \frac{1}{2} \cdot \frac{u_1(V)}{2}$.

[1]The abbreviation PMMS has already been used in the literature for 'pairwise maximin share fairness' (e.g., see (Amanatidis, Aziz, et al. 2023)). These two meanings of this abbreviation should not be confused.



	v_1	v_2	v_3	v_4	v_5	v_6
u_1	4	4	4	2	2	2
u_2	1	1	5	1	1	1

Fig. 1. A graph of goods and two utility functions.

In this paper we use standard graph theoretic definitions and notation. In particular, we write $V(G)$ for the set of vertices of a graph G. A subgraph H of a graph G is *induced by a set of vertices* $X \subseteq V(G)$ if $V(H) = X$ and a pair of vertices $x, y \in V(H)$ is an edge in H if and only if xy is an edge in G. For any $Y \subseteq V(G)$, we denote by $G - Y$ the subgraph of G induced by the set of vertices $V(G) - Y$. If Y has only one vertex, say x, then we write $G - x$ instead of $G - \{x\}$.

By $K_{1,m}$ we denote a star with m edges. We say that a graph G is *m-claw-free* if G does not contain a copy of $K_{1,m}$ as an induced subgraph.

For a graph G with the edge set E, we denote by $L(G)$ its *line graph*, which is the graph with vertex set E such that two vertices u and v are adjacent in $L(G)$ if and only if the edges u and v have a common vertex in G. A subgraph H of a graph G is *spanning* if $V(H) = V(G)$. A family of graphs $\mathcal{G}$ is *hereditary* if any induced subgraph of a graph in $\mathcal{G}$ is in $\mathcal{G}$, too.

3 Approximate Proportional Allocations

We consider the following general problem:

Problem 1. *Let C be an infinite class of connected graphs. For every real $\alpha > 0$ and $n \geq 1$ find the largest value $\beta = \beta_n(\alpha)$ such that for any set of n α-proportionally bounded agents with utility functions defined on a graph in C, there exists a β-proportional allocation.*

Consider a set of n α-proportionally bounded agents with utility functions $u_1, \ldots, u_n$ defined on a graph G with m vertices. Then, for each utility function u_i, $u_i(V(G)) = \sum_{v \in V(G)} u_i(v) \leq \alpha \frac{u_i(V(G))}{n} \cdot m$, so $m \geq \frac{n}{\alpha}$. Thus, if $m < \frac{n}{\alpha}$ then no set of n α-proportionally bounded agents can be defined on the graph G. Consequently, if the family C were not infinite, the parameter $\beta_n(\alpha)$ would be undefined for some values of n and α (specifically, when the fraction $\frac{n}{\alpha}$ exceeds the number of vertices in the largest graph in C). Since the family C in Problem 1 is infinite, it can be easily verified that the parameter $\beta_n(\alpha)$ is defined for every $n \geq 1$ and $\alpha > 0$.

For $n = 1$, the obvious solution to Problem 1 is $\beta_1(\alpha) = 1$ for every $\alpha > 0$. We include this trivial single-agent case for technical reasons, as doing so significantly simplifies the base step of the inductive proof of Theorem 3.7 presented later in this section.

Moreover, if $n \geq 2$ and $\alpha \geq \frac{n}{n-1}$ then $\beta_n(\alpha) = 0$. Indeed, let G be a graph in C with at least $n - 1$ vertices and let each of n agents have the same utility function assigning the value $\frac{1}{n-1}$ to some $n - 1$ vertices of G and 0 to the remaining ones. Clearly, the agents are α-proportionally bounded because $\frac{1}{n-1} \leq \alpha \cdot \frac{1}{n}$ and in any allocation of bundles to the agents, some agent receives a bundle of value 0. Hence, $\beta_n(\alpha) = 0$. Therefore, we will often assume that $\alpha \leq \frac{n}{n-1}$.

Another case when Problem 1 is not particularly interesting is covered by the following proposition.

Proposition 3.1. *If a class C of graphs contains stars of arbitrarily large sizes, then for every $\alpha > 0$ and $n \geq 2$, we have $\beta_n(\alpha) = 0$.*

Proof. Suppose $\beta = \beta_n(\alpha) > 0$. Consider a star with more than $\max(\frac{n}{\alpha}, \frac{n}{\beta})$ leaves and a set of n agents with identical utility functions assigning 0 to the center of the star and 1 to each of its leaves. Clearly, the agents are α-proportionally bounded because the total value of all vertices for each agent is $S > \frac{n}{\alpha}$, so $1 < \alpha\frac{S}{n}$. Since $n \geq 2$, in any allocation of connected bundles to n agents at least one of the agents receives at most one vertex. Obviously, this allocation is not β-proportional because $S > \frac{n}{\beta}$, so $1 < \beta\frac{S}{n}$. □

In view of Proposition 3.1, Problem 1 is trivial for graph classes containing arbitrarily large stars. In this paper, we focus on the class of connected $(d+1)$-claw-free graphs, where d is fixed, which obviously does not contain arbitrarily large stars.

Let us observe that the class of connected 2-claw-free graphs is simply the class of complete graphs. Thus, in this case, our Problem 1 reduces to a problem of proportional allocations without any connectivity restrictions. This problem was solved by (Hill 1987) and its polynomiality was proved by (Markakis and Psomas 2011).

According to the terminology used in the present paper, the solution to our Problem 1 for the class C of complete graphs looks as follows[2].

Theorem 3.2. *(Hill 1987, Theorem 1.2), (Markakis and Psomas 2011, Theorem 2) Let $n \geq 2$ be an integer and let $\alpha \in (0, n]$. Define $\gamma_n : (0, n] \longrightarrow [0, 1]$ to be the unique nonincreasing function such that*

$$\gamma = \gamma_n(\alpha) = \begin{cases} n - k(n-1)\alpha & \text{if } \frac{(k+1)n}{k((k+1)n-1)} \leq \alpha \leq \frac{n}{kn-1} \text{ for some } k \geq 1; \\ \frac{kn}{(k+1)n-1} & \text{if } \frac{n}{(k+1)n-1} < \alpha < \frac{(k+1)n}{k((k+1)n-1)} \text{ for some } k \geq 1. \end{cases}$$

(i) *For every set of n α-proportionally bounded agents with utility functions defined on a complete graph of goods, there exists a γ-proportional allocation. Moreover, this allocation can be computed in polynomial time.*

(ii) *For each α and $\gamma' > \gamma_n(\alpha)$, there is a set of n α-proportionally bounded agents with utility functions defined on a complete graph of goods, such that no γ'-proportional allocation exists.*

Here is the solution to Problem 1 for the class of connected $(d+1)$-claw-free graphs, where $d \geq 2$, which is the main result of this section.

Theorem 3.3. *Let $d \geq 2$ and $n \geq 1$ be integers and let α be a positive real number such that $(n-1)\alpha \leq n$. Define*

$$\beta = \beta_n(\alpha) = \begin{cases} \frac{n-(n-1)\alpha}{(n-1)(d-1)+1} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{n}{2n-1} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}. \end{cases} \tag{2}$$

(i) *For every connected $(d+1)$-claw-free graph of goods and every set of n α-proportionally bounded agents with utility functions defined on this graph, there exists a β-proportional allocation. Moreover, this allocation can be computed in polynomial time.*

(ii) *For each α and $\beta' > \beta_n(\alpha)$, there is a connected $(d+1)$-claw-free graph of goods and a set of n α-proportionally bounded agents with utility functions defined on this graph, such that no β'-proportional allocation exists.*

[2]The relationship between the function $\gamma_n(\cdot)$ appearing in Theorem 3.2 and the function $V_n(\cdot)$ defined by (Hill 1987) is as follows: $\gamma_n(n\alpha) = nV_n(\alpha)$, for every $\alpha \in (0, 1]$.

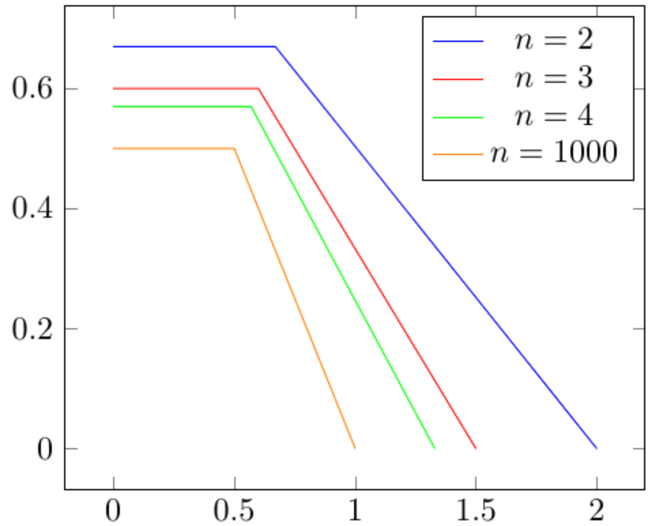


Fig. 2. The functions $\beta_n(\cdot)$ for $d = 2$ and $n \in \{2, 3, 4, 1000\}$.

In Figure 2, one can see the functions $\beta_n(\cdot)$ for $n \in \{2, 3, 4, 1000\}$.

It is perhaps interesting and surprising that for $d = 2$ and $\alpha = \frac{n}{2n-1}$, we have $\beta_n(\alpha) = \frac{n}{2n-1} = \gamma_n(\alpha)$. In other words, for these special values of α, the best proportionality fairness guarantee for the allocation problem without any connectivity constraints is the same as when the connectivity constants are captured by a 3-claw-free graph.

Before we prove Theorem 3.3, let us recall some basic definitions and facts from graph theory that we will use. A *cut vertex* of a connected graph is any vertex whose removal makes the graph disconnected. A connected graph is *biconnected* if it has no cut vertices. A *block* of a graph G is a maximal biconnected subgraph of G. It is easy to observe (Bondy and Murty 2008, Proposition 5.3) that each two blocks of a graph G have at most one vertex in common and this vertex, if it exists, is a cut vertex of G.

Following the terminology introduced in (Bondy and Murty 2008), for a connected graph G, we define $B(G)$ to be a bipartite graph with bipartition $(\mathcal{B}, C)$, where $\mathcal{B}$ is the set of blocks of G and C is the set of cut vertices of G. A block B and a cut vertex v are adjacent in $B(G)$ if B contains v. It is easy to see that the graph $B(G)$ is a tree, called the *block tree* of G. The blocks corresponding to the leaves of the tree $B(G)$ are called *end blocks* of G. If G is not biconnected then it has at least two end blocks. Clearly, each end block contains exactly one cut vertex of G.

Every tree with more than one vertex contains a vertex whose all neighbors except for possibly one, are leaves. If G is not biconnected then $B(G)$ has more than one vertex so it contains such a vertex, say v. Since leaves in $B(G)$ are end blocks in G, the vertex v is a cut vertex of G such that all blocks containing v except for possibly one, are end blocks. We call such a cut vertex of G *terminal*.

If G is not biconnected, then let v be a terminal cut vertex in G. We denote by $B_0, B_1, \ldots, B_s$ the blocks in G containing v indexed so that $B_1, \ldots, B_s$ are end blocks (B_0 may but does not have to be an end block). Obviously, v is the only cut vertex contained in the blocks $B_1, \ldots, B_s$. Let $B = V(B_1) \cup \ldots \cup V(B_s)$. We call such set B a *terminal set*. We observe that if the graph G is $(d+1)$-claw-free then $s + 1 \leq d$ because otherwise v is the center of an induced star $K_{1,d+1}$. Moreover, by the definition of v, both the graph induced in G by B and the graph $G - B$ are connected.

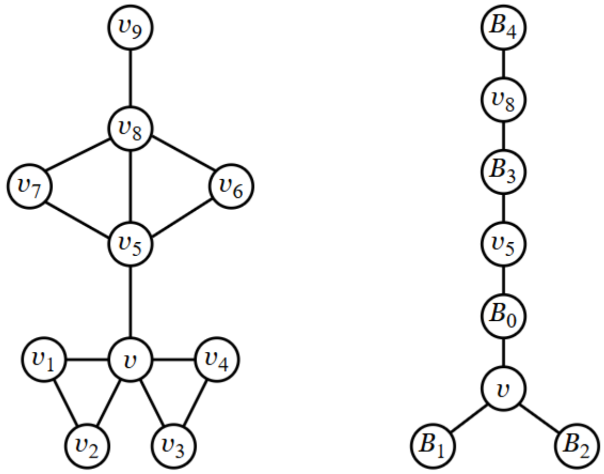


Fig. 3. A graph and its block tree.

To illustrate the concepts defined above, let us consider the graph G shown on the left side of Figure 3. It has three cut vertices v, v_5, v_8 and five blocks $B_0 = \{v, v_5\}$, $B_1 = \{v, v_1, v_2\}$, $B_2 = \{v, v_3, v_4\}$, $B_3 = \{v_5, v_6, v_7, v_8\}$, and $B_4 = \{v_8, v_9\}$. The block tree $B(G)$ of G is depicted on the right side of Figure 3. The graph G has three end blocks B_1, B_2, B_4, two terminal cut vertices v, v_8, and one non-terminal cut vertex v_5. The set $B = V(B_1) \cup V(B_2) = \{v, v_1, v_2, v_3, v_4\}$ is the terminal set containing the cut vertex v.

A *bipolar ordering* of a graph G is an ordering $v_1, \ldots, v_m$ of all vertices of G such that for each $i \in \{1, \ldots, m-1\}$ both the graph induced in G by the set of vertices $\{v_1, \ldots, v_i\}$ and the graph induced in G by the set of vertices $\{v_{i+1}, \ldots, v_m\}$ are connected. This concept as well as the concepts of a block and a block decomposition were already used before in the context of fair division (Bei et al. 2022; Bilò et al. 2022; Hummel and Igarashi 2024).

In the proof of Theorem 3.3 we will apply the following lemma. A somewhat stronger version of this lemma, formulated in a different terminology, was proved by (Lempel et al. 1967). Nevertheless, for the sake of completeness, we present here a simple proof of this lemma.

Lemma 3.4. *If G is a biconnected graph, then for every vertex v of G there exists a bipolar ordering starting with v. Moreover, this ordering can be constructed in a polynomial time.*

Proof. We will construct the required bipolar ordering $v_1, v_2, \ldots, v_m$ of vertices of G inductively.

Let $v_1 = v$ and suppose we have already defined, for $i < m$, the sequence $v_1, v_2, \ldots, v_i$ such that for every $j \in \{1, \ldots, i\}$ both the graph G_j induced in G by the set vertices $\{v_1, v_2, \ldots, v_j\}$ and the graph H_j induced in G by the set of the remaining vertices of G are connected. If $i = m-1$ then we define v_m to be the only vertex of H_{m-1} and we are done. Otherwise, we consider two cases.

First, let us assume that H_i is biconnected. Then, no vertex of H_i is a cut vertex. So, let v_{i+1} be any vertex of H_i joined by an edge with a vertex of G_i. Clearly, the graph G_{i+1} induced by the set of vertices $\{v_1, v_2, \ldots, v_{i+1}\}$ and the graph H_{i+1} induced by the set of the remaining vertices of G are connected.

Assume now that H_i is not biconnected. Consider an end block, say C, of H_i. By definition, this block contains exactly one cut vertex, say u, of H_i. We observe that there is an edge in G joining a vertex of G_i with a vertex, say w, of $C - u$ because otherwise u is a cut vertex of G, a contradiction. We define $v_{i+1} := w$. Clearly, the graph G_{i+1} is connected and the graph $H_{i+1} = H_i - w$ is also connected because w is not a cut vertex of H_i.

The lemma follows by induction. Obviously, this inductive proof can be turned into a polynomial algorithm constructing a bipolar ordering. □

Let us recall that we denote by S_i the total value of all goods for an agent i.

Let G be a $(d+1)$-claw-free graph of goods. An agent i with utility function u_i defined on the set of vertices of G is called *(α,β)-proportionally bounded* if $u_i(x) \le \alpha\frac{S_i}{n}$ whenever the vertex x is the center of an induced star $K_{1,d}$ in G, and $u_i(x) \le \beta\frac{S_i}{n}$ otherwise.

In the proof of the next lemma, we provide constructions of the bundles that will be assigned to agents to form a β-proportional allocation in the proof of Theorem 3.3 (i).

Lemma 3.5. *Let d, α and β be as defined in Theorem 3.3, and let $n \ge 2$ be an integer. Moreover, let G be a connected $(d+1)$-claw-free graph of goods and let $N = \{1,\dots,n\}$ be a set of agents. We assume that the graph G is biconnected or there is a terminal set B in G such that $u_i(B) > \max(\alpha,\beta)\cdot\frac{S_i}{n}$ for some agent i. Moreover, the agents are*

(i) *(α,β)-proportionally bounded, if* $0 < \alpha < \frac{n}{d(n-1)+1}$ *or*

(ii) *α-proportionally bounded, if* $\frac{n}{d(n-1)+1} \le \alpha \le \frac{n}{n-1}$.

Then, there is a connected bundle $L \subseteq V(G)$, which can be constructed in a polynomial time, such that the graph $G - L$ is connected too, and

$$u_k(L) \ge \beta\frac{S_k}{n}. \tag{3}$$

for some agent $k \in N$. Moreover,

$$u_j(L) \le \begin{cases} (\alpha + (d-1)\beta)\frac{S_j}{n} & \text{if } d \ge 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \ge \frac{n}{2n-1}; \\ \frac{2n}{2n-1}\frac{S_j}{n} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}, \end{cases} \tag{4}$$

for every agent $j \in N$.

Proof. We consider two cases.

Case 1. $0 < \alpha < \frac{n}{d(n-1)+1}$.

In this case the agents are (α,β)-proportionally bounded. One can easily verify that the inequalities $0 < \alpha < \frac{n}{d(n-1)+1}$ imply $\frac{n}{d(n-1)+1} < \beta < \frac{n}{(n-1)(d-1)+1} \le 1$. In particular, $\alpha < \beta$ and $2\beta \le n$.

To construct the required bundle L we consider two subcases.

Subcase 1.1. There is a terminal set B in G such that $u_i(B) > \max(\alpha,\beta)\cdot\frac{S_i}{n} = \beta\frac{S_i}{n}$ for some agent i.

Let us denote by $B_1,\dots,B_s$ the end blocks of G such that $B = V(B_1)\cup\dots\cup V(B_s)$ and by v the common vertex of these blocks. Recall that $s+1 \le d$ as G is $(d+1)$-claw-free.

Subcase 1.1.1. There is an agent i satisfying the inequality $u_i(B) > \beta\frac{S_i}{n}$ such that for some end block B_r containing v, we have $u_i(V(B_r - v)) \ge \beta\frac{S_i}{n}$.

By Lemma 3.4 applied for the graph B_r, there exists a bipolar ordering $v_1, v_2, \dots, v_m$, where $m = |V(B_r)|$, of vertices of B_r with $v_1 = v$ and this ordering can be constructed in polynomial time.

Let $\ell > 1$ be the largest t such that there is an agent $k \in N$, for whom the value of the set $\{v_t, v_{t+1}, \dots, v_m\}$ is at least $\beta\frac{S_k}{n}$. Clearly, ℓ is well-defined because the value for the agent i of the set $\{v_2, v_3, \dots, v_m\} = V(B_r - v)$ is at least $\beta\frac{S_i}{n}$. We define $L = \{v_\ell, v_{\ell+1}, \dots, v_m\}$. Clearly, the set L can be constructed in polynomial time and

$$u_k(L) \ge \beta\frac{S_k}{n}.$$

By the definition of a bipolar ordering, both the graph induced in G by the set L and the graph $G - L$ are connected.

By the definition of ℓ, the value of the set $\{v_{\ell+1}v_{\ell+2},\dots,v_m\}$ for each agent $j \in N$ is smaller than $\beta\frac{S_j}{n}$. Since the agents are (α,β)-proportionally bounded and $\alpha < \beta$, we have $u_j(v_\ell) \le \beta\frac{S_j}{n}$. Thus, if $d \ge 3$, then for every agent $j \in N$, we have

$$u_j(L) = u_j(v_\ell) + u_j(\{v_{\ell+1},\dots,v_m\}) \le 2\beta\frac{S_j}{n} \le (\alpha + (d-1)\beta)\frac{S_j}{n}.$$

If $d = 2$, then for every agent $j \in N$, we have

$$u_j(L) \le 2\beta\frac{S_j}{n} = \frac{2n}{2n-1}\frac{S_j}{n},$$

which completes the proof in this subcase.

Subase 1.1.2. For every agent i satisfying the inequality $u_i(B) > \beta\frac{S_i}{n}$ and for every end block B_r containing v, we have $u_i(V(B_r - v)) < \beta\frac{S_i}{n}$.

Let k be any agent such that $u_k(B) > \beta\frac{S_k}{n}$. We define $L := B$, so the inequality (3) is satisfied. Clearly, the set L can be constructed in polynomial time and both the graph induced in G by the set L and the graph $G - L$ are connected.

We observe that if $u_j(L) = u_j(B) \le \beta\frac{S_j}{n}$ for a certain agent j, then obviously the inequality $u_j(V(B_r - v)) \le \beta\frac{S_j}{n}$ holds for every end block B_r, too. Thus, this inequality holds for all agents $j \in N$.

Suppose first that the vertex v is the center of an induced star $K_{1,d}$ in G. Then, for every agent $j \in N$

$$u_j(L) = u_j(v) + \sum_{r=1}^{s} u_j(V(B_r - v)) \le \alpha\frac{S_j}{n} + s\cdot\beta\frac{S_j}{n} \le (\alpha + (d-1)\cdot\beta)\frac{S_j}{n}$$

because $s + 1 \le d$.

Assume now that v is not the center of an induced star $K_{1,d}$ in G. Then, $s + 1 \le d - 1$, so for every agent $j \in N$

$$u_j(L) = u_j(v) + \sum_{r=1}^{s} u_j(V(B_r - v)) \le \beta\frac{S_j}{n} + s\cdot\beta\frac{S_j}{n}$$
$$\le (d-1)\cdot\beta\frac{S_j}{n} \le (\alpha + (d-1)\cdot\beta)\frac{S_j}{n}.$$

We proved that the inequality $u_j(L) \le (\alpha + (d-1)\cdot\beta)\frac{S_j}{n}$ is true for $d \ge 2$, so in particular, (4) holds for $d \ge 3$.

Let $d = 2$. Since $\alpha < \frac{n}{2n-1}$, it follows that $\beta = \frac{n}{2n-1}$. So, for every agent j, $u_j(L) \le (\alpha + (d-1)\cdot\beta)\frac{S_j}{n} \le 2\beta\frac{S_j}{n} = \frac{2n}{2n-1}\frac{S_j}{n}$, which proves (4) for $d = 2$.

Subcase 1.2. G is biconnected.

The proof in this subcase is exactly the same as in subcase 1.1.1 with B_r replaced by G. The vertex v is now an arbitrary vertex of G and i is an arbitrary agent in N.

Moreover, applying the fact that the agents are (α,β)-proportionally bounded and the inequalities $\alpha < \beta$ and $2\beta \le n$, we get

$$u_i(V(G - v)) = S_i - u_i(v) \ge S_i - \beta\frac{S_i}{n} = (n - \beta)\frac{S_i}{n} \ge \beta\frac{S_i}{n}.$$

Thus, the condition analogous to the inequality $u_i(V(B_r - v)) \ge \beta\frac{S_i}{n}$ in subcase 1.1.1 is satisfied.

Case 2. $\frac{n}{d(n-1)+1} \le \alpha \le \frac{n}{n-1}$.

In this case the agents are α-proportionally bounded and the parameter $\beta = \frac{n-(n-1)\alpha}{(n-1)(d-1)+1}$ for every integer $d \ge 2$. Our assumption that $\frac{n}{d(n-1)+1} \le \alpha \le \frac{n}{n-1}$ implies that $0 \le \beta \le \frac{n}{d(n-1)+1}$. In particular, $\beta \le \alpha$.

The general idea of the proof in this case is the same as in case 1. We consider similar subcases as in case 1.

Subcase 2.1. There is a terminal set B in G such that $u_i(B) > \max(\alpha, \beta) \cdot \frac{S_i}{n} = \alpha \frac{S_i}{n}$ for some agent i.

We define the end blocks $B_1, \ldots, B_s$ in the same manner as in subcase 1.1.

Subase 2.1.1. There is an agent $i \in N$ satisfying the inequality $u_i(B) > \alpha \frac{S_i}{n}$ such that for some end block B_r containing i, we have $u_i(V(B_r - v)) \geq \beta \frac{S_i}{n}$.

The definition of L and the choice of agent k are the same as in subcase 1.1.1. In particular, $L = \{v_\ell, v_{\ell+1}, \ldots, v_m\}$, $u_k(L) \geq \beta \frac{S_k}{n}$ and $u_j(\{v_{\ell+1}, \ldots, v_m\}) < \beta \frac{S_j}{n}$ for every agent j. Since $u_j(v_\ell) \leq \alpha \frac{S_j}{n}$ for every agent j, we have

$$u_j(L) = u_j(v_\ell) + u_j(\{v_{\ell+1}, \ldots, v_m\}) \leq (\alpha + \beta) \frac{S_j}{n} \leq (\alpha + (d-1)\beta) \frac{S_j}{n},$$

which completes the proof in this subcase.

Subcase 2.1.2. For every agent i satisfying the inequality $u_i(B) > \alpha \frac{S_i}{n}$ and for every end block B_r containing v, we have $u_i(V(B_r - v)) < \beta \frac{S_i}{n}$.

Analogously to subcase 1.1.2, we define agent k as any agent such that $u_k(B) > \alpha \frac{S_k}{n}$, and we set $L := B$. Since $\alpha \geq \beta$, $u_k(L) > \beta \frac{S_k}{n}$.

If j is an agent satisfying $u_j(L) > \alpha \frac{S_j}{n}$, then we apply the fact that the agent is α-proportionally bounded and the inequality $s + 1 \leq d$ to derive

$$u_j(L) = u_j(v) + \sum_{r=1}^{s} u_j(V(B_r - v)) \leq \alpha \frac{S_j}{n} + s \cdot \beta \frac{S_j}{n} \leq (\alpha + (d-1)\beta) \frac{S_j}{n}.$$

For agents j satisfying $u_j(L) \leq \alpha \frac{S_j}{n}$, the inequality $u_j(L) \leq (\alpha + (d-1)\beta) \frac{S_j}{n}$ is obvious.

Subcase 2.2. G is biconnected.

The proof in this subcase is exactly the same as in subcase 2.1.1 with B_r replaced by G. The vertex v is now an arbitrary vertex of G and i is an arbitrary agent in N.

Moreover, applying the fact that all agents are α-proportionally bounded and the inequality $n \geq \alpha + \beta$ (following easily from the definition of β), we get

$$u_i(V(G - v)) = S_i - u_i(v) \geq S_i - \alpha \frac{S_i}{n} \geq \beta \frac{S_i}{n}.$$

Thus, the condition analogous to the inequality $u_i(V(B_r - v)) \geq \beta \frac{S_i}{n}$ in subcase 2.1.1 is satisfied too.

This completes the proof of the lemma. □

The next technical lemma will be applied in the inductive proof of part (i) of Theorem 3.3.

Lemma 3.6. *Let d, α and β be as defined in Theorem 3.3, and let $n \geq 2$ be an integer. Moreover, let*

$$\alpha' = \begin{cases} \frac{(n-1)\alpha}{(n-2)\alpha + ((n-2)(d-1)+1)\beta} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{n-1}{2n-3} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1} \end{cases} \tag{5}$$

and $\beta' = \beta_{n-1}(\alpha')$. Let u be a utility function defined on the set of vertices of a graph of goods G and let $L \subseteq V(G)$. We define S (respectively, S') to be the total value of all goods in G (respectively, in $G - L$) under the utility function u. If

$$u(L) \leq \begin{cases} (\alpha + (d-1)\beta) \frac{S}{n} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{2n}{2n-1} \frac{S}{n} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}, \end{cases} \tag{6}$$

then

$$\alpha\frac{S}{n} \leq \alpha'\frac{S'}{n-1} \tag{7}$$

and

$$\beta\frac{S}{n} \leq \beta'\frac{S'}{n-1}. \tag{8}$$

Proof. We consider two cases.

Case 1. $d \geq 3$ or $d = 2$ and $\alpha \geq \frac{n}{2n-1}$.

By the inequality (6) and the equality $n = (n-1)\alpha + ((n-1)(d-1)+1)\beta$ following from the definition of β,

$$\begin{aligned} S' = S - u(L) &\geq S - (\alpha + (d-1)\beta)\frac{S}{n} = (n - \alpha - (d-1)\beta)\frac{S}{n} \\ &= ((n-2)\alpha + ((n-2)(d-1)+1)\beta)\frac{S}{n}. \end{aligned} \tag{9}$$

By (9) and (5),

$$\alpha\frac{S}{n} \leq \alpha\frac{S'}{(n-2)\alpha + ((n-2)(d-1)+1)\beta} = \alpha'\frac{S'}{n-1},$$

which proves (7).

To prove (8), we show first that for $d = 2$, $\alpha' \geq \frac{n-1}{2n-3}$. Indeed, by the inequality $\alpha \geq \frac{n}{2n-1}$ and the fact that $\beta = \beta_n(\alpha)$ is a nonincreasing function of α, we get $\beta = \beta_n(\alpha) \leq \beta_n(\frac{n}{2n-1}) = \frac{n}{2n-1}$. Consequently $\frac{\beta}{\alpha} \leq 1$, so

$$\alpha' = \frac{(n-1)\alpha}{(n-2)\alpha + ((n-2)(d-1)+1)\beta} = \frac{n-1}{(n-2) + (n-1)\frac{\beta}{\alpha}} \geq \frac{n-1}{2n-3}.$$

Thus, applying the definition of β' and (5), for any $d \geq 2$, we get

$$\begin{aligned} \beta' = \beta_{n-1}(\alpha') &= \frac{n - 1 - (n-2)\frac{(n-1)\alpha}{(n-2)\alpha + ((n-2)(d-1)+1)\beta}}{(n-2)(d-1)+1} \\ &= (n-1)\frac{1 - \frac{(n-2)\alpha}{(n-2)\alpha + ((n-2)(d-1)+1)\beta}}{(n-2)(d-1)+1} \\ &= (n-1)\frac{\frac{((n-2)(d-1)+1)\beta}{(n-2)\alpha + ((n-2)(d-1)+1)\beta}}{(n-2)(d-1)+1} \\ &= \frac{(n-1)\beta}{(n-2)\alpha + ((n-2)(d-1)+1)\beta}. \end{aligned}$$

Hence, by this equality and the inequality (9),

$$\beta'\frac{S'}{n-1} = \frac{(n-1)\beta}{(n-2)\alpha + ((n-2)(d-1)+1)\beta} \cdot \frac{S'}{n-1} \geq \beta\frac{S}{n},$$

which proves (8).

Case 2. $d = 2$ and $\alpha < \frac{n}{2n-1}$.

By the inequality (6),

$$S' = S - u(L) \geq S - \frac{2n}{2n-1}\frac{S}{n} = \frac{2n-3}{2n-1}S. \tag{10}$$

From the inequality $\alpha < \frac{n}{2n-1}$, (10), and (5), we obtain

$$\alpha\frac{S}{n} < \frac{n}{2n-1}\frac{S}{n} \leq \frac{1}{2n-3}S' = \alpha'\frac{S'}{n-1},$$

which completes the proof of (7).

To prove (8), we observe that in this case $\beta' = \beta_{n-1}(\alpha') = \beta_{n-1}(\frac{n-1}{2n-3}) = \frac{n-1}{2n-3}$. Applying this equality, the inequality (10), and the equality $\beta = \beta_n(\alpha) = \frac{n}{2n-1}$, we get

$$\beta'\frac{S'}{n-1} = \frac{n-1}{2n-3}\frac{S'}{n-1} \geq \frac{n-1}{2n-3}\frac{2n-3}{2n-1}\frac{S}{n-1} = \beta\frac{S}{n},$$

which completes the proof of (8). □

We are ready to prove the main result of this section, i.e., Theorem 3.3. We start with the proof of part (i) of this theorem. In fact, we will show the following statement, which slightly strengthens Theorem 3.3 (i) for $0 < \alpha < \frac{n}{d(n-1)+1}$:

Theorem 3.7. *Let $d \geq 2$ and $n \geq 1$ be integers and let α be a positive real number such that $(n-1)\alpha \leq n$. Define*

$$\beta = \beta_n(\alpha) = \begin{cases} \frac{n-(n-1)\alpha}{(n-1)(d-1)+1} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{n}{2n-1} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}. \end{cases}$$

For every connected $(d+1)$-claw-free graph of goods and every set of n agents who are

(i) *(α, β)-proportionally bounded, if $0 < \alpha < \frac{n}{d(n-1)+1}$ or*
(ii) *α-proportionally bounded, if $\alpha \geq \frac{n}{d(n-1)+1}$,*

there exists a β-proportional allocation.

Moreover, this allocation can be computed in polynomial time.

Theorem 3.7 implies Theorem 3.3 (i). This is evident for $\alpha \geq \frac{n}{d(n-1)+1}$. If $0 < \alpha < \frac{n}{d(n-1)+1}$, then $\alpha < \beta$, so any α-proportionally bounded agent is (α, β)-proportionally bounded. Hence, indeed, Theorem 3.3 (i) follows from Theorem 3.7.

Proof of Theorem 3.7. We apply induction on $n + m$, where m is the number of vertices in the graph of goods. For $n = 1$ the theorem is trivially true because in this case $\beta = \beta_1(\alpha) = 1$ for every $\alpha > 0$ and the allocation assigning the agent the entire graph of goods is 1-proportional. So, we can assume that $n \geq 2$.

Consider a connected $(d+1)$-claw-free graph of goods G with m vertices and a set $N = \{1, \dots, n\}$ of agents who are (α, β)-proportionally bounded if $\alpha < \frac{n}{d(n-1)+1}$ or α-proportionally bounded otherwise. Assume that the theorem holds for instances with smaller than $m + n$ sum of the number of goods and the number of agents.

We consider two cases.

Case 1. G is not biconnected and for each terminal set B in G, $u_i(B) \leq \max(\alpha, \beta)\frac{S_i}{n}$, for each agent $i \in N$.

Since G is not biconnected, it contains terminal sets. Let B be one of them. We denote by G' the graph obtained from G by removing the vertices of $B - \{v\}$, where v is the terminal cut vertex contained in B. Clearly, the graph G' is connected and has less vertices than G because $B - \{v\} \neq \emptyset$. For each agent $i \in N$ we define a new utility function u'_i on the set $V(G')$ as follows: $u'_i(v) := u_i(B) \leq \max(\alpha, \beta)\frac{S_i}{n}$ and $u'_i(x) := u_i(x)$ for all vertices $x \neq v$. Clearly, the graph G' and the utility functions u'_i can be computed in polynomial time.

Assume first that $\alpha < \frac{n}{d(n-1)+1}$. Then, $\alpha < \beta$ and the agents are (α, β)-proportionally bounded. Moreover, $u'_i(v) \leq \beta\frac{S_i}{n}$ for each agent i. We note that since v is a cut vertex in a $(d+1)$-claw-free graph G, which belongs to at least one block removed from G, the vertex v is not the center of an induced star $K_{1,d}$ in G'. Thus, each agent $i \in N$ with the utility function u'_i defined on the set of vertices of G' is (α, β)-proportionally bounded.

If $\alpha \geq \frac{n}{d(n-1)+1}$ then $\alpha \geq \beta$. In this case, the agents are α-bounded. Moreover, $u'_i(v) \leq \max(\alpha,\beta)\frac{S_i}{n} \leq \alpha\frac{S_i}{n}$ for each agent i. Thus, obviously, each agent $i \in N$ with the utility function u'_i defined on the set of vertices of G' is α-proportionally bounded.

By the induction hypothesis, there exists a β-proportional allocation $(A_1, \ldots, A_n)$ for the graph of goods G' and the set of agents N with the utility functions u'_i, which assigns the bundle A_i to agent i for each $i \in N$. Moreover, this allocation can be computed in polynomial time. We add to the bundle, say A_j, containing v the vertices of $B - \{v\}$. This way we get a β-proportional allocation for the graph of goods G and the set of agents N with the utility functions u_i, which can be computed in polynomial time.

Case 2. G is biconnected or there is a terminal set B in G such that $u_i(B) > \max(\alpha,\beta)\frac{S_i}{n}$, for some agent $i \in N$.

By Lemma 3.5, we can construct in polynomial time a connected bundle $L \subseteq V(G)$ such that the graph $G' = G - L$ is connected too, and

$$u_k(L) \geq \beta\frac{S_k}{n},$$

for some agent $k \in N$. Moreover,

$$u_j(L) \leq \begin{cases} (\alpha + (d-1)\beta)\frac{S_j}{n} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{2n}{2n-1}\frac{S_j}{n} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}\ , \end{cases}$$

for every agent $j \in N$.

Let

$$\alpha' = \begin{cases} \frac{(n-1)\alpha}{(n-2)\alpha+((n-2)(d-1)+1)\beta} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{n}{2n-1}; \\ \frac{n-1}{2n-3} & \text{if } d = 2 \text{ and } \alpha < \frac{n}{2n-1}. \end{cases}$$

By Lemma 3.6 applied to the utility function u_j of each agent j, we have

$$\alpha\frac{S_j}{n} \leq \alpha'\frac{S'_j}{n-1} \tag{11}$$

and

$$\beta\frac{S_j}{n} \leq \beta'\frac{S'_j}{n-1}, \tag{12}$$

where $\beta' = \beta_{n-1}(\alpha')$, and S_j (respectively, S'_j) is the total value of all vertices of the graph G (respectively, G') to the agent j.

We will apply the induction hypothesis for the graph $G' = G - L$, the set of agents $N' = N - \{k\}$ and α'. Notice that $(n-2)\alpha' \leq n-1$.

Assume first that $\alpha' < \frac{n-1}{d(n-2)+1}$. Then, $\alpha' < \beta'$. If a vertex x is the center of an induced star $K_{1,d}$ in G' then it was the center of an induced star $K_{1,d}$ in G, too. Hence, by the inequality (11)

$$u_j(x) \leq \alpha\frac{S_j}{n} \leq \alpha'\frac{S'_j}{n-1},$$

for each agent $j \in N'$. If x is not the center of an induced star in G' then whether or not x was the center of an induced star in G, by the inequalities $\alpha' < \beta'$, (11), and (12), we obtain

$$u_j(x) \leq \max(\alpha,\beta)\frac{S_j}{n} \leq \max(\alpha',\beta')\frac{S'_j}{n-1} = \beta'\frac{S'_j}{n-1},$$

for each agent $j \in N'$. Thus, the utility functions u_j restricted to the set of vertices of G' are (α',β')-proportionally bounded for each agent $j \in N'$.

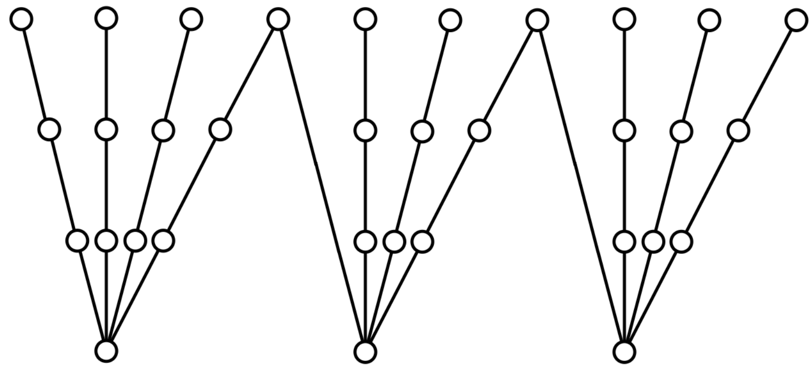


Fig. 4. The graph G for $d = 4$, $p = 4$ and $n = 4$.

Assume now that $\alpha' \geq \frac{n-1}{d(n-2)+1}$. Then, $\alpha' \geq \beta'$. If x is a vertex in G' then, by the inequalities (11) and (12) again,

$$u_j(x) \leq \max(\alpha, \beta)\frac{S_j}{n} \leq \max(\alpha', \beta')\frac{S_j}{n} = \alpha'\frac{S'_j}{n-1},$$

for each agent $j \in N'$. Thus, the utility functions u_j restricted to the set of vertices of G' are α'-proportionally bounded for each agent $j \in N'$.

By the induction hypothesis, there exists a β'-proportional allocation for the graph of goods G' and the set of agents N', which can be computed in polynomial time. In this allocation, each agent $j \in N'$ receives a bundle of value at least $\beta'\frac{S'_j}{n-1} \geq \beta\frac{S_j}{n}$. This allocation, together with the bundle L given to agent k, forms a β-proportional allocation for the graph of goods G and the set of agents N, which can be computed in polynomial time. □

It remains to prove part (ii) of Theorem 3.3.

Proof of Theorem 3.3 (ii). Clearly, it suffices to show that for every $n \geq 1$, $d \geq 2$, and positive α satisfying the inequality $(n-1)\alpha \leq n$, there exists a connected $(d+1)$-claw-free graph of goods and a collection of n α-proportionally bounded agents defined on this graph such that in each allocation of connected bundles to these n agents some agent i receives a bundle of value at most $\beta_n(\alpha) \cdot \frac{S_i}{n}$, where S_i is the total value of all goods for agent i. This statement is, obviously, true for $n = 1$ because $\beta_1(\alpha) = 1$, so let us assume that $n \geq 2$.

Case 1. $d \geq 3$ or $d = 2$ and $\alpha \geq \frac{n}{2n-1}$.

Let us consider first the case of $\alpha = \frac{n}{n-1}$. In this case $\beta_n(\alpha) = 0$. Let G be a complete graph on $n-1$ vertices and let each of n agents have the same utility function assigning the value 1 to each vertex of G. It is evident that the agents are α-proportional because $1 = \alpha\frac{S_i}{n}$ for each agent i, and in every allocation of the goods to the agents, at least one agent does not receive any goods.

Assume now that $\alpha < \frac{n}{n-1}$. Then, $\beta = \beta_n(\alpha) = \frac{n-(n-1)\alpha}{(n-1)(d-1)+1} > 0$. We construct a graph of goods G in the following way. Let T (respectively, T_1) be a tree consisting of $d-1$ paths (respectively, d paths), each with $p = \lceil \beta/\alpha \rceil + 1 > 1$ vertices, sharing one common vertex, which is an end of each path. We call this vertex *the center* of T (respectively, of T_1). We define G to be the tree obtained from one copy of T_1 and $n-2$ copies $T_2, \ldots, T_{n-1}$ of the tree T by joining with an edge the center of T_{i+1} with one of the leaves of T_i, for each $i \in \{1, \ldots, n-2\}$ (see Figure 4). Clearly, the graph G is connected and $(d+1)$-claw-free.

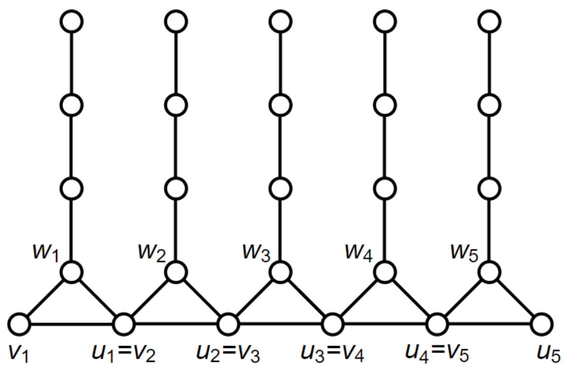


Fig. 5. The graph F for $p = 4$ and $n = 3$.

There are n agents with the same utility function assigning the value α to the $n-1$ vertices which are centers of the trees T_i and the value $\frac{\beta}{p-1} = \frac{\beta}{\lceil \beta/\alpha \rceil} \leq \alpha$ to the remaining $(p-1)((n-1)(d-1)+1)$ vertices of G. For each agent, the total value of all vertices of G is equal to $(n-1)\alpha + ((n-1)(d-1)+1)\beta = n$. Thus, it is evident that the agents are α-bounded.

We observe that the total value of every connected subgraph of G which does not contain the center of any tree T_i is at most β. Since there are $n-1$ centers only, in any allocation of bundles of G to n agents there is an agent who does not receive the center of any tree T_i. Thus, this agent gets a bundle of value at most β, which proves Theorem 3.3 (ii) in this case.

Case 2. $d = 2$ and $0 < \alpha < \frac{n}{2n-1}$.

In this case $\beta = \beta_n(\alpha) = \frac{n}{2n-1}$. We construct a graph of goods F in the following way. Let H be the graph obtained from a path with $p = \left\lceil \frac{n}{(2n-1)\alpha} \right\rceil + 1 > 1$ vertices and a triangle by identifying one end of the path with a vertex of the triangle. Consider $2n-1$ disjoint copies $H_1, \ldots, H_{2n-1}$ of H and label with w_i the common vertex of the path and the triangle in each graph H_i. Moreover, let v_i and u_i be the vertices of degree 2 in the triangle in each graph H_i. The graph F is obtained from the graphs $H_1, \ldots, H_{2n-1}$ by identifying vertices u_i and v_{i+1}, for every $i \in \{1, \ldots, 2n-2\}$ (see Figure 5). Clearly, the graph F is connected and 3-claw-free.

There are n agents with the same utility function assigning the value 0 to the $4n-1$ vertices of the triangles in F and the value $\frac{n}{(2n-1)(p-1)} = \frac{n}{(2n-1)\lceil n/((2n-1)\alpha) \rceil} \leq \alpha$ to the remaining $(2n-1)(p-1)$ vertices of F. The total value of all vertices of F is equal to n for each agent. Thus, the agents are α-proportionally bounded.

We observe that the total value of every connected subgraph of F which contains at most one of the vertices w_i is at most $(p-1)\frac{n}{(2n-1)(p-1)} = \frac{n}{2n-1}$. Since there are $2n-1$ vertices w_i only, in any allocation of bundles of F to n agents there is an agent who receives at most one vertex w_i. Thus, this agent gets a bundle of value at most $\frac{n}{2n-1} = \beta$, which completes the proof of Theorem 3.3 (ii). □

We will now derive several corollaries from Theorem 3.3. The first one is a simplified but weaker version of Theorem 3.3, which provides the best approximation ratio for a proportional allocation for arbitrarily large sets of α-proportionally bounded agents.

Corollary 3.8. *Let $d \geq 2$ and $0 < \alpha \leq 1$. Define*

$$\beta' = \beta'(\alpha) = \begin{cases} \frac{1-\alpha}{d-1} & \text{if } d \geq 3 \quad \text{or} \quad d = 2 \text{ and } \alpha \geq \frac{1}{2}; \\ \frac{1}{2} & \text{if } d = 2 \text{ and } \alpha < \frac{1}{2}. \end{cases} \tag{13}$$

(i) *For every every connected $(d + 1)$-claw-free graph of goods and every set of α-proportionally bounded agents with utility functions defined on this graph, there exists a β'-proportional allocation. Moreover, this allocation can be computed in polynomial time.*

(ii) *For each α and $\gamma > \beta'(\alpha)$, there exists a connected $(d+1)$-claw-free graph of goods and a set of α-proportionally bounded agents, with utility functions defined on this graph, such that no γ-proportional allocation exists.*

Proof. For a fixed $\alpha \in (0, 1]$, let $\beta_n(\alpha)$ be the sequence defined by the equality (2) in Theorem 3.3. One can easily verify that this sequence is nonincreasing. Moreover,

$$\lim_{n\to\infty} \beta_n(\alpha) = \beta'(\alpha) \tag{14}$$

for every $0 < \alpha \leq 1$. Hence,

$$\beta'(\alpha) \leq \beta_n(\alpha), \tag{15}$$

To prove (i) consider any connected $(d+1)$-claw-free graph of goods and any set of n α-proportionally bounded agents with utility functions defined on the set of vertices of this graph. By Theorem 3.3 (i), there exists a $\beta_n(\alpha)$-proportional allocation for these agents and it an be computed in polynomial time. This allocation is a $\beta'(\alpha)$-proportional allocation by the inequality (15), which completes the proof of (i).

To prove (ii), let us consider any $\gamma > \beta'(\alpha)$. By (14), we can find n' such that $\beta_{n'}(\alpha) < \gamma$. Therefore, Theorem 3.3 (ii) implies that there is a set of n' α-proportionally bounded agents with utility functions defined on some connected $(d + 1)$-claw-free graph of goods, such that no γ-proportional allocation exists. This completes the proof of (ii). □

Applying Theorem 3.3 (i) for $\alpha = \frac{n}{d(n-1)+1} = \beta$ we get the following result that will be of use when dealing with mms-allocations in Section 4.

Corollary 3.9. *Let $d \geq 2$ and $n \geq 1$ be integers. For every connected $(d + 1)$-claw-free graph of goods and every set of n $\frac{n}{d(n-1)+1}$-proportionally bounded agents with utility functions defined on this graph, there exists an $\frac{n}{d(n-1)+1}$-proportional allocation. Moreover, this allocation can be computed in polynomial time.* □

Since the function $\beta_n(\alpha)$ appearing in Theorem 3.3 is nonincreasing with respect to α, it follows easily from Theorem 3.3 (ii) that Corollary 3.9 is best possible. In other words, for any $c = c_n > \frac{n}{d(n-1)+1}$, there is a connected $(d + 1)$-claw-free graph of goods and a set of n c-proportionally bounded agents with utility functions defined on this graph, such that no c-proportional allocation exists.

Theorem 3.3 (i) can be generalized in the following way.

Corollary 3.10. *Let $d \geq 2$ and $n \geq 1$ be integers and let α be a positive real number such that $(n - 1)\alpha \leq n$. Moreover, let β be defined by the equality (2). If a graph G has a $(d+1)$-claw-free connected spanning subgraph, then for every set of n α-proportionally bounded agents with utility functions defined on G, there exists a β-proportional allocation for the graph of goods G.*

Proof. This corollary follows immediately from Theorem 3.3 (i) and the observation that for any β, a β-proportional allocation for a spanning connected subgraph of G is a β-proportional allocation for the graph G. □

Since a path is obviously a 3-claw-free graph, the following statement is a special case of Corollary 3.10.

Corollary 3.11. *Let $n \geq 1$ be an integer and let α be a positive real number such that $(n - 1)\alpha \leq n$. Moreover, let*

$$\beta = \beta_n(\alpha) = \begin{cases} \frac{n-(n-1)\alpha}{n} & \text{if } \alpha \geq \frac{n}{2n-1}; \\ \frac{n}{2n-1} & \text{if } \alpha < \frac{n}{2n-1}. \end{cases}$$

If a graph G has a Hamilton path, then for any set of n α-proportionally bounded agents with utility functions defined on G, there exists a β-proportional allocation for the graph of goods G. □

4 Approximate Mms-Allocations

The following theorem is the main result of this section.

Theorem 4.1. *Let $d \geq 2$ be an integer. For every connected $(d+1)$-claw-free graph of goods and any set of n agents with utility functions defined on this graph, there exists a complete $\frac{n}{d(n-1)+1}$-mms allocation. Moreover, this allocation can be computed in polynomial time.*

In fact, we will prove a stronger version of this theorem for graphs that do not have to be connected.

Theorem 4.1'. *Let $d \geq 2$ be an integer. For every $(d+1)$-claw-free graph of goods and any set of n agents with utility functions defined on this graph, there exists an $\frac{n}{d(n-1)+1}$-pmms allocation. Moreover, this allocation can be computed in polynomial time.*

Theorem 4.1 follows immediately from Theorem 4.1' by the following proposition which explains the relationship between pmms-allocations and mms-allocations for connected graphs of goods.

Proposition 4.2. *Let G be a connected graph of goods.*

(i) *For any utility function u defined on $V(G)$ and a positive integer n, $pmms^{(n)}(G,u) = mms^{(n)}(G,u)$.*
(ii) *For any $c > 0$ and any collection of agents, a complete c-mms allocation exists if and only if a c-pmms allocation exists. Moreover, if one of these allocations can be computed in polynomial time, then so can the other.*

Proof. (i) Since every (G,n)-split is a (G,n)-packing, $pmms^{(n)}(G,u) \geq mms^{(n)}(G,u)$.

To prove the opposite inequality, consider an mms-packing Π for G, u, and n. Clearly, the values of the bundles in Π under the utility function u are at least $pmms^{(n)}(G,u)$. Let W be the union of all bundles in Π. We observe that every component of the graph $G-W$ is joined by an edge in G with some bundle in Π. Otherwise, G is disconnected. Thus, we can adjoin vertex sets of the components of $G-W$ to the bundles of Π to get a (G,n)-split, say Π', of G. Clearly, the value of each bundle in Π' is at least $pmms^{(n)}(G,u)$. Hence, $mms^{(n)}(G,u) \geq pmms^{(n)}(G,u)$.

(ii) The "only if" part follows from the fact that, by (i), every c-mms allocation is also a c-pmms allocation.

To prove the "if" part let us consider a c-pmms allocation, say A, for a graph of goods G and some n agents. Applying a similar argument as in the proof of part (i) we observe that the bundles received by some of the agents can be extended to larger bundles so that every vertex in G is in some bundle. This way we get a complete allocation for the graph G and these n agents. By part (i) this allocation is a c-mms allocation. Clearly, the construction of this complete c-mms allocation from the allocation A can be done in polynomial time. □

We will also need the following simple observation.

Lemma 4.3. *Let G be a graph of goods and let u be a utility function on G. For every set of vertices $X \subseteq V(G)$ and any integer $n > |X|$,*

$$pmms^{(n-|X|)}(G-X,u) \geq pmms^{(n)}(G,u).$$

Proof. Let Π be an mms-packing for G, u, and n. Clearly, at least $n-|X|$ bundles of Π do not intersect X. Thus, the family of any $n-|X|$ such bundles is, obviously, a $(G-X, n-|X|)$-packing with the value of each bundle under u at least $pmms^{(n)}(G,n)$. Thus, $pmms^{(n-|X|)}(G-X,u) \geq pmms^{(n)}(G,u)$. □

The following auxiliary parameter $t^{(n)}(G,u)$ will play an important role in showing the polynomiality of computing the allocations in Theorems 4.1 and 4.1'.

For a graph of goods G with components $G_1, \ldots, G_q$, a positive integer n, a utility function u, and n agents, we denote by $t^{(n)}(G,u)$ the largest real number t such that

$$\left\lfloor \frac{m_1}{t} \right\rfloor + \ldots + \left\lfloor \frac{m_q}{t} \right\rfloor \geq n, \tag{16}$$

where $m_j := u(V(G_j))$, for each $j \in \{1, \ldots, q\}$.

The lemma below provides some basic properties of the parameter $t^{(n)}(G,u)$.

Lemma 4.4. *Let G be a graph of goods, n a positive integer, and u a utility function defined on $V(G)$.*

(i) *The parameter $t^{(n)}(G,u)$ is well-defined and can be computed in polynomial time.*

(ii)
$$pmms^{(n)}(G,u) \leq t^{(n)}(G,u) \leq \frac{u(V(G))}{n}.$$

Proof. Let $G_1, \ldots, G_q$ be the components of G of values $m_1, \ldots, m_q$ under u, respectively.

(i) We define
$$T := \left\{ \frac{m_j}{k} : j \in \{1, \ldots, q\} \text{ and } k \in \{1, \ldots, n\} \right\}.$$

Clearly, the set of elements of T satisfying the inequality (16) is nonempty because $t = \frac{m_1}{n} \in T$ satisfies (16). Let t_0 be the largest element of T that satisfies (16). We will show that t_0 is in fact the largest real number that satisfies (16). This will, obviously, complete the proof of (i).

Let $f(t) := \lfloor \frac{m_1}{t} \rfloor + \ldots + \lfloor \frac{m_q}{t} \rfloor$. Since $f(\frac{m_j}{n}) \geq n$, we have $t_0 \geq \frac{m_j}{n}$ for each j.

Let m_1 be the largest of the numbers $m_1, \ldots, m_q$. If $t_0 = m_1$, then the inequality (16) does not hold for $t > t_0$ because $f(t) = 0$ for such t. So, we are done in this case.

Let us assume now that $t_0 < m_1$. We denote by t_1 the least element of T that is larger than t_0. We will show that if $t \in (t_0, t_1]$, then $\lfloor \frac{m_j}{t} \rfloor = \lfloor \frac{m_j}{t_1} \rfloor$ for each j. This is obviously true if $t > m_j$. So, assume that $t \leq m_j$. For each such j we define $\ell_j := \lfloor \frac{m_j}{t} \rfloor$. Then, $0 < \ell_j \leq \frac{m_j}{t} < \ell_j + 1$ so, $\frac{m_j}{\ell_j+1} < t \leq \frac{m_j}{\ell_j}$, for each j such that $t \leq m_j$. By the inequality $t_0 \geq \frac{m_j}{n}$, we have $\ell_j = \lfloor \frac{m_j}{t} \rfloor \leq \frac{m_j}{t} < \frac{m_j}{t_0} \leq n$, so $\frac{m_j}{\ell_j} \in T$. Moreover, $\frac{m_j}{\ell_j} \geq t > t_0$. By the definition of t_1, $t_1 \leq \frac{m_j}{\ell_j}$, so $\ell_j \leq \frac{m_j}{t_1} \leq \frac{m_j}{t} < \ell_j + 1$ and, consequently, $\lfloor \frac{m_j}{t_1} \rfloor = \ell_j = \lfloor \frac{m_j}{t} \rfloor$.

We proved that $f(t) = f(t_1)$ for each $t \in (t_0, t_1]$. Since $t_1 \in T$ and $t_1 > t_0$, by the definitions of t_0 and t_1, we have $f(t_1) < n$, so $f(t) < n$ for $t \in (t_0, t_1]$. Since the function $f(t)$ is nonincreasing, $f(t) < n$ for every $t > t_0$. Thus, t_0 is the largest real number satisfying the inequality (16).

(ii) Let $pmms_u := pmms^{(n)}(G,u)$ and $t_u := t^{(n)}(G,u)$. Consider an mms-packing Π for G, n, and the utility function u. Let b_j be the number of bundles in Π contained in the component G_j, for $j \in \{1, \ldots, q\}$. Obviously, $b_j \cdot pmms_u \leq m_j$, for each j. Hence, $b_j \leq \left\lfloor \frac{m_j}{pmms_u} \right\rfloor$, so

$$n = b_1 + \ldots + b_q \leq \left\lfloor \frac{m_1}{pmms_u} \right\rfloor + \ldots + \left\lfloor \frac{m_q}{pmms_u} \right\rfloor.$$

Consequently, by the definition of $t^{(n)}(G,u)$,

$$pmms^{(n)}(G,u) = pmms_u \leq t^{(n)}(G,u).$$

Moreover, by the definition of $t^{(n)}(G,u)$ again,

$$\frac{u(V(G))}{t_u} = \frac{m_1 + \ldots + m_q}{t_u} \geq \left\lfloor \frac{m_1}{t_u} \right\rfloor + \ldots + \left\lfloor \frac{m_q}{t_u} \right\rfloor \geq n,$$

so

$$t^{(n)}(G, u) = t_u \leq \frac{u(V(G))}{n}.$$

□

We will now prove a general lemma that reduces the problem of the existence of a c-pmms allocation for arbitrary agents to the existence of a c-proportional allocation for c-proportionally bounded agents, provided that the graphs of goods belong to a hereditary class of graphs. We will apply this lemma in the proof of Theorem 4.1' for the class of $(d+1)$-claw-free graphs, which is obviously hereditary.

Lemma 4.5. *Let $\mathcal{G}$ be a fixed hereditary class of graphs, $n \geq 1$ an integer, and $c > 0$ a real number.*

(i) *If there exists a c-proportional allocation for any connected graph in $\mathcal{G}$ and any set of at most n c-proportionally bounded agents, then there exists a c-pmms allocation for any graph of goods in $\mathcal{G}$ and any set of n agents.*
(ii) *Moreover, if the former allocation can be computed in polynomial time, then the latter allocation can also be computed in polynomial time.*

Proof. Consider an arbitrary graph of goods $G \in \mathcal{G}$ and an arbitrary set $N = \{1, \ldots, n\}$ of agents. Let, for each agent $i \in N$, u_i be the utility function of i. We need to construct in polynomial time an allocation assigning each agent i a bundle of value at least $c \cdot pmms^{(n)}(G, u_i)$ to this agent.

In the first part of the algorithm, we eliminate agents that can be satisfied by a single vertex of the graph of goods. This part of the algorithm works in several steps.

In the first step, we choose, if possible, any agent $i_1 \in N$ and any vertex x_1 of $G_0 := G$ such that

$$u_{i_1}(x_1) \geq c \cdot t^{(n)}(G_0, u_{i_1}).$$

Similarly, in the jth step, where $j > 1$, we choose, if possible, any agent $i_j \in N - \{i_1, \ldots, i_{j-1}\}$ and any vertex x_j of $G_{j-1} := G - \{x_{i_1}, \ldots, x_{i_{j-1}}\}$ such that

$$u_{i_j}(x_j) \geq c \cdot t^{(n-(j-1))}(G_{j-1}, u_{i_j}). \tag{17}$$

We define k to be the largest j for which such a choice is possible, if it is possible at least once. Otherwise, $k = 0$.

We observe that, by (17), Lemma 4.4 (ii), and Lemma 4.3, for every $j = 1, \ldots, k$

$$u_{i_j}(x_j) \geq c \cdot pmms^{(n-(j-1))}(G_{j-1}, u_{i_j}) \geq c \cdot pmms^{(n)}(G, u_{i_j}), \tag{18}$$

so the vertex x_j satisfies the agent i_j. Each agent i_j receives the vertex x_j in our allocation and quits the game. By Lemma 4.4 (i), this part of the construction of our allocation can be done in polynomial time.

The resulting graph $G' := G - \{x_{i_1}, \ldots, x_{i_k}\}$ has the property that for every vertex $v \in G'$ and for every agent $i \in N' := N - \{i_1, \ldots, i_k\}$ (i.e., for every agent who is still in the game),

$$u_i(v) < c \cdot t^{(n')}(G', u_i), \tag{19}$$

where $n' := n - k$.

By Lemmas 4.4 (ii) and 4.3, for each agent $i \in N'$,

$$t^{(n')}(G', u_i) \geq pmms^{(n')}(G', u_i) \geq pmms^{(n)}(G, u_i).$$

Thus, to prove our lemma, it suffices to show that there is an allocation assigning to each agent $i \in N'$ a bundle inducing a connected subgraph of G' that is of value at least $c \cdot t^{(n')}(G', u_i)$ to this agent. To simplify notation we define $t_i := t^{(n')}(G', u_i)$.

Algorithm *allocate*(N', G', c)

1 $T := N'$;

2 **for** $j = 1, 2, \ldots, q$ **do**

3 sort the agents $i \in T$ nonincreasingly according to the key $f(i, j)$: $i^j_1, i^j_2, \ldots, i^j_{|T|}$;

4 $k_j :=$ the largest p such that $f(i^j_p, j) \geq p$;

5 $I_j := \{i^j_1, i^j_2, \ldots, i^j_{k_j}\}$;

6 agents in I_j distribute the vertices of G'_j to themselves using the protocol $A(I_j, G'_j)$;

7 $T := T - I_j$;

8 **if** $T = \emptyset$ **then**

9 **return**

Fig. 6. Algorithm *allocate*(N', G', c)

Let $G'_1, \ldots, G'_q$ be the components of G' and let m^i_j be the value of the component G'_j for agent $i \in N'$. We define $f(i, j) := \left\lfloor \frac{m^i_j}{t_i} \right\rfloor$. Then,

$$t^{(n')}(G', u_i) = t_i \leq \frac{m^i_j}{f(i, j)} \tag{20}$$

for each agent i.

By the definition of t_i,

$$\sum_{j=1}^{q} f(i, j) = \left\lfloor \frac{m^i_1}{t_i} \right\rfloor + \ldots + \left\lfloor \frac{m^i_q}{t_i} \right\rfloor \geq n'. \tag{21}$$

Claim. Let $I_j \subseteq N'$ be a set of agents such that $f(i, j) \geq |I_j|$ for each agent $i \in I_j$. Then, there exists a c-proportional allocation for the graph of goods G'_j and the set of agents I_j.

Proof of the Claim. We observe that the agents of I_j are c-proportionally bounded with respect to the graph of goods G'_j and the utility functions restricted to $V(G'_j)$. Indeed, by the inequalities (19) and (20), for every vertex $v \in V(G'_j)$ and every agent $i \in I_j$

$$u_i(v) < c \cdot t^{(n')}(G', u_i) \leq c \cdot \frac{m^i_j}{f(i, j)} \leq c \cdot \frac{m^i_j}{|I_j|} = c \cdot \frac{u_i(V(G'_j))}{|I_j|}.$$

Since $\mathcal{G}$ is a hereditary class of graphs, $G'_j \in \mathcal{G}$. Moreover, $|I_j| \leq n' \leq n$, so by the premise in part (i) of our lemma, there is a c-proportional allocation for the graph of goods G'_j and the set of agents I_j. ∎

Obviously, under the premise in part (ii) of our lemma, the c-proportional allocation described in the *Claim* can be computed in polynomial time. So, we can assume that a protocol, say $A(I_j, G'_j)$, which for a component $G'_j \in \mathcal{G}$ of the graph G' and a set of agents $I_j \subseteq N'$ constructs a c-proportional allocation of the vertices of G'_j to the agents in I_j, works in polynomial time.

We shall prove that the algorithm *allocate* shown in Figure 6 produces the required allocation of the vertices of G' to the agents of N'. In the jth pass of the loop of this algorithm, bundles of vertices from the component G'_j of G' are distributed to agents. First, we sort (line 3) the agents i who have not yet received their bundles in nonincreasing order according to the values of $f(i,j)$. We denote the ℓth agent in this sorting by i_ℓ^j. Next, we define the set I_j which consists of k_j initial agents in the sorting (line 5), where k_j is the largest p such that $f(i_p^j, j) \geq p$ (line 4) . We observe that, by the definition of k_j, $f(i,j) \geq k_j = |I_j|$ for every agent $i \in I_j$. Thus, it follows from the *Claim* that there is a c-proportional allocation of vertices of G'_j to agents of I_j assigning to each agent $i \in I_j$ a bundle of value at least

$$c \cdot \frac{u_i(V(G'_j))}{|I_j|} = c \cdot \frac{m_j^i}{k_j} \geq c \cdot \frac{m_j^i}{f(i,j)} \geq c \cdot t^{(n')}(G', u_i) \geq c \cdot pmms^{(n')}(G', u_i).$$

We applied here the inequality (20) and Lemma 4.4 (ii). We distribute the vertices of G'_j to agents of I_j according to the protocol $A(I_j, G'_j)$ (line 6) and the agents of I_j quit the game (line 7). Clearly, under the premise in part (ii) of our lemma, this allocation can be computed in polynomial time.

It remains to show that when the algorithm *allocate* stops, the set T is empty, i.e., all agents received their bundles. Suppose otherwise. Let i be an agent who is still in the set T when the algorithm stops. Let T_j be the set T at start of the jth pass of the loop of our algorithm. By the definition of k_j, for all agents $t \in T_j$ which are not included in the set I_j (and thus do not receive a bundle in the jth pass of the loop), we have $f(t,j) < k_j + 1$, i.e., $f(t,j) \leq k_j$. Since the agent i was not allocated a bundle in any pass of the loop, we have $f(i,j) \leq k_j$ for all j's. Clearly, $\sum_{j=1}^q k_j$ is the total number of agents who received their bundles when the algorithm stops, so $\sum_{j=1}^q k_j < n'$. Hence, by the inequality (21),

$$n' \leq \sum_{j=1}^{q} f(i,j) \leq \sum_{j=1}^{q} k_j < n',$$

a contradiction. Thus, all agents received their bundles when the algorithm stops. □

As we have already mentioned, we will apply Lemma 4.5 to the class of $(d+1)$-claw-free graphs, which is obviously hereditary. However, it is worth noting that there are many other important hereditary classes of graphs (e.g., the class of all graphs, the class of planar graphs, the class of bipartite graphs, the class of perfect graphs, etc.). Therefore, it seems that Lemma 4.5 is of independent interest and could potentially be used to show approximation results for pmms allocations for some other hereditary classes of graphs.

Theorem 4.1' follows now from Lemma 4.5 and Corollary 3.9.

Proof of Theorem 4.1'. We apply Lemma 4.5 to the class $\mathcal{G}$ of $(d+1)$-claw-free graphs, which is obviously hereditary, for a fixed positive integer n, and $c = \frac{n}{d(n-1)+1}$. To this end we consider any connected $(d+1)$-claw-free graph of goods G' and any set N' of $n' \leq n$ c-proportionally bounded agents with utility functions defined on the set of vertices of G'. To apply Lemma 4.5, we need to show that there exists a c-proportional allocation for the graph G' and the set of agents N'. Since $c' := \frac{n'}{d(n'-1)+1} \geq \frac{n}{d(n-1)+1} = c$, the agents in N' are also c'-proportionally bounded. By Corollary 3.9 applied to the graph G' and the integer n', there exists a c'-proportional allocation for the graph G' and the agents in N', and it can be computed in polynomial time. Clearly, this allocation is a c-proportional allocation, by the inequality $c' \geq c$ again. Thus, by Lemma 4.5, for any $(d+1)$-claw-free graph of goods and any set of n agents, there exists an $\frac{n}{d(n-1)+1}$-pmms allocation, and it can be computed in polynomial time. □

Since $\frac{n}{d(n-1)+1} \geq \frac{1}{d}$, for $d \geq 2$, Theorem 4.1 implies the following weaker but simpler statement.

Corollary 4.6. *Let $d \geq 2$ be an integer. For every connected $(d+1)$-claw-free graph of goods and any set of agents, there exists a complete $\frac{1}{d}$-mms allocation. Moreover, this allocation can be computed in polynomial time.* □

As we have mentioned in the Introduction, our results can be applied to the following "edge variant" of our problem. Let G be a connected graph and let N be a set of n agents with additive utility functions u_i defined on the set E of edges of G. Find a fair allocation of sets of edges of G (called *edge G-bundles*) which induce connected subgraphs.

Let $L(G)$ be the *line graph* of G, i.e., a graph whose vertices are the edges of G and two vertices in $L(G)$ are adjacent if the corresponding edges in G have a common vertex. Clearly, the problem described in the preceding paragraph is equivalent to the problem of finding a fair allocation of (connected) $L(G)$-bundles of vertices for the graph of goods $L(G)$ and the utility functions u_i defined on the set of vertices E of the graph $L(G)$.

For any $c > 0$, let us call an allocation $(E_1, \ldots, E_n)$ of edge G-bundles to agents in N an *edge c-mms allocation* if this allocation is a c-mms allocation for the graph of goods $L(G)$. Moreover, let an allocation $(E_1, \ldots, E_n)$ be *complete* if every edge in E is assigned to some agent.

It is a well-known fact (e.g. see (Faudree et al. 1997, p. 88)) that line graphs are 3-claw-free. Therefore, Corollary 3.9 holds for line graphs with $d = 2$. Furthermore, it follows directly from the definition of a line graph that the class of line graphs is hereditary. Consequently, Lemma 4.5 holds for the class $\mathcal{G}$ of line graphs. By applying a reasoning similar to that in the proof of Theorem 4.1', we obtain the following result.

Corollary 4.7. *For any connected graph G and any set of n agents with utility functions defined on the set of edges of G, there exists a complete edge $\frac{n}{2n-1}$-mms allocation. Moreover, this allocation can be computed in polynomial time.* □

5 Final Remarks and Open Problems

We investigated the problem of fair allocation of indivisible good in the setting where the goods are vertices of a graph and the bundles acceptable by agents must form connected subgraphs. We proved two main results concerning the existence and efficient computation of approximate proportional allocations and approximate mms-allocations in the case where the graph of goods does not contain a star with $d + 1$ edges as an induced subgraph.

Our main result on approximate proportional allocations (Theorem 3.3) is, in a sense, complete. For any set of n agents whose valuations of individual goods are bounded by an α-fraction of their proportional shares, we determined the largest value $\beta = \beta_n(\alpha)$ such that there exists an allocation guaranteeing each agent at least a β-fraction of her proportional share. Moreover, we observed (Proposition 3.1) that for any class of graphs containing stars of arbitrarily large size, it is impossible to guarantee any non-zero fraction of the proportional share to agents whose valuations for individual vertices are bounded by an α-fraction of that share, for any $\alpha > 0$. Thus, there is little hope of extending our result to any natural class of graphs that is not contained in the class of $(d + 1)$-claw-free graphs for some fixed d.

By contrast, our main result on approximate mms-allocations (Theorem 4.1) is far from complete. It remains an open question whether the approximation ratio $\frac{1}{d}$ for mms-allocations given in Corollary 4.6 can be improved, even in the case of 3-claw-free graphs (i.e., for $d = 2$). Therefore, we pose the following problem.

Problem 2. *What is the largest constant c such that for each connected 3-claw-free graph of goods and any set of agents, there exists a c-mms allocation?*

It was shown (Lonc and Truszczynski 2020, p. 639) that if the graph of goods is a cycle with 12 vertices (which is obviously 3-claw-free), then for some 6 agents no c-mms allocation exists for any $c > \frac{3}{4}$. This statement and Theorem 4.1 imply that the constant c in Problem 2 satisfies the inequalities $\frac{1}{2} \leq c \leq \frac{3}{4}$.

Unlike in the case of our main result on approximate proportional allocations, we do not see any reason why our main result on approximate mms-allocations could not be extended to some more general classes of graphs than $(d + 1)$-claw-free graphs. Let us recall that the class of trees includes stars of arbitrarily large size, yet mms-allocations always exist when the graph of goods is a tree, regardless of the number of agents. These remarks motivate the following question.

Problem 3. *Is there an absolute constant $c > 0$ such that for each connected graph of goods and any set of agents, there exists a c-mms allocation?*

We do not know any example of a graph and a collection of agents such that for this graph and these agents no $\frac{3}{4}$-mms allocation exists. Thus, it is conceivable that the constant c in Problem 3 can be as large as $\frac{3}{4}$.

We believe that the answer to the question posed in Problem 3 is positive. Nevertheless, proving it will require methods different from those used in this paper, as the proof of our main result on the approximation of mms-allocations relies on our main result on proportional allocations, which, as observed earlier, cannot be extended to classes of graphs that contain stars of arbitrarily large size.

Acknowledgments

The author thanks Miroslaw Truszczynski for his comments, which significantly improved the presentation of the results in this paper..

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Received 23 September 2024; accepted 14 September 2025