Probability of a Condorcet Winner for Large Electorates: An Analytic Combinatorics Approach

EMMA CAIZERGUES, Université Paris Dauphine-PSL, France. Nokia Bell Labs, France
FRANÇOIS DURAND, Nokia Bell Labs, France
MARC NOY, Universitat Politècnica de Catalunya, Spain
ÉLIE DE PANAFIEU, Nokia Bell Labs, France
VLADY RAVELOMANANA, Université Paris-Cité, France

We study the probability that a given candidate is an $\boldsymbol{\alpha}$-*winner*, that is, preferred over every other candidate j by at least a fraction α_j of the voters. This general framework encompasses the classical notion of a *Condorcet winner*, corresponding to $\boldsymbol{\alpha} = (\frac{1}{2}, \ldots, \frac{1}{2})$. Assuming voters have independent preferences, we derive general results and apply them to prominent models such as Impartial Culture and the Mallows model. While previous work relies on probabilistic arguments to derive the limiting probability for a fixed number of candidates m as the number of voters n tends to infinity, we use techniques from the field of analytic combinatorics to compute not only this limit but also convergence rates, and we provide a method for obtaining higher-order terms in the asymptotic expansion. In particular, we refine existing results on the Condorcet winner under Impartial Culture by establishing the following asymptotic behavior for its probability: $a_{0,m} + a_{1,m,n}n^{-1/2} + O(n^{-1})$, where we give explicit expressions for the coefficients $a_{0,m}$ and $a_{1,m,n}$. The latter depends on the number of voters n only through its parity, while the number of candidates m is fixed in our asymptotic regime. Along the way, we derive technical results in multivariate analytic combinatorics that may be of independent interest.

JAIR Track: AI & Society

JAIR Associate Editor: Reshef Meir

JAIR Reference Format:
Emma Caizergues, François Durand, Marc Noy, Élie de Panafieu, and Vlady Ravelomanana. 2026. Probability of a Condorcet Winner for Large Electorates: An Analytic Combinatorics Approach. *Journal of Artificial Intelligence Research* 87, Article 21 (November 2026), 41 pages. DOI: 10.1613/jair.1.19712

1 Introduction

This introduction presents the motivation for our work, our main contributions, related work, the limitations of our approach, and a roadmap of the paper.

1.1 Motivation

A long-standing tradition in social choice theory is to study the probability that a *Condorcet winner* exists, *i.e.,* a candidate preferred to every other candidate by a majority of voters. A comprehensive overview of this research is presented in Gehrlein (2006), which is entirely dedicated to this topic. Niemi and Weisberg (1968) derived the

Authors' Contact Information: Emma Caizergues, ORCID: 0009-0003-2760-9879, emma.caizergues@gmail.com, Université Paris Dauphine-PSL, Paris, France. and Nokia Bell Labs, Massy, France; François Durand, ORCID: 0000-0002-0861-0028, francois.durand@nokia-bell-labs.com, Nokia Bell Labs, Massy, France; Marc Noy, ORCID: 0000-0002-2399-1359, marc.noy@upc.edu, Universitat Politècnica de Catalunya, Barcelona, Spain; Élie de Panafieu, ORCID: 0009-0002-1386-971X, elie.de_panafieu@nokia-bell-labs.com, Nokia Bell Labs, Massy, France; Vlady Ravelomanana, ORCID: 0000-0003-1702-7317, vlad@irif.fr, Université Paris-Cité, Paris, France.



This work is licensed under a Creative Commons Attribution International 4.0 License.

© 2026 Copyright held by the owner/author(s).
DOI: 10.1613/jair.1.19712

limit of this probability as the number of voters tends to infinity, under only the assumption of independent preferences. More recently, Krishnamoorthy and Raghavachari (2005) revisited and modernized this result.

Beyond limiting probabilities, understanding convergence rates is also valuable. For instance, in epistemic democracy, one seeks to recover a hidden ground truth by aggregating the votes of agents who each receive noisy information. Knowing how fast the probability of electing the correct candidate converges to 1, or how fast the probability of a wrong outcome vanishes, provides a principled way to estimate the epistemic error as a function of the number of voters, or to determine how many voters are needed to meet a desired confidence level.

To achieve this objective, we propose a methodological shift. To the best of our knowledge, all previous studies on this question have relied on classical combinatorial or probabilistic techniques. For instance, the results of Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005) are primarily based on Gaussian approximations. In contrast, this paper introduces tools from *analytic combinatorics* into this domain, drawing on the approach outlined in the reference book by Flajolet and Sedgewick (2009). These methods encode combinatorial problems into analytic objects, allowing the application of powerful analytical tools to extract information, especially limits, but also speeds of convergence, and more generally asymptotic expansions as certain parameters tend to infinity.

1.2 Contributions

Similarly to Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005), we consider a general probability distribution over voter preferences which we call *General Independent Culture (GIC)*, with the primary assumption that individual preferences are independent. Additionally, we assume that the distribution is *generic*, in the sense that every possible ranking over the candidates has a positive probability.

To obtain results as general as possible, we introduce the notion of an $\boldsymbol{\alpha}$*-winner*, a generalization of the Condorcet winner, where the required victory threshold for each pairwise comparison is not necessarily one-half of the voters.

We analyze the probability that a given candidate is an $\boldsymbol{\alpha}$-winner, with a particular focus on its asymptotic behavior as the number of candidates m remains fixed while the number of voters n tends to infinity. Our main contributions lie in providing a method that not only recovers previous results on limiting probabilities using a different approach but also enables the computation of the rate of convergence and higher-order terms in the asymptotic expansion.

These general theoretical results are illustrated through applications to the usual notion of Condorcet winner under probabilistic models known as the *Impartial Culture* and the *Mallows model*. In particular, we establish that the probability of a given candidate being the Condorcet winner in Impartial Culture is $a_{0,m} + a_{1,m,n} n^{-1/2} + O(n^{-1})$, where we explicitly provide the values of the constant $a_{0,m}$ and the coefficient $a_{1,m,n}$. The latter depends on the number of voters n only through its parity, while the number of candidates m is fixed in our asymptotic regime. Although we focus primarily on these two illustrative models, our theoretical results apply directly to a broad class of probabilistic models, including the Perturbed Culture model (Gehrlein 2006; Williamson and Sargent 1967), the Plackett–Luce model (Luce 1959; Plackett 1975), Thurstonian models (Thurstone 1927), and, more generally, random utility models (RUMs) (Azari et al. 2012). While we do not develop these applications in detail in this paper, they fall within the scope of our framework.

In the process, we also establish results in multivariate analytic combinatorics that may be of independent interest. These are detailed in the paragraph "Road map and contributions" of Appendix A.

Beyond the specific problem we study, our approach opens new perspectives for the analysis of other problems in social choice theory. The techniques we develop offer a blueprint for encoding such problems into an analytic framework, with the explicit goal of serving as a tutorial for researchers interested in applying analytic combinatorics in this domain. We highlight some of the possible directions in the conclusion (Section 9).

1.3 Related Work

Regarding the probability of a Condorcet winner, the standard reference is Gehrlein (2006). In the General Independent Culture (GIC), which assumes independent voter preferences, the cases for finite values of voters n and candidates m were explored by Gehrlein and Fishburn (1976b), Gillett (1978), and Gehrlein (1983). The most relevant works to ours are Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005), which derive the limiting probability as n tends to infinity in this general setting. Our approach differs in methodology and results: we use analytic combinatorics and provide more precise asymptotic behaviors.

Among the specific cases of GIC, the most studied is *Impartial Culture (IC)*, where all rankings are equally likely (Gehrlein 2006). For finite values of n and m or for the asymptotic regime where $m \to \infty$, see Sauermann (2022). Other works, like ours, focus on large electorates (*i.e.,* $n \to \infty$) and a fixed number m of candidates. An explicit formula exists for $m = 3$ (Garman and Kamien 1968; Guilbaud 1952; Niemi and Weisberg 1968), while the limit for $m = 4$ follows from a recurrence relation (Fishburn 1973; May 1971). Explicit formulas for $m \in \{5, 6, 7\}$ have been derived in (Gehrlein 1983; Gehrlein and Fishburn 1978). The general case of arbitrary fixed m with $n \to \infty$ was addressed by Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005), whose results we recall in Theorem 5. Finally, the case where both the numbers n of voters and m of candidates tend to infinity has been investigated by Pittel (2025). In IC, as in the broader GIC framework, our main contribution is to provide the rate of convergence and a method for computing higher-order terms in the asymptotic expansion.

Other specific cases of GIC have been studied, notably the *Dual Culture* (Gehrlein 2006) and the *Perturbed Culture* (Gehrlein 2006; Williamson and Sargent 1967). Although the *Mallows model* (Mallows 1957) is popular, the probability of a Condorcet winner in this framework has received little attention. This is likely because, for large electorates, the limiting behavior is trivial: the candidate favored by the culture becomes the Condorcet winner with probability tending to 1. The question of the convergence rate, which we explore here, is much more interesting.

More distant from our work, other models of culture fall outside the GIC framework as they do not assume voter independence. Notable examples include *Impartial Anonymous Culture (IAC)* (Gehrlein 2006; Gehrlein and Fishburn 1976a; Lepelley 1989) and, more generally, *Pólya-Eggenberger models* (Berg and Lepelley 1992; Gehrlein 2006).

One can also study the notion of a *weak Condorcet winner*, a candidate who wins or ties in every pairwise comparison. Krishnamoorthy and Raghavachari (2005) note that the probability of a tie in a pairwise comparison tends to zero as the electorate grows, implying that the limiting probability of a weak Condorcet winner is the same as for a Condorcet winner. However, the convergence speeds may differ, as discussed in our Section 4.2. Other works examine the probability that the *pairwise majority relation* is transitive (Bell 1981, 1978; Gehrlein 2006; Kelly 1974). While this could be addressed with analytic combinatorics, it is beyond the scope of this paper.

Our analysis is based on representing voting configurations by polynomials, expressing probabilities via coefficient extraction, and deriving asymptotics using the saddle-point method. In Impartial Anonymous Culture, many voting questions (Gehrlein 2002; Huang and Chua 2000; Lepelley et al. 2008; Wilson and Pritchard 2007) have been studied using Ehrhart polynomials (Barvinok 1994), reducing the enumeration of voting configurations to counting integral points in a polyhedron. Maassen and Bezembinder (2002) use generating functions to express the probability of a Condorcet winner in Impartial Culture or its variant with weak orders, obtaining exact expressions and asymptotics for several cases. However, none of these works apply complex analysis to derive asymptotics. May (1971) uses the saddle-point method to study the asymptotics as $m \to \infty$ of the limit as $n \to \infty$ of the probability that a candidate is the Condorcet winner in Impartial Culture. This double limit problem is not addressed in this paper and leads to very different computations (see Pittel (2025)).

1.4 Limitations

A limitation of our work is that we focus on the probability that a *given* candidate is an $\boldsymbol{\alpha}$-winner, rather than the probability that at least one exists. For certain values of $\boldsymbol{\alpha}$, multiple candidates may satisfy this condition simultaneously, requiring inclusion-exclusion techniques to compute the probability of existence. We exclude this issue from the scope of this paper, as it does not arise for the Condorcet winner, who, if existent, is unique; thus, its probability is simply the sum of individual probabilities.

Moreover, our main results concern the asymptotic behavior as n tends to infinity. While our approach also provides exact expressions for finite n, these expressions are essentially the same as those obtained using classical combinatorial methods. The main advantage of our method, therefore, lies in its ability to analyze asymptotic behavior. Indeed, although the expressions we obtain often involve complex integrals that cannot be exactly solved, they remain amenable to efficient numerical evaluation. This contrasts with exact combinatorial computations, which tend to become intractable as n increases—see, for instance, the discussion in Section 6.2 about Figure 5.

We assume that voters have independent preferences, excluding cases with more complex interactions. However, De Finetti's theorem (De Finetti 1929; Hewitt and Savage 1955) asserts that *exchangeable* random variables (i.e., roughly speaking, symmetric ones) are conditionally independent given a latent variable. In our case, the preferences of voters are exchangeable up to random relabeling. Therefore, the problem can, in principle, be reduced to independent preferences by conditioning on the latent variable and integrating over its possible values.

To avoid degenerate cases, we also assume that the preference distribution is *generic*, meaning that all rankings have positive probability. While analytic combinatorics methods could be applied without this assumption, doing so would introduce additional subcases that we prefer to avoid, where the *saddle point* (which we will define shortly) has null or infinite coordinates.

Finally, in some situations, our method requires computing a sum that, in the worst case, may contain up to $O(2^m)$ terms (cf. Section 7). However, it is worth noting that merely specifying the full probability distribution already requires $m!$ real numbers in general.

1.5 Roadmap

Section 2 presents the necessary preliminaries. Section 3 translates our combinatorial problem into an analytic question and introduces the notion of *saddle point*. Depending on the coordinates of the saddle point, the analysis then branches into different cases, detailed in Sections 4 to 7. Section 8 extends the analysis by showing how to derive higher-order terms in the asymptotic expansion. Section 9 concludes.

Appendix A establishes the technical results of analytic combinatorics used throughout the paper. Appendix B proves general properties of the GIC model. Appendix C focuses on properties specific to Impartial Culture and the Mallows model.

The paper is accompanied by the Python package `Actinvoting` (Analytic Combinatorics Tools In Voting) (Durand 2025), which was used for symbolic mathematics and numerical simulations.

2 Preliminaries

In this section, we begin by introducing the main definitions and notations, and we formulate our research question. We then briefly present the field of analytic combinatorics.

2.1 Definitions, Notations, and Research Question

The number of voters is denoted by n. We define the set of candidates as $\{1, \dots, m\}$, where m denotes the number of candidates. A *profile* represents the preferences of the voters and is defined as a list of n rankings over the set of candidates. A *culture* refers to a probability distribution over the $(m!)^n$ possible profiles. We assume that voters have independent preferences, meaning the culture is characterized by a probability p_r for each preference

ranking r. Surprisingly, there is no standard terminology for this common assumption, which we refer to as a *General Independent Culture (GIC)*. We also assume that the culture is *generic*, in the sense that $p_r > 0$ for every ranking r. The notation $\mathbb{P}$ refers to the probability under the given culture.

A *Condorcet winner* is a candidate who is preferred to every other candidate by more than $\frac{n}{2}$ voters. Without loss of generality, we study the probability $\mathbb{P}(m \text{ is CW})$ that candidate m is the Condorcet winner. Our theoretical results extend this question to a more general notion, which we call an $\boldsymbol{\alpha}$*-winner*, defined as follows. Let $\mathcal{A} := \{1, \dots, m-1\}$ be the set of adversaries of candidate m. Consider a vector $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_{m-1}) \in (0,1)^{m-1}$, where each α_j represents the proportion of voters that candidate m needs on their side to win the pairwise comparison against candidate j. For an adversary $j \in \mathcal{A}$, we write $m \succ_{\boldsymbol{\alpha}} j$, and we say that m wins against j in the sense of $\boldsymbol{\alpha}$, if candidate m is preferred to j by more than $\alpha_j n$ voters. For a subset of adversaries $\mathcal{X} = \{j_1, \dots, j_k\} \subseteq \mathcal{A}$, we write $m \succ_{\boldsymbol{\alpha}} \mathcal{X}$, or equivalently $m \succ_{\boldsymbol{\alpha}} j_1, \dots, j_k$, if $m \succ_{\boldsymbol{\alpha}} j$ holds for every $j \in \mathcal{X}$. Similarly, we write $m \preccurlyeq_{\boldsymbol{\alpha}} j$, and we say that m does not win against j in the sense of $\boldsymbol{\alpha}$, if candidate m is preferred to j by at most $\alpha_j n$ voters. We say that candidate m is an $\boldsymbol{\alpha}$*-winner* if $m \succ_{\boldsymbol{\alpha}} \mathcal{A}$, *i.e.,* m is preferred to each adversary j by more than $\alpha_j n$ voters. The standard notion of a Condorcet winner corresponds to the special case $\boldsymbol{\alpha} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. The notion of a *generalized Condorcet winner*, as introduced by Sertel and Sanver (2004), is recovered by considering a vector $\boldsymbol{\alpha}$ whose coordinates are equal.

The goal of this paper is to study $\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner})$, the probability that candidate m is an $\boldsymbol{\alpha}$-winner, with a particular focus on its asymptotic behavior as the number of voters n tends to infinity while the number m of candidates is fixed. Note that the probability of the existence of a Condorcet winner can be obtained as the sum of the probabilities for all m candidates.

Our theoretical results will be illustrated using the *Mallows model* (Mallows 1957). This model is characterized by a reference ranking r_0 over the candidates and a concentration parameter $\rho \in \mathbb{R}_{\geq 0}$. The probability of a ranking r is given by

$$p_r := \gamma e^{-\rho d(r, r_0)},$$

where γ is a normalization constant ensuring $\sum_r p_r = 1$, and $d(r, r_0)$ denotes the Kendall-tau distance (Kendall 1938), which counts the minimum number of adjacent swaps needed to transform r into r_0. The Mallows model is commonly used in the field of *epistemic democracy* (Brandt et al. 2016, Chapters 8 and 10), where noisy evaluations from voters are collected with the aim of uncovering a hidden truth. The reference ranking models the hidden truth, and the concentration parameter ρ indicates the skill level of the voters. When $\rho = 0$, all rankings are equally probable, recovering the classical *Impartial Culture* model. We denote by $\mathcal{M}_{m \text{ last}}$ (resp. $\mathcal{M}_{m \text{ first}}$) a Mallows culture with parameter ρ, where the reference ranking r_0 places candidate m last (resp. first). We can assume, without loss of generality, that the reference ranking is $(1, \dots, m)$ (resp. $(m, \dots, 1)$).

Finally, we typeset vectors in boldface, *e.g.,* $\boldsymbol{x} := (x_1, \dots, x_{m-1})$. If $\mathcal{X}$ is a set of indices, we define $\boldsymbol{x}_{\mathcal{X}} := (x_j)_{j \in \mathcal{X}}$. We define $\log(\boldsymbol{x}) := (\log(x_1), \dots, \log(x_{m-1}))$, and similarly for the exponential. We write $\boldsymbol{u} \leq \boldsymbol{v}$ if $u_j \leq v_j$ for every coordinate j, with similar notation for strict inequalities. We denote $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := (1, \dots, 1)$, where the vector's size is understood from context. The diagonal matrix with diagonal elements $u_1, u_2, \dots$ is denoted $\mathrm{diag}(\boldsymbol{u})$. Finally, for convenience, we set $\boldsymbol{\beta} := \mathbf{1} - \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the vector of victory thresholds in the definition of an $\boldsymbol{\alpha}$-winner.

2.2 A Short Introduction to Analytic Combinatorics

The techniques employed in this paper come from the field of analytic combinatorics (Flajolet and Sedgewick 2009). Within this framework, formal variables are introduced and linked to specific parameters that define a particular counting problem. Once these variables are properly introduced, a formal power series is defined as a function of these variables, called the *generating function*. It serves as a formal notation that succinctly represents the combinatorial structure of the problem. Furthermore, it can be interpreted as a function of complex

variables, which can be studied with the powerful tools of complex analysis, hence giving insights into the original combinatorial problem.

To illustrate this, consider a classical combinatorial example: counting binary trees with k vertices. The problem is first encoded as a formal power series by introducing a variable z marking the number of vertices. The generating function $T(z)$ is then defined as

$$T(z) := \sum_{k=0}^{\infty} T_k z^k,$$

where T_k is the number of binary trees with k vertices. The *symbolic method* then translates the combinatorial structure into analytic properties. In this example, whose detailed analysis is beyond the scope of this paper, the combinatorial property that a binary tree is either an isolated leaf or a root with two binary subtrees translates into the equation $T(z) = z + zT(z)^2$. To determine T_k, we solve this equation and extract the coefficient of z^k in the Taylor expansion of $T(z)$, which is denoted by the *coefficient extraction* $[z^k]T(z)$.

While this example is solved using an algebraic equation, other cases may involve different analytical techniques, such as differential equations or complex calculus.

3 From Our Combinatorial Problem to Its Analytic Formulation

As for binary trees, we first encode the problem of determining the probability that a candidate is an $\boldsymbol{\alpha}$-winner. However, instead of a univariate infinite series, we now work with a multivariate polynomial. We then show that the coefficient extractions can be achieved through *Cauchy integrals*. Finally, to analyze their asymptotic behavior for large electorates, we introduce the general principle of the *saddle-point method*. Its different sub-cases are later developed in Sections 4 to 7.

3.1 Symbolic Method

As a warm-up, consider the case $m = 3$. To determine whether candidate 3 is the Condorcet winner, or more generally an $\boldsymbol{\alpha}$-winner, it suffices to know whether each voter ranks candidate 1 and/or 2 above candidate 3, rather than their full ranking. Thus, we introduce a formal variable x_1 (resp. x_2) that indicates when a voter prefers candidate 1 (resp. 2) over candidate 3.

To represent the probability distribution governing the preferences of a single voter, we introduce the characteristic polynomial $P(x_1, x_2)$, as illustrated in Figure 1. In this polynomial, the probability of each ranking is multiplied by the appropriate formal variables, depending on whether candidate 1 and/or 2 is preferred over candidate 3. At this stage, the characteristic polynomial may appear to be merely a compact way of representing the probability distribution of a single voter.

Note that the characteristic polynomial can be rewritten as

$$P(x_1, x_2) = p_{\emptyset} + p_{\{1\}}x_1 + p_{\{2\}}x_2 + p_{\{1,2\}}x_1x_2, \tag{1}$$

where, for example, $p_{\{1,2\}} := p_{123}+p_{213}$ is the probability that a voter ranks both candidates 1 and 2 above candidate 3. This probability can also be expressed as a multivariate coefficient extraction, denoted by $[x_1{}^1x_2{}^1]P(x_1, x_2)$.

More generally, for an arbitrary value of m and a subset of adversaries $\mathcal{X} \subseteq \mathcal{A}$, we overload the notation p by defining $p_{\mathcal{X}}$ as the total probability that in a random ranking, the set of adversaries above m is exactly $\mathcal{X}$. We then encode the probability distribution for a single voter as follows.

Definition 1. *The* characteristic polynomial *is defined as*

$$P(\boldsymbol{x}) := \sum_{\mathcal{X}\subseteq\mathcal{A}} \Big(p_{\mathcal{X}} \prod_{j\in\mathcal{X}} x_j\Big).$$

p_{123}	p_{132}	p_{213}	p_{231}	p_{312}	p_{321}
1 2 [3]	1 [3] 2	2 1 [3]	2 [3] 1	[3] 1 2	[3] 2 1
x_1x_2	x_1	x_1x_2	x_2	1	1

$$P(x_1, x_2) := p_{123} \cdot x_1x_2 + p_{132} \cdot x_1 + p_{213} \cdot x_1x_2 + p_{231} \cdot x_2 + p_{312} \cdot 1 + p_{321} \cdot 1$$

Fig. 1. Definition of the characteristic polynomial $P(x_1, x_2)$ encoding the probability distribution for a single voter when $m = 3$. The notation p_{123}, for example, is a shorthand for $p_{(1,2,3)}$. The formal variable x_j marks rankings where candidate j is preferred to candidate 3.

Note that the sum over all rankings $\sum_r p_r = 1$ implies that the evaluation of P at the all-ones vector is $P(\mathbf{1}) = 1$.

Now, let us examine what happens with several voters, beginning with the case with $m = 3$ candidates and $n = 2$ voters. The core observation, illustrated in Figure 2, is that the algebraic operation of expanding $P(x_1, x_2)^2$ is isomorphic to the probability tree associated with the preferences of two voters. For example, the probability that, among the two voters, exactly two prefer candidate 1 over candidate 3, and exactly one prefers candidate 2 over candidate 3 is given by the coefficient extraction $[x_1{}^2x_2{}^1]P(x_1, x_2)^2 = 2p_{\{1\}}p_{\{1,2\}}$.

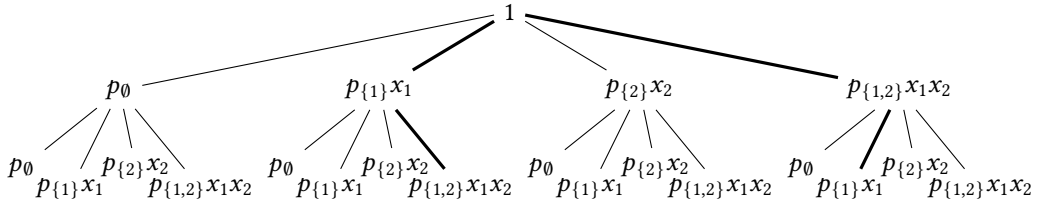


Fig. 2. Tree representing the algebraic expansion of $P(x_1, x_2)^2$. The edges correspond to multiplications, and each path from the root to a leaf represents a term in the expansion. For example, the highlighted paths correspond to the terms involving $x_1{}^2x_2{}^1$, with a total coefficient given by $[x_1{}^2x_2{}^1]P(x_1, x_2)^2 = 2p_{\{1\}}p_{\{1,2\}}$. This coefficient extraction corresponds to the probability that, among the two voters, exactly two prefer candidate 1 over candidate 3, and exactly one prefers candidate 2 over candidate 3.

More generally, when m and n are arbitrary, for a vector $\boldsymbol{\ell} \in \mathbb{N}^{m-1}$, the coefficient extraction

$$[\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})^n := [{x_1}^{\ell_1} \cdots {x_{m-1}}^{\ell_{m-1}}]P(x_1, \ldots, x_{m-1})^n$$

corresponds to the probability that each adversary j is preferred to candidate m by exactly ℓ_j voters.

Now, for m to be an $\boldsymbol{\alpha}$-winner, every adversary j must be preferred to m by less than $(1 - \alpha_j)n = \beta_j n$ voters. Summing over all such cases, the probability that candidate m is an $\boldsymbol{\alpha}$-winner is given by

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) = [\boldsymbol{x}^{<\boldsymbol{\beta}n}]P(\boldsymbol{x})^n := \sum_{\boldsymbol{\ell}<\boldsymbol{\beta}n} [\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})^n.$$

However, the coordinates of $\boldsymbol{\beta}n$ are not necessarily integers, and we prefer to express the summation bounds in terms of integers:

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) = [\boldsymbol{x}^{\leq\lceil\boldsymbol{\beta}n\rceil-1}]P(\boldsymbol{x})^n := \sum_{\boldsymbol{\ell}\leq\lceil\boldsymbol{\beta}n\rceil-1} [\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})^n. \tag{2}$$

Note that if we define a *weak $\boldsymbol{\alpha}$-winner* as a candidate preferred to any other candidate j by *at least* $\alpha_j n$ voters, then all the results of this paper extend to weak $\boldsymbol{\alpha}$-winners by replacing $\lceil \boldsymbol{\beta} n \rceil - 1$ with $\lfloor \boldsymbol{\beta} n \rfloor$. In particular, our findings for the Condorcet winner naturally extend to the *weak Condorcet winner* (defined analogously) by replacing all occurrences of $\lceil n/2 \rceil - 1$ with $\lfloor n/2 \rfloor$.

3.2 Coefficient Extraction via Cauchy Integrals

In analytic combinatorics, it is common to extract the coefficient f_ℓ of a series $F(z) = \sum_k f_k z^k$ with a positive radius of convergence by expressing it as a Cauchy integral (Flajolet and Sedgewick 2009, Th. IV.4):

$$[z^\ell]F(z) = \frac{1}{2i\pi} \oint F(z) \frac{dz}{z^{\ell+1}}. \tag{3}$$

The symbol $\oint$ indicates that the integral is taken over a positively oriented loop around 0 of sufficiently small radius (say smaller than the radius of convergence of $F(z)$) in the complex plane. To intuitively understand this equality, one can expand $F(z)$ into its series, interchange sum and integral, and integrate over a circle of small radius centered at 0. The residue theorem (see *e.g.,* Flajolet and Sedgewick (2009, Th. IV.3)) then implies that all terms of the form $z^{k-\ell-1}$ result in a zero integral unless $k = \ell$. Now, to extract $[z^{\leq L}]F(z) := f_0 + \cdots + f_L$, we apply

$$[z^{\leq L}]F(z) = [z^L]\frac{F(z)}{1-z}, \tag{4}$$

which is obtained by remarking that $\frac{F(z)}{1-z} = \left(\sum_k f_k z^k\right)\left(\sum_k z^k\right) = \sum_k \left(\sum_{\ell=0}^k f_\ell\right) z^k$.

Equations (3) and (4) extend to the multivariate case (see *e.g.,* Krantz (2001)). Applying them to Equation (2) gives

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) = \frac{1}{(2i\pi)^{m-1}} \oint \frac{P(\boldsymbol{x})^n}{\prod_{j=1}^{m-1}(1-x_j)} \frac{d\boldsymbol{x}}{\prod_{j=1}^{m-1} x_j^{\lceil \beta_j n \rceil}}, \tag{5}$$

where $\oint d\boldsymbol{x}$ is a shorthand for $\oint \cdots \oint dx_1 \dots dx_{m-1}$.

3.3 Saddle Point Method

We study the asymptotic behavior of our complex integrals as $n \to +\infty$. To build intuition, let us momentarily disregard the ceiling function in Equation (5) and observe that

$$\oint \frac{P(\boldsymbol{x})^n}{\prod_{j=1}^{m-1}(1-x_j)} \frac{d\boldsymbol{x}}{\prod_{j=1}^{m-1} x_j^{\beta_j n}} = \oint \frac{e^{-n\left(-\log(P(\boldsymbol{x})) + \boldsymbol{\beta}^\intercal \log(\boldsymbol{x})\right)}}{\prod_{j=1}^{m-1}(1-x_j)} d\boldsymbol{x} = \oint A(\boldsymbol{x}) e^{-n\psi(\log(\boldsymbol{x}))} d\boldsymbol{x},$$

for an appropriately defined function of $m-1$ variables $A(\boldsymbol{x})$ and $\psi(\boldsymbol{t}) = -\log(P(e^{\boldsymbol{t}})) + \boldsymbol{\beta}^\intercal \boldsymbol{t}$, where $\boldsymbol{t} = \log(\boldsymbol{x}) = (\log(x_1), \dots, \log(x_{m-1}))$.

To analyze the asymptotic behavior of such integrals, it is standard to apply the *saddle-point method* (Flajolet and Sedgewick 2009, Chapter VIII). The key idea is to find a contour of integration where, as n approaches infinity, the integral's dominant contribution arises from the neighborhood of a specific point, while the contribution from the rest of the contour becomes negligible in comparison.

For this approach to be valid, the chosen point must satisfy the condition that the gradient of the function ψ vanishes. In the univariate case, since a holomorphic function cannot exhibit local extrema in modulus, the graph of the function's modulus at such a point resembles a saddle. This is why it is called a *saddle point*, even in the general multivariate case.

In our case, the presence of the integer parts $\lceil \beta_j n \rceil$ does not fundamentally alter the method. The necessary adaptations are detailed in Appendix A. Moreover, Lemma 9 in Appendix B.1 ensures the existence of a unique

saddle point. The absolute value of $e^{-n\psi(\log(\boldsymbol{x}))}$ at the saddle point is minimal on the real line and maximal on the integration circle. These considerations lead us to define the following objects, which we will use extensively in the rest of the paper.

Definition 2. *The* cumulant generating function *of P is*

$$K : \boldsymbol{t} \in \mathbb{R}^{m-1} \mapsto \log\left(P(e^{\boldsymbol{t}})\right).$$

Observing that the function $\psi : \boldsymbol{t} \mapsto -K(\boldsymbol{t}) + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{t}$ is strictly concave on $\mathbb{R}^{m-1}$ (an immediate consequence of Lemma 6 in Appendix A.1), we define the log saddle point $\boldsymbol{\tau}$ *and the* saddle point $\boldsymbol{\zeta}$ *as*

$$\boldsymbol{\tau} := \underset{\boldsymbol{t}\in\mathbb{R}^{m-1}}{\arg\max}\left(-K(\boldsymbol{t}) + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{t}\right), \qquad \boldsymbol{\zeta} = e^{\boldsymbol{\tau}}.$$

Observe that this implies that $\boldsymbol{\zeta}$ minimizes the function $\boldsymbol{x} \in (\mathbb{R}_{>0})^{m-1} \mapsto \frac{P(\boldsymbol{x})}{\prod_j x_j^{\beta_j}}$.

To approximate the integral in Equation (5), a key ingredient is the Taylor expansion of ψ at its saddle point. Consequently, we will need the Hessian of ψ at $\boldsymbol{\tau}$. Since the term $\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{t}$ is linear, this Hessian differs only by a sign from $\mathcal{H}_K(\boldsymbol{\tau})$, the Hessian of $K(\boldsymbol{t})$ at $\boldsymbol{\tau}$, which can be computed either directly or as

$$\mathcal{H}_K(\boldsymbol{\tau}) = \operatorname{diag}(\boldsymbol{\zeta})\frac{\mathcal{H}_P(\boldsymbol{\zeta})}{P(\boldsymbol{\zeta})}\operatorname{diag}(\boldsymbol{\zeta}) + \operatorname{diag}(\boldsymbol{\beta}) - \boldsymbol{\beta}\boldsymbol{\beta}^{\mathsf{T}}. \tag{6}$$

Probabilistic interpretation. There is a nice probabilistic interpretation for the cumulant generating function. For any vector parameter $\boldsymbol{s} \in \mathbb{R}^{m-1}$, we define the random vector $\boldsymbol{X}^{(\boldsymbol{s})}$ as follows. For any subset $\mathcal{X} \subseteq \mathcal{A}$ of adversaries whose indicator vector is denoted $\boldsymbol{\ell} \in \{0,1\}^{m-1}$:

$$\mathbb{P}(\boldsymbol{X}^{(\boldsymbol{s})} = \boldsymbol{\ell}) := \frac{[\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})e^{\boldsymbol{\ell}^{\mathsf{T}}\boldsymbol{s}}}{P(e^{\boldsymbol{s}})} = \frac{p_{\mathcal{X}}e^{\boldsymbol{\ell}^{\mathsf{T}}\boldsymbol{s}}}{P(e^{\boldsymbol{s}})}.$$

Note that $\boldsymbol{X}^{(\boldsymbol{0})}$ corresponds to the original probability distribution, as encoded by P. For $\boldsymbol{s} \neq \boldsymbol{0}$, the distribution of $\boldsymbol{X}^{(\boldsymbol{s})}$ represents a perturbation of the original culture.

The *cumulant generating function* of $\boldsymbol{X}^{(\boldsymbol{s})}$ is classically defined as

$$K_{\boldsymbol{s}} : \boldsymbol{t} \mapsto \log(\mathbb{E}(e^{\boldsymbol{t}^{\mathsf{T}}\boldsymbol{X}^{(\boldsymbol{s})}})).$$

Remark that $K = K_{\boldsymbol{0}}$. It is well known that the gradient and Hessian matrix of this function at $\boldsymbol{t} = \boldsymbol{0}$ are equal to the mean and covariance matrix of $\boldsymbol{X}^{(\boldsymbol{s})}$. Actually, all the information contained in $K_{\boldsymbol{s}}$ can be retrieved through K thanks to the following observation.

$$\begin{aligned} K_{\boldsymbol{s}}(\boldsymbol{t}) &= \log\left(\sum_{\boldsymbol{\ell}} \frac{\left([\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})\right)e^{\boldsymbol{\ell}^{\mathsf{T}}\boldsymbol{s}}}{P(e^{\boldsymbol{s}})} e^{\boldsymbol{\ell}^{\mathsf{T}}\boldsymbol{t}}\right) = \log\left(\sum_{\boldsymbol{\ell}} \left([\boldsymbol{x}^{\boldsymbol{\ell}}]P(\boldsymbol{x})\right)e^{\boldsymbol{\ell}^{\mathsf{T}}(\boldsymbol{s}+\boldsymbol{t})}\right) - \log(P(e^{\boldsymbol{s}})) \\ &= \log(P(e^{\boldsymbol{s}+\boldsymbol{t}})) - \log(P(e^{\boldsymbol{s}})) = K(\boldsymbol{s}+\boldsymbol{t}) - \log(P(e^{\boldsymbol{s}})), \end{aligned}$$

so the mean and covariance matrix of $\boldsymbol{X}^{(\boldsymbol{s})}$ are respectively equal to the gradient and the Hessian of $K(\boldsymbol{t})$ at $\boldsymbol{t} = \boldsymbol{s}$.

Since $\boldsymbol{\tau}$ is the vector where the gradient of $\psi : \boldsymbol{t} \mapsto -K(\boldsymbol{t}) + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{t}$ vanishes, it follows that the gradient of K at $\boldsymbol{\tau}$, *i.e.,* the mean of $\boldsymbol{X}^{(\boldsymbol{\tau})}$, is equal to $\boldsymbol{\beta}$. In other words, the distribution of $\boldsymbol{X}^{(\boldsymbol{\tau})}$, *i.e.,* the perturbation of the original culture induced by $\boldsymbol{\tau}$, is such that, in expectation, candidate m is precisely at the threshold for being an $\boldsymbol{\alpha}$-winner.

Criticality. In Equation (5), the terms of the form $\frac{1}{1-x_j}$ require special precautions. If $\boldsymbol{\zeta}_j < 1$, there is no issue, as we can integrate over a circle of radius ζ_j that loops around 0 without enclosing the singularity at 1. In this case, we say that the coordinate $\boldsymbol{\zeta}_j$ is *subcritical*. However, if $\boldsymbol{\zeta}_j$ is *critical*, i.e., if $\boldsymbol{\zeta}_j = 1$, then the integration path must be adjusted to slightly bypass the singularity. Finally, if $\boldsymbol{\zeta}_j$ is *supercritical*, i.e., if $\boldsymbol{\zeta}_j > 1$, then a circle of radius ζ_j necessarily encloses a singularity, and the issue remains even with slight deformations of the integration path, requiring the singularity at 1 to be explicitly accounted for in the analysis.

In Section 4, we analyze the case where all coordinates are subcritical, while Section 5 focuses on the case where all coordinates are critical. Section 6 extends the analysis to the mixed case, where each coordinate is either subcritical or critical. To address supercritical coordinates, we introduce the necessary techniques in Section 7, which makes it possible to tackle the most general setting. Finally, Section 8 provides higher-order terms in the asymptotic expansion of the previous formulas. Together, these sections establish the theoretical asymptotic behavior of an $\boldsymbol{\alpha}$-winner in the GIC, followed by an application to the Condorcet winner in a particular culture.

4 Subcritical Case

In this section, we study the case where all coordinates of the saddle point are subcritical. We then illustrate this scenario with $\mathcal{M}_{3\text{ last}}$, a Mallows culture where the reference ranking is $(1, 2, 3)$, making candidate 3 particularly unlikely to be the Condorcet winner.

4.1 Theoretical Result in the Subcritical Case

Since $P(\boldsymbol{x})$ has strictly positive coefficients and $\boldsymbol{\beta} \in (0, 1)^{m-1}$, it follows from Appendix B.1 that all the assumptions of Theorem 13 in Appendix A.2 are satisfied, allowing its application. Recall that $P(\boldsymbol{x})$ is introduced in Definition 1, and that $\boldsymbol{\tau}$, $\boldsymbol{\zeta}$, and $\mathcal{H}_K(\boldsymbol{\tau})$ are defined in Definition 2.

Theorem 1. *Assume that all coordinates of the saddle point are subcritical, i.e., $\zeta_j < 1$ for every adversary $j \in \mathcal{A}$. Then:*

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) \underset{n\to+\infty}{\sim} \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{A}} \left((1-\zeta_j)\zeta_j^{\lceil \beta_j n\rceil - 1}\right)} \frac{1}{\sqrt{(2\pi n)^{m-1}\det(\mathcal{H}_K(\boldsymbol{\tau}))}}.$$

To interpret this asymptotic behavior, recall that $\boldsymbol{\zeta}$ is the unique global minimizer of $\boldsymbol{x} \mapsto \frac{P(\boldsymbol{x})}{\prod_j x_j{}^{\beta_j}}$. Hence, $\frac{P(\boldsymbol{\zeta})}{\prod_j \zeta_j{}^{\beta_j}} < \frac{P(\mathbf{1})}{\prod_j 1^{\beta_j}} = 1$. Thus, Theorem 1 indicates that, in the subcritical case, $\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner})$ tends to 0 exponentially fast.

4.2 Application: Mallows Culture $\mathcal{M}_{3\text{ last}}$

In the model $\mathcal{M}_{3\text{ last}}$, defined in Section 2.1, candidate 3 is ranked last among the three options in the reference ranking $(1, 2, 3)$, which can be interpreted as representing the hidden truth. We now analyze the probability of the *a priori* undesirable event in which candidate 3 nevertheless emerges as the Condorcet winner.

To find this probability, the first step is to determine the saddle point. We provide the general formula for an arbitrary number of candidates m. The proof is detailed in Appendix C.2.

Lemma 1. *Under the Mallows culture $\mathcal{M}_{m\text{ last}}$, the log saddle point $\boldsymbol{\tau}$ is given by*

$$\boldsymbol{\tau} = \left(\frac{-m\rho}{2}, \frac{(-m+2)\rho}{2}, \ldots, \frac{(m-4)\rho}{2}\right),$$

where ρ is the concentration parameter of the culture, defined in Section 2.1.

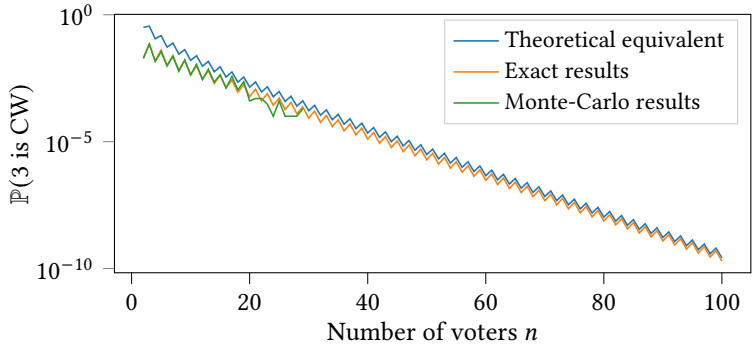


Fig. 3. Probability that candidate 3 is the Condorcet winner as a function of n in a culture $\mathcal{M}_{3\,\text{last}}$ with $\rho = \log(2)$, shown on a semilog scale. The theoretical equivalent is based on Theorem 2, while exact results rely on Equation (2). Monte Carlo simulations use 10,000 profiles per point, yielding an error of order 10^{-2}. For $n \geq 30$, they return zero, which is not visible due to the logarithmic scale.

In the case of three candidates, Lemma 1 yields $\boldsymbol{\tau} = (\frac{-3\rho}{2}, \frac{-\rho}{2})$, ensuring that both coordinates of the saddle point are subcritical and allowing the application of Theorem 1. Standard algebraic calculations then lead to the following result.

Theorem 2. *Under the Mallows culture $\mathcal{M}_{3\,\text{last}}$, the probability that candidate 3 is the Condorcet winner has asymptotic behavior*

$$\mathbb{P}(3 \text{ is CW}) \underset{n\to+\infty}{\sim} \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{A}} \left((1-\zeta_j)\zeta_j^{\lceil n/2\rceil-1}\right)} \frac{1}{2\pi n\sqrt{\det(\mathcal{H}_K(\boldsymbol{\tau}))}},$$

where $\boldsymbol{\zeta} = (e^{-3\rho/2}, e^{-\rho/2})$, $P(\boldsymbol{\zeta}) = 2\frac{e^{-2\rho}(1+e^{-\rho/2}+e^{-\rho})}{(1+e^{-\rho})(1+e^{-\rho}+e^{-2\rho})}$ *and* $\det(\mathcal{H}_K(\boldsymbol{\tau})) = \frac{1}{4}\frac{e^{-\rho/2}(1+e^{-\rho})}{(1+e^{-\rho/2}+e^{-\rho})^2}$.

As we saw in Section 4.1, this probability tends exponentially fast to 0. This can be viewed as a generalization of Condorcet's Jury Theorem (Condorcet 1785), which gives the same result for 2 candidates.

As mentioned in Section 3.1, the counterpart of Theorem 2 for the weak Condorcet winner is obtained by replacing $\lceil n/2\rceil - 1$ with $\lfloor n/2\rfloor$. Notably, for even n, the former simplifies to $n/2-1$, while the latter becomes $n/2$. Thus, although the limit remains 0 in both cases, the rates of convergence for even n differ by a factor $\prod_{j\in\mathcal{A}} \zeta_j$.

Numerical simulations. To validate our results numerically, we developed the Python package `Actinvoting` (Durand 2025). This package notably compares our theoretical results with Monte Carlo simulations, which estimate probabilities by generating a large number of random profiles. Ranking samples in the Mallows model are obtained using the algorithms of Doignon et al. (2004) and Lu and Boutilier (2014).

Figure 3 illustrates the case of $\mathcal{M}_{3\,\text{last}}$, showing the probability that candidate 3 is the Condorcet winner as a function of the number of voters n. We provide three estimations: the theoretical approximation given by Theorem 2, the exact result obtained via coefficient extraction from Equation (2), and a Monte Carlo estimate based on 10,000 profiles per value of n. The concentration parameter is set to $\rho = \log(2)$. This choice balances two opposing constraints observed in practice. Lower concentration values require larger n for the theoretical approximation to be accurate, making direct comparison with exact results difficult, as their computational cost is lower bounded by the number of monomials in the algebraic expansion of $P(\boldsymbol{x})^n$, which is $\Omega(n^{m-1})$. Conversely, higher concentration values cause probabilities to decay rapidly, rendering Monte Carlo estimation unreliable.

The computational costs of these methods vary significantly. For instance[1], at $n = 100$, Monte Carlo simulations take 46 seconds, the exact computation 17 seconds, and the theoretical approximation only 338 μs. As previously mentioned, the cost of the exact computation is $\Omega(n^{m-1})$. Monte Carlo simulations are also computationally expensive, as their error scales as $O(1/\sqrt{N})$, where N is the sample size. In contrast, the complexity of the theoretical approximation is polynomial in $\log(n)$, making it essentially constant time in practice.

A striking feature of Figure 3 is the non-monotonicity of the curves, which stems from parity effects in the number of voters n. For small n, the exact and Monte Carlo curves are nearly indistinguishable, while the theoretical approximation slightly overestimates the probability. When probabilities drop significantly below the order of magnitude of the Monte Carlo error, $\frac{1}{\sqrt{N}} = 10^{-2}$, the Monte Carlo results naturally become more unstable. Below $\frac{1}{N} = 10^{-4}$, they become undetectable due to sample size limitations. As expected, for a large number of voters n, the theoretical approximation converges to the exact result, while being much faster computationally.

5 Critical Case

In this section, we study the case where all coordinates of the saddle point are critical, which is exemplified by the notion of Condorcet winner in the Impartial Culture.

5.1 Theoretical Result in the Critical Case

When $\boldsymbol{\zeta} = \mathbf{1}$, it means that the expected proportion of voters who prefer candidate m to candidate j is exactly α_j, as explained in the probabilistic interpretation of Section 3.3. In other words, in expectation, candidate m is precisely at the threshold of being an $\boldsymbol{\alpha}$-winner.

As soon as one coordinate ζ_j is equal to 1, the term $\frac{1}{1-x_j}$ in Equation (5) prevents integration along a circle of radius ζ_j. Theorem 14 in Appendix A.3 is based on the idea that this difficulty can be circumvented by choosing an integration path that slightly deforms around the singularity at 1. Applying this result to the particular case where all coordinates are critical, we obtain the following theorem.

Theorem 3. *Assume that all coordinates of the saddle point are critical, i.e., $\zeta_j = 1$ for every adversary $j \in \mathcal{A}$. Then:*

$$\lim_{n\to+\infty} \mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) = \frac{1}{\sqrt{(2\pi)^{m-1}\det(\mathcal{H}_K(\boldsymbol{\tau}))}} \int_{(0,+\infty)^{m-1}} e^{-\boldsymbol{u}^\intercal \mathcal{H}_K(\boldsymbol{\tau})^{-1}\boldsymbol{u}/2} d\boldsymbol{u}.$$

In this case, Equation (6) simplifies to $\mathcal{H}_K(\boldsymbol{\tau}) = \mathcal{H}_P(\mathbf{1}) + \mathrm{diag}(\boldsymbol{\beta}) - \boldsymbol{\beta}\boldsymbol{\beta}^\intercal$ because $\boldsymbol{\zeta} = \mathbf{1}$ and $P(\mathbf{1}) = 1$.

For the critical case, Theorem 3 generalizes to $\boldsymbol{\alpha}$-winners the result of Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005, Equation (12)) on Condorcet winners. While we derive this result via analytic combinatorics, their approach relies on a standard probabilistic method: the Gaussian approximation. This method allows interpretation of the vector $\boldsymbol{u}$ as the standardized outcome of each pairwise comparison between candidate m and their adversaries. Integrating over the positive orthant corresponds to requiring that each comparison outcome exceeds its expectation $\alpha_j n$.

The two approaches differ in the matrix involved. In their case, it is the correlation matrix of the pairwise comparisons against each adversary, denoted R_m by both Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005). Appendix B.2 recalls its definition and establishes the relation $R_m = 4D\mathcal{H}_K(\boldsymbol{\tau})D$, where D is a diagonal matrix, which is the identity matrix when considering the notion of Condorcet winner. Since the result of Theorem 3 is divided by the square root of the determinant, this rescaling has no impact. However, our analytic combinatorics approach makes it easier to deal with non-critical cases and extends naturally to the

[1]All of our numerical simulations were performed on an 11th Gen Intel Core i9-11980HK processor (8 cores, 16 logical processors) with 64 GB of RAM.

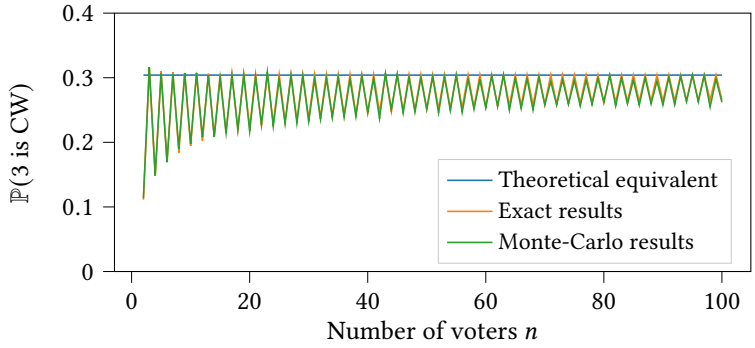


Fig. 4. Probability that a given candidate is the Condorcet winner as a function of n in the Impartial Culture with three candidates. The theoretical equivalent is based on Theorem 4, while exact results rely on Equation (2). Monte Carlo simulations use 10,000 profiles per point, yielding an error of order 10^{-2}. When the exact results curve is not visible, it is overlapped by the Monte Carlo results curve.

computation of an asymptotic expansion (see Section 8). For instance, in the subcritical case, their formula only establishes that the limit is zero, whereas Theorem 1 additionally provides the convergence rate.

5.2 Application: Impartial Culture

Computing the limiting probability for a candidate to be the Condorcet winner in Impartial Culture corresponds to applying Theorem 3 with $\boldsymbol{\beta} = \mathbf{1}/2$. We obtain the following result, proved in Appendix C.1.

Theorem 4. *Under Impartial Culture with m candidates, the probability that candidate m is the Condorcet winner has the following limit:*

$$\lim_{n\to+\infty} \mathbb{P}(m \text{ is CW}) = \frac{1}{\sqrt{(2\pi)^{m-1}\det(\mathcal{H}_K(\mathbf{0}))}} \int_{(0,+\infty)^{m-1}} e^{-\boldsymbol{u}^{\mathsf{T}}\mathcal{H}_K(\mathbf{0})^{-1}\boldsymbol{u}/2} d\boldsymbol{u}$$

where, letting I denote the identity matrix and J the matrix where all elements are 1s, both of dimension $(m-1)\times(m-1)$,

$$\mathcal{H}_K(\mathbf{0}) = \frac{1}{6}\left(I + \frac{1}{2}J\right), \qquad \mathcal{H}_K(\mathbf{0})^{-1} = 6\left(I - \frac{1}{m+1}J\right), \qquad \det(\mathcal{H}_K(\mathbf{0})) = \frac{m+1}{2}\frac{1}{6^{m-1}}. \tag{7}$$

We show in Appendix B.2 that this result is the same as Equation (10) from Niemi and Weisberg (1968) and Equation (12) from Krishnamoorthy and Raghavachari (2005). For completeness, we recall a simpler univariate integral formulation introduced by Ruben (1954) to study moments of Gaussian distributions, and linked to the voting setting first implicitly by Niemi and Weisberg (1968), then explicitly by May (1971) (Equation (7)).

Theorem 5 (May (1971)). *Under Impartial Culture,*

$$\lim_{n\to+\infty} \mathbb{P}(m \text{ is CW}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} (1 - \Phi(u))^{m-1} du \tag{8}$$

where $\Phi(u) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{u}{\sqrt{2}}\right)\right)$ and erf *denotes the error function:* $\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2} dv$.

Numerical simulations. Figure 4 illustrates our results for the Impartial Culture with three candidates. Computational times vary significantly: for $n = 100$, Monte Carlo simulations take 5 seconds (faster than for the Mallows model due to quicker random profile generation in IC), the exact computation takes 22 seconds, and the theoretical approximation requires only 358 μs. As before, the Monte Carlo and exact result curves exhibit sawtooth patterns due to parity effects in the number of voters n and remain nearly indistinguishable. In contrast, the theoretical approximation, which is just a numerical limit here, is simply represented as a horizontal line. Notably, for odd n, the exact values closely match the limit, whereas for even n, they are significantly lower, although they converge to the same value. In Section 8, we present the next term in the asymptotic expansion of the probability, which improves the fit between the theoretical approximation and the exact curve, and explains why the limit approximation is already very accurate for an odd number of voters n.

6 Mixed Case: Subcritical and Critical

We study the case where some of the coordinates are subcritical and some are critical, combining the ideas developed in Sections 4 and 5.

6.1 Theoretical Result in the Mixed Case

As in the critical case, we apply Theorem 14 from Appendix A.3, now in its more general form.

THEOREM 6. *Let $\mathcal{S} := \{j \in \mathcal{A},\ \zeta_j < 1\}$ and $\mathcal{C} := \{j \in \mathcal{A},\ \zeta_j = 1\}$ respectively denote the subset of subcritical and critical coordinates of the saddle point. Assume $\mathcal{A} = \mathcal{S} \cup \mathcal{C}$. Then:*

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) \underset{n\to+\infty}{\sim} \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{S}} \left((1-\zeta_j)\zeta_j^{\lceil \beta_j n\rceil - 1}\right)} \frac{1}{\sqrt{(2\pi)^{m-1} n^{|\mathcal{S}|} \det(\mathcal{H}_K(\boldsymbol{\tau}))}} \int_{(0,+\infty)^{|\mathcal{C}|}} e^{-\boldsymbol{u}^\intercal M \boldsymbol{u}/2} d\boldsymbol{u},$$

where M is the submatrix of $\mathcal{H}_K(\boldsymbol{\tau})^{-1}$ that corresponds to the rows and columns of $\mathcal{C}$.

As in the subcritical case, this result establishes that the probability converges exponentially fast to zero whenever at least one coordinate is subcritical. The Gaussian integral associated with the critical coordinates is just a multiplicative constant in this asymptotic behavior.

6.2 Application: Mallows Culture $\mathcal{M}_{4\,\text{last}}$

In Section 4.2, we established that for the Mallows culture $\mathcal{M}_{3\,\text{last}}$, all coordinates of the saddle point are subcritical. In the case of 4 candidates, *i.e.,* for $\mathcal{M}_{4\,\text{last}}$, Lemma 1 yields $\boldsymbol{\zeta} = (e^{-2\rho}, e^{-\rho}, 1)$. The first two coordinates are subcritical, while the last one is critical. We then apply Theorem 6. Since the Gaussian integral is univariate in that case, we compute it explicitly.

THEOREM 7. *Under the Mallows culture $\mathcal{M}_{4\,\text{last}}$, the probability that candidate 4 is the Condorcet winner has asymptotic behavior*

$$\mathbb{P}(4 \text{ is CW}) \underset{n\to+\infty}{\sim} \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{S}} \left((1-\zeta_j)\zeta_j^{\lceil n/2\rceil - 1}\right)} \frac{1}{4\pi n \sqrt{\det(\mathcal{H}_K(\boldsymbol{\tau}))M_{33}}},$$

where M_{33} is the bottom right coefficient of $\mathcal{H}_K(\boldsymbol{\tau})^{-1}$.

In this expression, computing $P(\boldsymbol{\zeta})$ by hand remains relatively straightforward and yields the elegant result: $P(\boldsymbol{\zeta}) = 8\gamma e^{-3\rho}(1+e^{-\rho})\left(\frac{1}{2}+e^{-\rho}\right)^2$. However, since the expressions of $\mathcal{H}_K(\boldsymbol{\tau})$ and its inverse are large, we omit them in this paper. More generally, as m increases, the computation of $P(\boldsymbol{\zeta})$ and $\mathcal{H}_K(\boldsymbol{\tau})$ is best left to the computer. The package `Actinvoting` (Durand 2025) provides symbolic algorithms to efficiently compute these quantities.

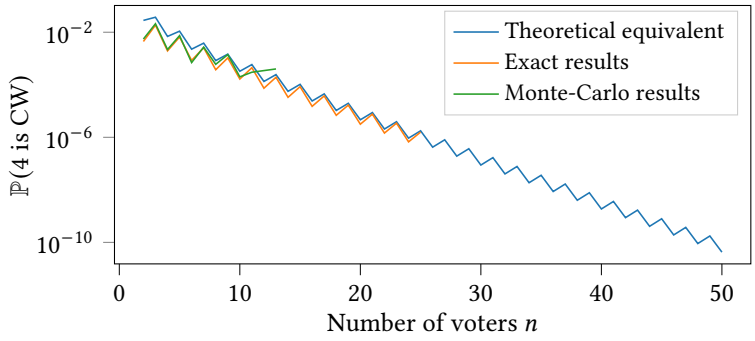


Fig. 5. Probability that candidate 4 is the Condorcet winner as a function of n in a culture $\mathcal{M}_{4\,\text{last}}$ with $\rho = \log(2)$, shown on a semilog scale. The theoretical equivalent is based on Theorem 7, while exact results rely on Equation (2). Monte Carlo simulations use 10,000 profiles per point, yielding an error of order 10^{-2}. For $n \geq 14$, they return zero, which is not visible due to the logarithmic scale. Exact results are not computed for $n \geq 26$ due to prohibitive runtime (e.g., 7 minutes for $n = 25$).

Numerical simulations. Figure 5 illustrates our results for $\mathcal{M}_{4\,\text{last}}$. The exact computation requires eliciting the polynomial $P(\boldsymbol{x})$, which has 2^{m-1} terms corresponding to all possible subsets of adversaries. This is followed by the algebraic expansion of $P(\boldsymbol{x})^n$, resulting in a computational cost of $\Omega(n^{m-1})$. For $m = 4$, the runtime quickly becomes prohibitive: for instance, for $n = 25$, the exact computation takes over 7 minutes. Therefore, we did not perform this calculation for larger values of n. In contrast, Monte Carlo simulations remain manageable, taking 24 seconds for $n = 25$. However, as usual, their limitation lies in the precision of the calculation: in particular, when the exact probabilities are small, Monte Carlo simulations often return an empirical probability of zero. Finally, the theoretical approximation fits the exact curve well, while remaining computationally inexpensive: it takes only 437 μs for $n = 25$. This illustrates that our asymptotic approach is not merely a means to accelerate computations, but a way to make them practically feasible in regimes where exact methods become intractable.

7 Dealing With Supercriticality

To compute $\mathbb{P}(m \succ_{\boldsymbol{\alpha}} \mathcal{A})$, our general approach relies on calculating the characteristic polynomial $P(\boldsymbol{x})$ and using the saddle point method. In Sections 4 to 6, the singularity at 1 arising from the terms $\frac{1}{1-x_j}$ in Equation (5) was not an issue, as each coordinate ζ_j of the saddle point was either subcritical or critical. Specifically, when ζ_j is subcritical, the corresponding integration contour is simply a circle that passes through ζ_j, and hence does not encircle 1. When ζ_j is critical, the integration contour for that coordinate can be chosen to slightly bypass the singularity at 1.

We now consider the case where at least one coordinate ζ_j is supercritical, *i.e.*, $\zeta_j > 1$. In this scenario, the integration contour, which must pass sufficiently close to ζ_j, inevitably encloses the singularity at 1. In general, the contribution from this singularity is non-negligible and can be determined via a residue calculation. However, a purely analytic treatment would obscure the underlying combinatorial meaning carried by the series. We therefore present an equivalent, more transparent interpretation based on the complement rule.

7.1 Theoretical Result for Supercriticality

Assume that the coordinate ζ_j is supercritical. The complement rule gives

$$\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{A}\big) = \mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{A} \setminus \{j\}\big) - \mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{A} \setminus \{j\} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} j\big). \tag{9}$$

In the first term on the right-hand side, the event $\{m \succ_{\boldsymbol{\alpha}} \mathcal{A} \setminus \{j\}\}$ means that m is an $\boldsymbol{\alpha}$-winner after removing adversary j, reducing the competition to $m-2$ adversaries. As we will elaborate, the associated culture arises by setting $x_j = 1$ in $P(\boldsymbol{x})$, allowing the application of the saddle point method in a lower-dimensional space.

The second term is the probability that m defeats all candidates in $\mathcal{A} \setminus \{j\}$ but not j in the sense of $\boldsymbol{\alpha}$. As we will see, this probability is expressed by using a modified characteristic polynomial and a new saddle point derived from $\boldsymbol{\zeta}$, which is modified solely by inverting its j-th coordinate, thereby making it subcritical.

At this stage, the saddle point of the new characteristic polynomial may still have supercritical coordinates. Thus, the process might need to be iterated, using the following generalization of Equation (9). Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets of adversaries, and let $j \in \mathcal{X}$. Then,

$$\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big) = \mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \setminus \{j\} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big) - \mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \setminus \{j\} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y} \cup \{j\}\big). \tag{10}$$

Once again, the first term reduces the dimensionality of the problem, while the second term enables the inversion of a saddle point coordinate, which will be used in practice to transform a supercritical coordinate into a subcritical one. By iterating this process, we systematically reduce the problem to terms of the form $\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big)$, for well-chosen sets $\mathcal{X}$ and $\mathcal{Y}$.

To compute such terms, we define a new characteristic polynomial $P^{\mathcal{X}}_{\mathcal{Y}}(\boldsymbol{x}_{\mathcal{X}}, \boldsymbol{y}_{\mathcal{Y}})$, with variables $\boldsymbol{x}_{\mathcal{X}} = (x_j)_{j\in\mathcal{X}}$ and $\boldsymbol{y}_{\mathcal{Y}} = (y_j)_{j\in\mathcal{Y}}$, derived from the original polynomial $P(\boldsymbol{x})$ through the following transformations. For each adversary $j \in \mathcal{X}$, the formal variable x_j keeps its interpretation and does not lead to a modification in the polynomial. For each adversary $j \in \mathcal{Y}$, instead of using x_j, which encodes when j is preferred to m, we introduce a new formal variable y_j that encodes when m is preferred to j. This variable will appear in all monomials that do not contain x_j in $P(\boldsymbol{x})$. Finally, for each adversary $j \in \mathcal{A} \setminus (\mathcal{X} \cup \mathcal{Y})$, we disregard the outcome of the pairwise comparison, hence we eliminate the corresponding variable by setting $x_j = 1$. This corresponds to the algebraic operation:

$$P^{\mathcal{X}}_{\mathcal{Y}}(\boldsymbol{x}_{\mathcal{X}}, \boldsymbol{y}_{\mathcal{Y}}) = \Big(\prod_{j\in\mathcal{Y}} y_j\Big) P\Big(\boldsymbol{x}_{\mathcal{X}}, \frac{\mathbf{1}}{\boldsymbol{y}_{\mathcal{Y}}}, \mathbf{1}_{\mathcal{A}\setminus(\mathcal{X}\cup\mathcal{Y})}\Big),$$

where $\frac{\mathbf{1}}{\boldsymbol{y}_{\mathcal{Y}}} := ({y_j}^{-1})_{j\in\mathcal{Y}}$. As an example, consider $m = 3$, $\mathcal{X} = \{1\}$, and $\mathcal{Y} = \{2\}$. From the expression of P in Equation (1), we deduce:

$$P^{\{1\}}_{\{2\}}(x_1, y_2) = p_{\emptyset} y_2 + p_{\{1\}} x_1 y_2 + p_{\{2\}} + p_{\{1,2\}} x_1.$$

Note that $P^{\mathcal{X}}_{\mathcal{Y}}\big(\boldsymbol{x}_{\mathcal{X}}, \boldsymbol{y}_{\mathcal{Y}}\big)$ generalizes $P(\boldsymbol{x})$, since $P(\boldsymbol{x}) = P^{\mathcal{A}}_{\emptyset}\big(\boldsymbol{x}_{\mathcal{A}}, \boldsymbol{y}_{\emptyset}\big)$. As in Equation (2), the probability of interest is expressed as a coefficient extraction:

$$\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big) = \Big[\prod_{j\in\mathcal{X}} {x_j}^{<\beta_j n} \prod_{j\in\mathcal{Y}} {y_j}^{\leq\alpha_j n}\Big] \Big(P^{\mathcal{X}}_{\mathcal{Y}}\big(\boldsymbol{x}_{\mathcal{X}}, \boldsymbol{y}_{\mathcal{Y}}\big)\Big)^n.$$

Given two disjoint sets of adversaries $\mathcal{X}$ and $\mathcal{Y}$, Definition 2 generalizes to the cumulant generating function $\tilde{K}(\boldsymbol{t}_{\mathcal{X}\cup\mathcal{Y}}) = \log\big(P^{\mathcal{X}}_{\mathcal{Y}}(\exp(\boldsymbol{t}_{\mathcal{X}\cup\mathcal{Y}}))\big)$, the log saddle point $\tilde{\boldsymbol{\tau}} = \arg\max_{\boldsymbol{t}_{\mathcal{X}\cup\mathcal{Y}}\in\mathbb{R}^{|\mathcal{X}\cup\mathcal{Y}|}} \big(-\tilde{K}(\boldsymbol{t}_{\mathcal{X}\cup\mathcal{Y}}) + (\boldsymbol{\beta}_{\mathcal{X}}, \boldsymbol{\alpha}_{\mathcal{Y}})^{\intercal}\boldsymbol{t}_{\mathcal{X}\cup\mathcal{Y}}\big)$, and the saddle point $\tilde{\boldsymbol{\zeta}} = e^{\tilde{\boldsymbol{\tau}}}$.

As previously mentioned, transferring an adversary from $\mathcal{X}$ to $\mathcal{Y}$ results in the inversion of the corresponding coordinate in the saddle point. The following lemma formalizes this process and is proved in Appendix B.3.

Lemma 2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two disjoint sets of adversaries. Let $\tilde{\boldsymbol{\zeta}}$ be the saddle point associated with $\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big)$. Let $j \in \mathcal{X}, \mathcal{X}' = \mathcal{X}\setminus\{j\}$, and $\mathcal{Y}' = \mathcal{Y}\cup\{j\}$. Then the saddle point associated with $\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X}' \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}'\big)$ has its k-th coordinate equal to $\tilde{\zeta}_k$ if $k \neq j$ and $\frac{1}{\tilde{\zeta}_j}$ if $k = j$.*

This lemma, combined with the iterated application of Equation (10), allows expressing the probability $\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner})$ as a sum of terms of the form $\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\big)$, where all the coordinates of all

saddle points are either subcritical or critical. The following theorem provides the asymptotic behavior of such terms. It directly follows from Theorem 14 in Appendix A.3.

Theorem 8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets of adversaries. Let $\tilde{P}(\boldsymbol{x}_\mathcal{X}, \boldsymbol{y}_\mathcal{Y}) := P_\mathcal{Y}^\mathcal{X}(\boldsymbol{x}_\mathcal{X}, \boldsymbol{y}_\mathcal{Y})$ and $\tilde{K}$ its cumulant generating function. Let $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\zeta}}$ respectively be the log saddle point and saddle point associated with $\mathbb{P}(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y})$. We assume that (1) $\mathcal{X} = \mathcal{S} \cup C$, where $\tilde{\zeta}_j < 1,\ \forall j \in \mathcal{S}$ and $\tilde{\zeta}_j = 1,\ \forall j \in C$, and (2) $\tilde{\zeta}_j < 1,\ \forall j \in \mathcal{Y}$. Then*

$$\mathbb{P}(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}) \underset{n\to+\infty}{\sim} \frac{\tilde{P}(\tilde{\boldsymbol{\zeta}})^n}{\prod_{j\in\mathcal{S}} \tilde{\zeta}_j^{\lceil \beta_j n\rceil - 1} \prod_{j\in\mathcal{Y}} \tilde{\zeta}_j^{\lfloor \alpha_j n\rfloor}} \frac{1}{\prod_{j\in\mathcal{S}\cup\mathcal{Y}} (1-\tilde{\zeta}_j)} \frac{1}{\sqrt{(2\pi)^{|\mathcal{X}\cup\mathcal{Y}|} n^{|\mathcal{S}\cup\mathcal{Y}|} \det\left(\mathcal{H}_{\tilde{K}}(\tilde{\boldsymbol{\tau}})\right)}} \int_{(0,+\infty)^{|C|}} e^{-\boldsymbol{u}^\intercal \tilde{M} \boldsymbol{u}/2} d\boldsymbol{u},$$

where $\tilde{M}$ is the submatrix of $\mathcal{H}_{\tilde{K}}(\tilde{\boldsymbol{\tau}})^{-1}$ that corresponds to the rows and columns of C.

In summary, to handle supercriticality, the process consists in applying Equation (10) to one of the supercritical coordinates of the original saddle point, and then iterating this step on each term whose saddle point still contains supercritical coordinates, until every term satisfies the assumptions of Theorem 8. The procedure is guaranteed to terminate, as each iteration either reduces the dimension by one or decreases the number of supercritical coordinates. Once Theorem 8 is used to compute the asymptotics of each term, some may be negligible compared to others, leading to a simplified sum with fewer terms, as illustrated in the following application.

7.2 Application: Mallows Culture $\mathcal{M}_{3\text{ first}}$

Analogously to Lemma 1 for the culture $\mathcal{M}_{m\text{ last}}$, Lemma 3 provides the expression of the saddle point for $\mathcal{M}_{m\text{ first}}$. The proof is similar and is provided in Appendix C.2.

Lemma 3. *Under the culture $\mathcal{M}_{m\text{ first}}$, the log saddle point $\boldsymbol{\tau}$ is given by*

$$\boldsymbol{\tau} = \left(\frac{m\rho}{2}, \frac{(m-2)\rho}{2}, \ldots, \frac{(-m+4)\rho}{2}\right),$$

where ρ is the concentration parameter of the culture, defined in Section 2.1.

In the case of three candidates, Lemma 3 gives $\boldsymbol{\zeta} = \left(e^{3\rho/2}, e^{\rho/2}\right)$, hence both coordinates are supercritical. Consequently, we apply the procedure outlined in Section 7.1. In this specific case, we show in Appendix C.3 that this approach effectively reduces to applying the inclusion-exclusion principle and identifying the dominant term in the sum. Unsurprisingly, the dominant term that may prevent candidate 3 from being the Condorcet winner corresponds to the event of not winning the pairwise comparison against the strongest adversary, candidate 2.

Theorem 9. *Under $\mathcal{M}_{3\text{ first}}$, the probability that candidate 3 is the Condorcet winner has asymptotic behavior*

$$1 - \mathbb{P}(3 \text{ is CW}) \underset{n\to+\infty}{\sim} \sqrt{\frac{2}{\pi n}} \frac{e^{-\rho\lceil n/2\rceil}}{1-e^{-\rho}} \left(\frac{2}{1+e^{-\rho}}\right)^n.$$

Numerical simulations. Figure 6 illustrates these results. Since the probability that candidate 3 is the Condorcet winner tends towards 1, we now represent the complement probability on a semi-logarithmic scale to visualize the difference with the limit. The computation times are similar to those in Figure 3 as they involve the same culture and parameters. As in the previous cases, Monte Carlo simulations become unreliable when probabilities are small. We observe good agreement between the theoretical approximation, the exact results, and the Monte Carlo estimate when the latter is relevant.

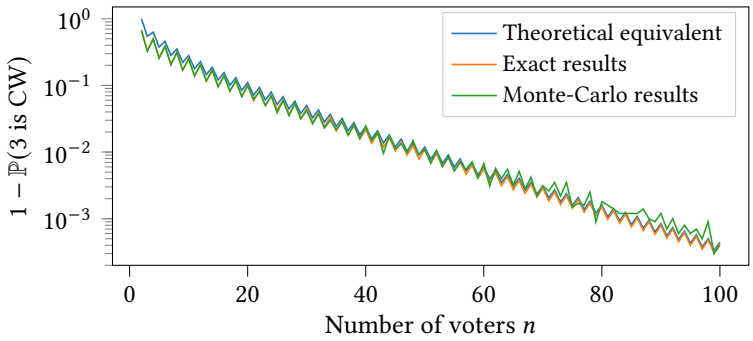


Fig. 6. Probability that candidate 3 fails to be the Condorcet winner as a function of n in a culture $\mathcal{M}_{3\,\text{first}}$ with $\rho = \log(2)$, shown on a semilog scale. The theoretical equivalent is based on Theorem 9, while exact results rely on Equation (2). Monte Carlo simulations use 10,000 profiles per point, yielding an error of order 10^{-2}. When the exact results curve is not visible, it is overlapped by the Monte Carlo results curve.

8 Asymptotic Expansion

We now explain how to improve our asymptotic results. Both Theorem 1 for the subcritical case and Theorem 3 for the critical case are encompassed by Theorem 6, which addresses the *mixed case*. As detailed in Section 7.1, the supercritical case also reduces to the mixed case. Therefore, we focus on this scenario, involving a set $\mathcal{S}$ of subcritical coordinates and a set C of critical coordinates. Essentially, Theorem 6 provides a value $a_{0,m}$, constant with respect to n, such that

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) \underset{n\to+\infty}{\sim} \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{S}} \zeta_j^{\lceil \beta_j n\rceil - 1}} \frac{1}{n^{|\mathcal{S}|/2}} a_{0,m}.$$

Theorem 15 from Appendix A.4 extends this result, establishing the existence of coefficients $(a_{k,m,n})_{k,n\geq 1}$ such that, for any $r \geq 1$, we have

$$\mathbb{P}(m \text{ is } \boldsymbol{\alpha}\text{-winner}) = \frac{P(\boldsymbol{\zeta})^n}{\prod_{j\in\mathcal{S}} \zeta_j^{\lceil \beta_j n\rceil - 1}} \frac{1}{n^{|\mathcal{S}|/2}} \left(a_{0,m} + a_{1,m,n} n^{-1/2} + \cdots + a_{r-1,m,n} n^{-(r-1)/2} + O(n^{-r/2})\right).$$

In general, the coefficients $a_{k,m,n}$ may depend on n, but only through the bounded vector $\lceil \boldsymbol{\beta} n\rceil - \mathbf{1} - \boldsymbol{\beta} n$. For example, in the case of the Condorcet winner, they depend only on the parity of n.

Such an approximation is called an *asymptotic expansion*. In the subcritical case, it is well known (Flajolet and Sedgewick 2009, Section VIII.3), (Pemantle et al. 2024, Section 5.5) and proven in Theorem 15, that $a_{k,m,n} = 0$ for odd k, leading to an expansion in powers of n^{-1} instead of $n^{-1/2}$. However, this does not hold in general, as the following result illustrates.

For the Impartial Culture, Theorem 4 gives the limiting probability but not the convergence rate, which we now address by computing $a_{1,m,n}$.

Theorem 10. *Under Impartial Culture with m candidates, the probability that candidate m is the Condorcet winner has asymptotic behavior*

$$\mathbb{P}(m \text{ is CW}) = a_{0,m} + a_{1,m,n} n^{-1/2} + O(n^{-1}),$$

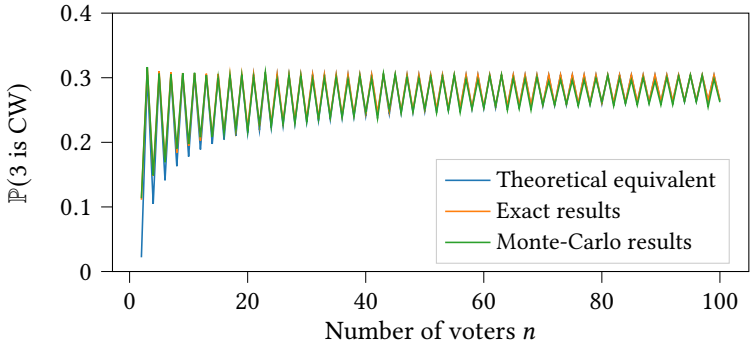


Fig. 7. Probability that a given candidate is the Condorcet winner as a function of n in the Impartial Culture with three candidates. The theoretical equivalent is based on Theorem 10, while exact results rely on Equation (2). Monte Carlo simulations use 10,000 profiles per point, yielding an error of order 10^{-2}. Not all curves being visible means that they overlap.

where $a_{0,m}$ *and* $a_{1,m,n}$ *are given by*

$$a_{0,m} = \frac{1}{\sqrt{(2\pi)^{m-1}\det(\mathcal{H}_K(\mathbf{0}))}} \int_{(0,+\infty)^{m-1}} e^{-\boldsymbol{u}^\intercal \mathcal{H}_K(\mathbf{0})^{-1}\boldsymbol{u}/2} d\boldsymbol{u},$$

$$a_{1,m,n} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -\frac{6}{m+1}\frac{1}{\sqrt{(2\pi)^{m-1}\det(\mathcal{H}_K(\mathbf{0}))}} \int_{(0,+\infty)^{m-1}} \mathbf{1}^\intercal \boldsymbol{u} e^{-\boldsymbol{u}^\intercal \mathcal{H}_K(\mathbf{0})^{-1}\boldsymbol{u}/2} d\boldsymbol{u}. & \text{if } n \text{ is even,} \end{cases}$$

and $\mathcal{H}_K(\mathbf{0})$*, its inverse and determinant are given in Equation (7).*

Alternatively, this result could be proven using multivariate Edgeworth expansions (see *e.g.*, Skovgaard (1986)). The coefficient $a_{0,m}$ tends to 0 when $m \to +\infty$ and its asymptotic behavior has been investigated by May (1971) and Pittel (2025).

Numerical simulations. In Figure 4, we observed that the approximation by the limit value was highly accurate for odd values of n. Theorem 10 provides an explanation for this: the term of the asymptotic expansion in $n^{-1/2}$ vanishes in this case. In contrast, for even n, the term is non-zero and negative, which is consistent with the previous observation that the exact values are significantly lower than the limit in these cases. To illustrate this, Figure 7 shows the same curves as Figure 4 for the Monte Carlo simulations and exact values. However, this time, the theoretical approximation includes the term in $n^{-1/2}$. As before, this approximation is computationally inexpensive: for instance, it takes only 55 μs for $n = 100$. Now, the three curves become nearly indistinguishable visually: the addition of this term in $n^{-1/2}$ significantly improves the accuracy of the approximation, the error term being $O(n^{-1})$.

9 Conclusion

Summary of the Contributions. In this paper, we present a method for calculating the probability that a candidate is an $\boldsymbol{\alpha}$-winner under the GIC model, assuming each ranking has a strictly positive probability. Our approach, based on analytic combinatorics, first computes the characteristic polynomial, the cumulant generating function, and the saddle point. The asymptotic analysis then splits into cases based on whether each coordinate of the saddle point is subcritical, critical, or supercritical. In all cases, we derive the limiting probability and the rate of convergence. Additionally, our method computes higher-order terms in the asymptotic expansion. We apply

this approach to the Impartial Culture and Mallows models, providing explicit formulas supported by numerical simulations.

Future Work. A natural direction for future work is to investigate cases where the saddle point has zero or infinite coordinates, which may arise in non-generic cultures. Additionally, it would be valuable to perform a finer analysis of the complexity of the algorithm presented in Section 7 for handling supercritical coordinates, particularly to identify conditions under which it runs in polynomial time. Another avenue is to examine the limiting behavior of the probability as the number of candidates m tends to infinity, along the lines of Pittel (2025). Finally, it would be insightful to extend our method to other events, such as the transitivity of the majority relation, different kinds of monotonicity failures, or the manipulability, *i.e.,* the susceptibility to strategic voting.

Acknowledgments

The authors would like to thank Stephen Melczer for helping us navigate the literature of analytic combinatorics in several variables. We also thank the reviewers and the editor for valuable comments that helped improve the paper.

This work was supported by the RandNET project, MSCA-RISE - Marie Skłodowska-Curie Research and Innovation Staff Exchange Programme (RISE), Grant agreement 101007705.

References

H. Azari, D. Parks, and L. Xia. 2012. "Random utility theory for social choice." *Advances in Neural Information Processing Systems*, 25.

A. I. Barvinok. 1994. "A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed." *Mathematics of Operations Research*, 19, 4, 769–779.

C. Bell. 1981. "A random voting graph almost surely has a Hamiltonian cycle when the number of alternatives is large." *Econometrica*, 1597–1603.

C. Bell. 1978. "What happens when majority rule breaks down?" *Public Choice*, 33, 2, 121–126.

S. Berg and D. Lepelley. 1992. "Note sur le calcul de la probabilité des paradoxes du vote." *Mathématiques et Sciences humaines*, 120, 33–48.

F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia. 2016. *Handbook of computational social choice*. Cambridge University Press.

M. Condorcet. 1785. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Imprimerie royale.

B. De Finetti. 1929. "Funzione caratteristica di un fenomeno aleatorio." In: *Atti del Congresso Internazionale dei Matematici: Bologna del 3 al 10 de settembre di 1928*, 179–190.

A. Dembo and O. Zeitouni. 2009. *Large deviations techniques and applications*. Vol. 38. Springer Science & Business Media.

J.-P. Doignon, A. Pekeč, and M. Regenwetter. 2004. "The repeated insertion model for rankings: Missing link between two subset choice models." *Psychometrika*, 69, 1, 33–54.

P. Duchon, P. Flajolet, G. Louchard, and G. Schaeffer. 2004. "Boltzmann Samplers For The Random Generation Of Combinatorial Structures." *Combinatorics, Probability and Computing*, 13, 2004.

F. Durand. 2025. *Actinvoting: Analytic Combinatorics Tools in Voting*. https://github.com/francois-durand/actinvoting. (2025).

P. Fishburn. 1973. "A proof of May's Theorem P(m, 4) = 2 P(m, 3)." *Behavioral Science*, 18, 3, 212.

P. Flajolet, B. Salvy, and P. Zimmermann. 1991. "Automatic average-case analysis of algorithms." *Theoretical Computer Science*, 79, 1, 37–109.

P. Flajolet and R. Sedgewick. 2009. *Analytic combinatorics*. Cambridge University Press.

M. Garman and M. Kamien. 1968. "The paradox of voting: Probability calculations." *Behavioral Science*, 13, 306–316.

W. Gehrlein. 2006. *Condorcet's Paradox*. Springer.

W. Gehrlein. 1983. "Condorcet's paradox." *Theory and Decision*, 15, 2, 161–197.

W. Gehrlein. 2002. "Obtaining representations for probabilities of voting outcomes with effectively unlimited precision integer arithmetic." *Social Choice and Welfare*, 19, 503–512, 3.

W. Gehrlein and P. Fishburn. 1976a. "Condorcet's paradox and anonymous preference profiles." *Public Choice*, 1–18.

W. Gehrlein and P. Fishburn. 1978. "Probabilities of election outcomes for large electorates." *Journal of Economic Theory*, 19, 1, 38–49.

W. Gehrlein and P. Fishburn. 1976b. "The probability of the paradox of voting: A computable solution." *Journal of Economic Theory*, 13, 1, 14–25.

R. Gillett. 1978. "A recursion relation for the probability of the paradox of voting." *Journal of Economic Theory*, 18, 2, 318–327.

G.-T. Guilbaud. 1952. "Les théories de l'intérêt général et le problème logique de l'agrégation." *Économie appliquée*, 5, 501–584.

E. Hewitt and L. J. Savage. 1955. "Symmetric measures on Cartesian products." *Transactions of the American Mathematical Society*, 80, 2, 470–501.
H. Huang and V. C. Chua. 2000. "Analytical representation of probabilities under the IAC condition." *Social Choice and Welfare*, 17, 1, 143–155.
J. Kelly. 1974. "Voting anomalies, the number of voters, and the number of alternatives." *Econometrica*, 239–251.
M. G. Kendall. 1938. "A new measure of rank correlation." *Biometrika*, 30, 1-2, 81–93.
S. G. Krantz. 2001. *Function theory of several complex variables*. Vol. 340. American Mathematical Soc.
M. Krishnamoorthy and M. Raghavachari. 2005. *Condorcet Winner Probabilities - A Statistical Perspective*. (2005).
D. Lepelley. 1989. "Contribution a l'analyse des procedures de decision collective." Ph.D. Dissertation. Caen.
D. Lepelley, A. Louichi, and H. Smaoui. 2008. "On Ehrhart polynomials and probability calculations in voting theory." *Social Choice and Welfare*, 30, 3, 363–383.
T. Lu and C. Boutilier. 2014. "Effective sampling and learning for mallows models with pairwise-preference data." *J. Mach. Learn. Res.*, 15, 1, 3783–3829.
R. D. Luce. 1959. *Individual choice behavior: A theoretical analysis*. Wiley.
H. Maassen and T. Bezembinder. 2002. "Generating random weak orders and the probability of a Condorcet winner." *Social Choice and Welfare*, 19, 3, 517–532.
C. L. Mallows. 1957. "Non-null ranking models. I." *Biometrika*, 44, 1/2, 114–130.
R. May. 1971. "Some mathematical remarks on the paradox of voting." *Behavioral Science*, 16, 2, 143–151.
S. Melczer. 2021. *An Invitation to Analytic Combinatorics*. Springer.
R. Niemi and H. Weisberg. 1968. "A mathematical solution for the probability of the paradox of voting." *Behavioral Science*, 13, 4, 317–323.
R. Pemantle, M. C. Wilson, and S. Melczer. 2024. *Analytic combinatorics in several variables, 2nd Edition*. Vol. 212. Cambridge University Press.
B. Pittel. 2025. "On likelihood of a Condorcet winner for uniformly random and independent voter preferences." *arXiv preprint arXiv:2506.20613*.
R. Plackett. 1975. "The analysis of permutations." *Journal of the Royal Statistical Society Series C: Applied Statistics*, 24, 2, 193–202.
H. Ruben. June 1954. "On the Moments of Order Statistics in Samples from Normal Populations." *Biometrika*, 41, 1/2, (June 1954), 200–227.
L. Sauermann. 2022. "On the probability of a Condorcet winner among a large number of alternatives." *arXiv preprint arXiv:2203.13713*.
M. R. Sertel and M. R. Sanver. 2004. "Strong equilibrium outcomes of voting games are the generalized Condorcet winners." *Social Choice and Welfare*, 22, 2, 331–347.
I. M. Skovgaard. 1986. "On multivariate Edgeworth expansions." *International Statistical Review/Revue Internationale de Statistique*, 169–186.
L. Thurstone. 1927. "A law of comparative judgment." *Psychological review*, 34.
O. Williamson and T. Sargent. 1967. "Social choice: A probabilistic approach." *The Economic Journal*, 77, 308, 797–813.
M. C. Wilson and G. Pritchard. 2007. "Probability calculations under the IAC hypothesis." *Mathematical Social Sciences*, 54, 3, 244–256.

A Multivariate Coefficient Extraction of Large Powers

This Appendix presents the main technical contributions of the paper in analytic combinatorics. We assume here that the reader is familiar with the main concepts of this field. For an introduction, less familiar readers may refer to Flajolet and Sedgewick (2009) or Sections 2 and 3 of this paper.

The *large power theorem* (Flajolet and Sedgewick 2009, Section VIII.8) provides sufficient conditions to derive the asymptotics of coefficients extraction of the form

$$[z^{\lambda n}]A(z)B(z)^n$$

as n tends to infinity. Our goal here is to extend this result to the multivariate setting

$$[\boldsymbol{z}^{\boldsymbol{\lambda} n}]A(\boldsymbol{z})B(\boldsymbol{z})^n.$$

We also cover the variant where $A(\boldsymbol{z})$ has a singularity at the *saddle point* (defined below), and derive *asymptotic expansions*, improving the precision of our results.

To this end, we rely on the following multivariate version of the Laplace method, presented by Pemantle et al. (2024, Lemma 5.16).

Theorem 11 (Multivariate Laplace method). *Consider a compact neighborhood $C \subset \mathbb{R}^d$ of $\mathbf{0}$ and two functions $A(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ analytic on C. Suppose that the real part of $\phi(\boldsymbol{x})$ is strictly positive except at the origin, that its*

gradient vanishes at the origin and that its Hessian matrix $\mathcal{H}$ is non-singular there. Assume $A(\mathbf{0}) \neq 0$, then

$$\int_C A(\boldsymbol{x}) e^{-n\phi(\boldsymbol{x})} d\boldsymbol{x} \sim \left(\frac{2\pi}{n}\right)^{d/2} \frac{A(\mathbf{0})}{\sqrt{\det(\mathcal{H})}}$$

and the choice of sign of the square root is defined by taking the product of the principal square roots of the eigenvalues of $\mathcal{H}$.

Variants of this theorem exist for the case where $A(\mathbf{0}) = 0$ and where the Hessian of $\phi(\boldsymbol{x})$ is singular at $\mathbf{0}$.

If the assumptions of the theorem are satisfied, except $\phi(\mathbf{0}) \neq 0$, then we rewrite

$$\int_C A(\boldsymbol{x}) e^{-n\phi(\boldsymbol{x})} d\boldsymbol{x} = e^{-n\phi(\mathbf{0})} \int_C A(\boldsymbol{x}) e^{-n(\phi(\boldsymbol{x})-\phi(\mathbf{0}))} d\boldsymbol{x}$$

and the theorem is applicable with $\phi(\boldsymbol{x})$ replaced by $\phi(\boldsymbol{x}) - \phi(\mathbf{0})$.

If the point where the gradient of $\phi(\boldsymbol{x})$ vanishes is some other point than $\mathbf{0}$, as long as the Hessian of ϕ is non-singular at this point and A does not vanish there, the theorem is applicable after a translation sending this point to $\mathbf{0}$.

Road map and contributions. Our analytical results are developed at a more general level than required by our application. This abstraction both clarifies the methodological framework and enables future extensions to other settings.

Our first contribution is to formulate general sufficient conditions to reduce the multivariate coefficient extraction of large powers to a Laplace integral. This extends to the multivariate setting results known on the univariate case (see (Flajolet and Sedgewick 2009, Section VIII.8)). We first establish a few lemmas in Appendix A.1 ensuring that the assumptions of the Laplace method are satisfied, then derive the multivariate large powers theorem in Appendix A.2. It is used in this article for the subcritical case from Section 4. Our second contribution is to consider in Appendix A.3 the case where the saddle point meets a singularity, which is applied both in Section 5 and Section 6. Finally, Appendix A.4 presents our third contribution which is the computation of an asymptotic expansion improving the precision of our results.

Notation. We use bold letters to denote vectors, *e.g.*, $\boldsymbol{x} = (x_1, \dots, x_d)$ with $d \in \mathbb{N}_{>0}$. Given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same dimension d and a scalar z, we define the notations

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{y}} &:= {x_1}^{y_1} \cdots {x_d}^{y_d}, \\ \boldsymbol{x}z^{\boldsymbol{y}} &:= (x_1 z^{y_1}, \dots, x_d z^{y_d}), \\ \log(\boldsymbol{x}) &:= (\log(x_1), \dots, \log(x_d)). \end{aligned}$$

In particular, we write $\boldsymbol{\zeta} e^{i\boldsymbol{\theta}}$ for the vector $(\zeta_1 e^{i\theta_1}, \dots, \zeta_d e^{i\theta_d})$. The *support* of a multivariate power series $B(\boldsymbol{z})$ is the set

$$S_B := \{\boldsymbol{n},\ [\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z}) \neq 0\}$$

and we define

$$\Delta(S_B) := \{\boldsymbol{n} - \boldsymbol{m},\ \boldsymbol{n}, \boldsymbol{m} \in S_B\}.$$

The *cumulant generating function* associated to the generating function $B(\boldsymbol{z})$ is

$$K(\boldsymbol{t}) := \log\left(B(e^{\boldsymbol{t}})\right) = \log\left(B(e^{t_1}, \dots, e^{t_d})\right).$$

The gradient of a function B at $\boldsymbol{x}$ is denoted by $\nabla_B(\boldsymbol{x})$.

We write $r + q\mathbb{Z}_{\geq 0}$ for the set $\{r + q\ell,\ \ell \in \mathbb{Z}_{\geq 0}\}$.

A *polydisc* is a Cartesian product of discs. A *torus* is a Cartesian product of circles.

A.1 Rank and Periodicity of a Multivariate Generating Function

This section presents general results on multivariate formal power series, defining their *rank* and *period*. We will use them to prove the existence and uniqueness of the saddle point, defined in the following section. In particular, we prove a multivariate version of the Daffodil lemma (Flajolet and Sedgewick 2009, Lemma IV.1, page 266). For a broader treatment of this topic, we refer the reader to (Flajolet and Sedgewick 2009) for the univariate case and to (Melczer 2021) for the multivariate case.

Lemma 4. *The kernel of any integer matrix has a basis composed of integer vectors.*

Proof. Let M denote our matrix and let its Smith normal form be

$$M = UDV,$$

where U and V are unimodular integer matrices and D is an integer diagonal matrix. The dimension of the kernel of M is equal to the number of zero elements on the diagonal of D. Consider such an element of index j and let $\boldsymbol{e}^{(j)}$, denote the vector with a 1 in position j and 0's everywhere else. Set $\boldsymbol{p}^{(j)} = V^{-1}\boldsymbol{e}^{(j)}$. Since V is unimodular, $\boldsymbol{p}^{(j)}$ has integer coefficients and is nonzero. We have

$$M\boldsymbol{p}^{(j)} = UDV\boldsymbol{p}^{(j)} = UD\boldsymbol{e}^{(j)} = \mathbf{0},$$

so $\boldsymbol{p}^{(j)}$ belongs to the kernel of M. The invertibility of V ensures that the vectors $\boldsymbol{p}^{(j)}$ are linearly independent, hence they form a basis of the kernel. □

Definition 3. *We define the* rank *of a formal power series $B(\boldsymbol{z})$ as the dimension of the real vector space generated by $\Delta(S_B)$. We say that $B(\boldsymbol{z})$ has* full rank *if its rank is equal to its number of variables.*

In particular, a formal power series on at least one variable and that has full rank cannot be a monomial.

Lemma 5. *If the formal power series $B(z_1, \dots, z_d)$ has rank k, then there exist $d-k$ integer vectors $\boldsymbol{p}^{(1)}, \dots, \boldsymbol{p}^{(d-k)}$ generating a real vector space of dimension $d-k$ and integers $r_1, \dots, r_{d-k}$ such that for all $j \in [1, d-k]$*

$$B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}^{(j)}}) = y^{r_j}B(\boldsymbol{z}).$$

Reciprocally, if such vectors $\boldsymbol{p}^{(j)}$ and integers r_j exist, then $B(\boldsymbol{z})$ has rank at most k.

Proof. If $B(z_1, \dots, z_d)$ has rank k, by Lemma 4, the vector space orthogonal to $\Delta(S_B)$ has a basis of $d-k$ integer vectors $(\boldsymbol{p}^{(1)}, \dots, \boldsymbol{p}^{(d-k)})$. Fix $\boldsymbol{m}$ in the support S_B of $B(\boldsymbol{z})$ and define $r_j = \boldsymbol{m}^\intercal\boldsymbol{p}^{(j)}$, then for all $\boldsymbol{n}$ in S_B, we have

$$\boldsymbol{n}^\intercal\boldsymbol{p}^{(j)} = r_j,$$

so

$$B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}^{(j)}}) = y^{r_j}B(\boldsymbol{z}).$$

Reciprocally, if there are $d-k$ linearly independent integer vectors $(\boldsymbol{p}^{(1)}, \dots, \boldsymbol{p}^{(d-k)})$ and integers $r_1, \dots, r_{d-k}$ such that

$$B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}^{(j)}}) = y^{r_j}B(\boldsymbol{z}),$$

then each $\boldsymbol{p}^{(j)}$ is in the vector space orthogonal to $\Delta(S_B)$. This implies that $\Delta(S_B)$ has dimension at most k. □

In the current article, we will consider a polynomial P having full rank. It is however worth noticing that coefficient extraction in a power series that does not have full rank can be simplified as follows. Say we are interested in the coefficient extraction $[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z})$ of a power series $B(\boldsymbol{z})$ that does not have full rank. Let $\boldsymbol{p}$ and r denote a nonzero integer vector and an integer such that

$$B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}}) = y^{r}B(\boldsymbol{z}).$$

Since $\boldsymbol{p}$ is nonzero, there exists j such that $p_j \neq 0$. Then for any $\boldsymbol{n} \in S_B$, we have

$$n_j = \frac{r - \sum_{\ell \neq j} p_\ell n_\ell}{p_j},$$

so n_j is uniquely determined by $(n_\ell)_{\ell \neq j}$ and

$$[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z}) = \begin{cases} 0 & \text{if } \boldsymbol{n}^\intercal \boldsymbol{p} \neq r, \\ [z_1^{n_1} \cdots z_{j-1}^{n_{j-1}} z_{j+1}^{n_{j+1}} \cdots z_d^{n_d}] B(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_d) & \text{otherwise.} \end{cases}$$

Thus, coefficient extraction in a power series that does not have full rank reduces to a coefficient extraction in a power series with fewer variables.

Consider a univariate probability generating function $B(z) = \sum_n b_n z^n$, meaning $b_n \geq 0$ for all n and $B(1) = 1$. Its *cumulant generating function* $K(t)$ is defined as

$$K(t) = \log(B(e^t)).$$

A classic result of probability theory states that this function is convex on its domain of definition (direct application of Hölder's inequality, see *e.g.,* (Dembo and Zeitouni 2009)). Our next lemma is a multivariate variant of this result. The proof is adapted from (Flajolet and Sedgewick 2009, Note IV.46).

Lemma 6 (Convexity of the cumulant generating function). *Consider a series $B(\boldsymbol{z})$ with nonnegative coefficients, full rank, and analytic on a neighborhood Ω of $\mathbf{0}$. Then its cumulant generating function*

$$K : \boldsymbol{t} \mapsto \log\left(B(e^{\boldsymbol{t}})\right)$$

is strictly convex on $\{\boldsymbol{t} \in \mathbb{R}^d,\ e^{\boldsymbol{t}} \in \Omega\}$.

Proof. Let $\boldsymbol{X}$ denote the vector of random variables parametrized by $\boldsymbol{x} \in \mathbb{R}^d_{>0} \cap \Omega$ with distribution

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{n}) = \frac{[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z})\boldsymbol{x}^{\boldsymbol{n}}}{B(\boldsymbol{x})}.$$

By construction, those probabilities sum to 1. This is the so-called *Boltzmann distribution* associated to the generating function $B(\boldsymbol{z})$ (see (Duchon et al. 2004)). We now follow the classical proof of convexity of the cumulant generating function of a multivariate random variable, then prove that the assumption that $B(\boldsymbol{z})$ has full rank implies strict convexity.

The moment generating function of $\boldsymbol{X}$ is

$$\mathbb{E}(e^{\boldsymbol{t}^\intercal X}) = \frac{B(\boldsymbol{x}e^{\boldsymbol{t}})}{B(\boldsymbol{x})}.$$

Observe that the cumulant generating function of $\boldsymbol{X}$ is equal to

$$\log\left(\mathbb{E}(e^{\boldsymbol{t}^\intercal X})\right) = \log\left(B(\boldsymbol{x}e^{\boldsymbol{t}})\right) - \log\left(B(\boldsymbol{x})\right) = K\big(\boldsymbol{t} + \log(\boldsymbol{x})\big) - \log\left(B(\boldsymbol{x})\right).$$

The vector of the means is

$$\mathbb{E}(\boldsymbol{X}) = \frac{\operatorname{diag}(\boldsymbol{x})\nabla_B(\boldsymbol{x})}{B(\boldsymbol{x})}.$$

It is equal to the gradient of $K(\boldsymbol{t})$ at $\boldsymbol{t} = \log(\boldsymbol{x})$. The covariance matrix is

$$\operatorname{Var}(\boldsymbol{X}) = \mathbb{E}\big((\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\intercal\big) = \mathbb{E}\big(\boldsymbol{X}\boldsymbol{X}^\intercal\big) - \mathbb{E}(\boldsymbol{X})\mathbb{E}(\boldsymbol{X})^\intercal.$$

It is equal to the Hessian of $K(\boldsymbol{t})$ at $\boldsymbol{t} = \log(\boldsymbol{x})$.

For any vector $\boldsymbol{u} \in \mathbb{R}^d$,

$$\boldsymbol{u}^\intercal \operatorname{Var}(\boldsymbol{X})\boldsymbol{u} = \mathbb{E}\Big(\big((\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\intercal \boldsymbol{u}\big)^2\Big),$$

so $\mathrm{Var}(\boldsymbol{X})$ is positive semi-definite. Consider a vector $\boldsymbol{u}$ such that

$$0 = \boldsymbol{u}^\intercal \mathrm{Var}(\boldsymbol{X})\boldsymbol{u} = \mathbb{E}\Big(\big((\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\intercal \boldsymbol{u}\big)^2\Big).$$

This implies $(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\intercal \boldsymbol{u} = \boldsymbol{0}$ so, for any $\boldsymbol{n}, \boldsymbol{m}$ in the support of $B(\boldsymbol{z})$,

$$(\boldsymbol{n} - \boldsymbol{m})^\intercal \boldsymbol{u} = 0.$$

Our assumption that $B(\boldsymbol{z})$ has full rank implies $\boldsymbol{u} = \boldsymbol{0}$, which proves that $\mathrm{Var}(\boldsymbol{X})$ is positive definite. Since $\mathrm{Var}(\boldsymbol{X})$ is equal to the Hessian of $K(\boldsymbol{t})$ at $\log(\boldsymbol{x})$, we deduce that $K(\boldsymbol{t})$ is strictly convex on $\{\boldsymbol{t} \in \mathbb{R}^d,\ e^{\boldsymbol{t}} \in \Omega\}$. □

We recall below the definition of q-periodicity from Flajolet, Salvy, et al. (1991, Section 4.2) for univariate formal power series.

Definition 4. *A univariate generating function $B(z)$ is said to be* q-periodic *if there exist an integer r and a formal power series $C(z)$ such that*

$$B(z) = z^r C(z^q).$$

Equivalently, the support S_B of $B(z)$ satisfies

$$S_B \subseteq r + q\mathbb{Z}_{\geq 0}.$$

The series $B(z)$ is said to be aperiodic *if it is not q-periodic for any $q \geq 2$.*

Let us now extend this definition to multivariate formal power series.

Definition 5. *A multivariate generating function $B(z_1, \ldots, z_d)$ is said to be* q-periodic *if there exist nonnegative integers $p_1, \ldots, p_d$, not all divisible by q, such that $B(y^{p_1}, \ldots, y^{p_d})$ is a q-periodic univariate formal power series. An equivalent formulation on the support S_B of $B(\boldsymbol{z})$ is that there exists a nonnegative integer r such that*

$$\{\boldsymbol{p}^\intercal \boldsymbol{n},\ \boldsymbol{n} \in S_B\} \subseteq r + q\mathbb{Z}_{\geq 0}. \tag{11}$$

Yet another equivalent formulation is that there exists a formal power series $E(\boldsymbol{z}, y)$, integers $p_1, \ldots, p_d$ not all divisible by q and an integer r such that

$$B(z_1 y^{p_1}, \ldots, z_d y^{p_d}) = y^r E(\boldsymbol{z}, y^q).$$

We say that $B(\boldsymbol{z})$ is aperiodic *if it is not q-periodic for any $q \geq 2$.*

Note that our definition of aperiodicity is related but distinct from the one from (Melczer 2021, Definition 5.9). Our next result presents a simple sufficient condition for a formal multivariate power series to have full rank and be aperiodic.

Proposition 1. *Let $\boldsymbol{e}_j$ denote the vector with a 1 at position j, and 0s everywhere else. Consider a formal power series $B(z_1, \ldots, z_d)$ whose support contains $\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. Then $B(\boldsymbol{z})$ has full rank and is aperiodic.*

Proof. Since the support of $B(\boldsymbol{z})$ contains $\boldsymbol{0}$, we have $S_B \subseteq \Delta(S_B)$. Thus, $\Delta(S_B)$ contains d independent vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$, so it has rank d and $B(\boldsymbol{z})$ has full rank. Consider an integer vector $\boldsymbol{p}$ and integers r and $q \geq 2$. Assume

$$\{\boldsymbol{p}^\intercal \boldsymbol{n},\ \boldsymbol{n} \in S_B\} \subseteq r + q\mathbb{Z}_{\geq 0}.$$

For $\boldsymbol{n} = \boldsymbol{0}$, we deduce that q divides r. For $\boldsymbol{n} = \boldsymbol{e}_j$, we deduce that q divides $p_j - r$, so p_j is a multiple of q. Thus, $\boldsymbol{p}$ is necessarily a multiple of q and $B(\boldsymbol{z})$ cannot be q-periodic, so $B(\boldsymbol{z})$ is aperiodic. □

The Daffodil lemma (Flajolet and Sedgewick 2009, Lemma IV.1, page 266) states that the absolute value of a univariate generating function with nonnegative coefficients reaches its maximum at a unique point on any disk contained in its domain of convergence if and only if it is aperiodic. Our following lemma extends this result to the multivariate setting. This is reminiscent of (Melczer 2021, Proposition 5.5) that provides sufficient conditions for the *minimal point* (a central concept of singularity analysis, defined there) of rational functions to have positive real coordinates.

Lemma 7 (Multivariate daffodil lemma). *Consider a multivariate power series $B(z_1, \dots, z_d)$ with nonnegative coefficients and a vector $\boldsymbol{\zeta} \in \mathbb{R}^d_{>0}$ such that the domain of convergence of $B(\boldsymbol{z})$ contains the closed polydisc of radius $\boldsymbol{\zeta}$*

$$D(\boldsymbol{\zeta}) := \{\boldsymbol{z}, |z_1| \le \zeta_1 \wedge \ldots \wedge |z_d| \le \zeta_d\}.$$

Then for all $\boldsymbol{z} \in D(\boldsymbol{\zeta})$, we have $|B(\boldsymbol{z})| \le B(\boldsymbol{\zeta})$.

Assume furthermore that $B(\boldsymbol{z})$ has full rank. Then $|B(\boldsymbol{z})|$ has a unique maximum on $D(\boldsymbol{\zeta})$ if and only if it is aperiodic.

Proof. Using the positivity of the coefficients of $B(\boldsymbol{z})$, triangular inequality yields for any $\boldsymbol{x} \in D(\boldsymbol{\zeta})$

$$|B(\boldsymbol{x})| = \left|\sum_{\boldsymbol{n}} b_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}\right| \le \sum_{\boldsymbol{n}} b_{\boldsymbol{n}} |\boldsymbol{x}|^{\boldsymbol{n}} \le \sum_{\boldsymbol{n}} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}} = B(\boldsymbol{\zeta}).$$

Assume $|B(\boldsymbol{x})| = B(\boldsymbol{\zeta})$, then $(|x_1|, \dots, |x_d|) = \boldsymbol{\zeta}$, so there exists a vector $\boldsymbol{t} \in [0,1)^d$ such that for all j, we have $x_j = \zeta_j e^{2i\pi t_j}$. The case $\boldsymbol{t} = \boldsymbol{0}$ corresponds to $\boldsymbol{x} = \boldsymbol{\zeta}$. Let us assume first $\boldsymbol{t} \neq \boldsymbol{0}$, and prove that there exists $q \ge 2$ such that $B(\boldsymbol{z})$ is q-periodic. The equality $|B(\boldsymbol{x})| = B(\boldsymbol{\zeta})$ implies

$$\left|\sum_{\boldsymbol{n}} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}} e^{2i\pi \boldsymbol{n}^\intercal \boldsymbol{t}}\right| = \sum_{\boldsymbol{n}} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}},$$

so by the triangular inequality, for all $\boldsymbol{n}$ in the support S_B of $B(\boldsymbol{z})$, the complex numbers $e^{2i\pi \boldsymbol{n}^\intercal \boldsymbol{t}}$ are aligned. This is equivalent with the existence of $t_0 \in [0,1)$ such that for all $\boldsymbol{n} \in S_B$, we have

$$\boldsymbol{n}^\intercal \boldsymbol{t} = t_0 \mod 1.$$

Then for all $\boldsymbol{n} \in \Delta(S_B)$, we have

$$\boldsymbol{n}^\intercal \boldsymbol{t} = 0 \mod 1.$$

Thus, for all $\boldsymbol{n} \in \Delta(S_B)$, there exist integers $k_{\boldsymbol{n}}$ such that

$$\boldsymbol{n}^\intercal \boldsymbol{t} = k_{\boldsymbol{n}}.$$

Let $\Delta(S_B)'$ denote the set of vectors $\boldsymbol{n}$ from $\Delta(S_B)$ with an additional element $-k_{\boldsymbol{n}}$ at the bottom, and let $\boldsymbol{t}'$ denote the vector $\boldsymbol{t}$ with an additional element 1 at the bottom, then the previous equality is equivalent with

$$\boldsymbol{n}^\intercal \boldsymbol{t}' = 0$$

for all $\boldsymbol{n} \in \Delta(S_B)'$. Since $B(\boldsymbol{z})$ has full rank, $\Delta(S_B)$ generates a vector space of dimension d, so the vector space orthogonal to $\Delta'(S_B)$ has dimension at most 1. It contains a nonzero vector, so the dimension is exactly 1. By Lemma 4, there exists a nonzero integer vector $\boldsymbol{u}$ orthogonal to $\Delta(S_B)'$. We have $t'_{d+1} = 1$ and $\boldsymbol{t}'$ proportional to the integer vector $\boldsymbol{u}$, so $\boldsymbol{t}$ is a vector of rational values. Recall that $\boldsymbol{t} \in [0,1)^d$ and is nonzero. Thus, there exist $q \in \mathbb{Z}_{\ge 2}$ and $\boldsymbol{p} \in \mathbb{Z}^d_{\ge 0}$ with q that does not divide all p_j, such that $\boldsymbol{t} = \boldsymbol{p}/q$, so for all $\boldsymbol{n} \in \Delta(S_B)$,

$$\boldsymbol{n}^\intercal \boldsymbol{p} = q\, k_{\boldsymbol{n}}.$$

Fix $r = \min_{\boldsymbol{n} \in S_B}(\boldsymbol{n}^\intercal \boldsymbol{p})$, then for all $\boldsymbol{n} \in S_B$,

$$\boldsymbol{n}^\intercal \boldsymbol{p} - r = q\, k_{\boldsymbol{n}}.$$

Thus, q divides $\boldsymbol{n}^\intercal \boldsymbol{p} - r$. This implies the existence of a formal univariate power series $C(y)$ such that

$$B(y^{p_1}, \dots, y^{p_d}) = y^r C(y^q), \tag{12}$$

so $B(\boldsymbol{z})$ is q-periodic.

Reciprocally, if $B(\boldsymbol{z})$ is q-periodic for some $q \geq 2$, there exist nonnegative integers $r, p_1, \dots, p_d$, with q not dividing all p_j, and a formal power series $C(y)$ such that Equation (12) holds. Then for all $\boldsymbol{n} \in S_B$, the integer q divides $\boldsymbol{n}^\intercal \boldsymbol{p} - r$, so $e^{2i\pi \boldsymbol{n}^\intercal \boldsymbol{p}/q} = e^{2i\pi r/q}$. For any $\boldsymbol{\zeta} \in \mathbb{R}^d_{>0}$ such that the domain of convergence of $B(\boldsymbol{z})$ contains the polydisc of radius $\boldsymbol{\zeta}$, setting $\boldsymbol{x} = \boldsymbol{\zeta} e^{2i\pi \boldsymbol{p}/q}$, we have

$$|B(\boldsymbol{x})| = \left| \sum_{\boldsymbol{n} \in S_B} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}} e^{2i\pi \boldsymbol{p}^\intercal \boldsymbol{n}/q} \right| = \left| \sum_{\boldsymbol{n} \in S_B} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}} e^{2i\pi r} \right| = \left| \sum_{\boldsymbol{n} \in S_B} b_{\boldsymbol{n}} \boldsymbol{\zeta}^{\boldsymbol{n}} \right| = B(\boldsymbol{\zeta}),$$

where the last equality comes from the fact that the coefficients of $B(\boldsymbol{z})$ are nonnegative. Thus, on the torus of radius $\boldsymbol{\zeta}$, at the point $\boldsymbol{x}$, which is distinct from $\boldsymbol{\zeta}$, we have $|B(\boldsymbol{x})| = B(\boldsymbol{\zeta})$. □

In the current article, we will consider a polynomial P that is aperiodic. It is however worth noticing that coefficient extraction in a periodic power series can be simplified as follows. Suppose we are interested in the coefficient extraction $[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z})$. If $B(\boldsymbol{z})$ does not have full rank, we saw after Lemma 5 how to simplify the coefficient extraction. Assume now $B(\boldsymbol{z})$ has full rank, is analytic at $\mathbf{0}$, and is q-periodic for some $q \geq 2$. Then there exist nonnegative integers $r, p_1, \dots, p_d$, with q not dividing all p_j, and a power series $E(\boldsymbol{z}, y)$ such that

$$B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}}) = y^r E(\boldsymbol{z}, y^q).$$

We have

$$[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z}) = [\boldsymbol{z}^{\boldsymbol{n}} y^{\boldsymbol{p}^\intercal \boldsymbol{n}}]B(z_1 y^{p_1}, \dots, z_d y^{p_d}) = [\boldsymbol{z}^{\boldsymbol{n}} y^{\boldsymbol{p}^\intercal \boldsymbol{n}}] y^r E(\boldsymbol{z}, y^q).$$

Let j be such that $p_j \neq 0$ (which must exist, otherwise q would divide all p_i). Let $\boldsymbol{n}_{\setminus j}$ denote the vector $(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_d)$ and similarly for $\boldsymbol{z}_{\setminus j}$. Then $B(\boldsymbol{z})$ having full rank implies

$$[\boldsymbol{z}^{\boldsymbol{n}} y^{\boldsymbol{p}^\intercal \boldsymbol{n}}]B(\boldsymbol{z}\boldsymbol{y}^{\boldsymbol{p}}) = [\boldsymbol{z}_{\setminus j}^{\boldsymbol{n}_{\setminus j}} y^{\boldsymbol{p}^\intercal \boldsymbol{n}}]B(z_1 y^{p_1}, \dots, z_{j-1} y^{p_{j-1}}, y^{p_j}, z_{j+1} y^{p_{j+1}}, \dots z_d y^{p_d}).$$

Define

$$D(\boldsymbol{z}_{\setminus j}, y) := E(z_1, \dots, z_{j-1}, 1, z_{j+1}, \dots, z_d, y),$$

then

$$[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z}) = \begin{cases} 0 & \text{if } q \text{ does not divide } \boldsymbol{p}^\intercal \boldsymbol{n} - r, \\ [\boldsymbol{z}_{\setminus j}^{\boldsymbol{n}_{\setminus j}} y^{(\boldsymbol{p}^\intercal \boldsymbol{n} - r)/q}] D(\boldsymbol{z}_{\setminus j}, y) & \text{otherwise.} \end{cases}$$

Thus, when $|B(\boldsymbol{z})|$ reaches its maximum on a polydisc at several points, the coefficient extraction $[\boldsymbol{z}^{\boldsymbol{n}}]B(\boldsymbol{z})$ can be simplified.

Corollary 1. *Consider a power series $B(\boldsymbol{z})$ with nonnegative coefficients and two complex tori T_1, T_2 centered at the origin and contained in the domain of convergence of $B(\boldsymbol{z})$. If $|B(\boldsymbol{z})|$ reaches its maximum on T_1 at a unique point, then it reaches its maximum on T_2 at a unique point as well.*

A.2 Large Powers

Theorem 13 is the key result used in Section 4, which corresponds to the case where the saddle point has all coordinates subcritical. Consequently, the contour of integration can pass through the saddle point without intersecting the singularity at 1.

We first present a preliminary theorem, which corresponds to extracting the coefficient of index $\boldsymbol{\lambda} n$ for some fixed vector $\boldsymbol{\lambda}$ of rational values. Then, we state the theorem in its more general form, considering a sequence of vectors of real values $\boldsymbol{\kappa}_n = \boldsymbol{\lambda} n + O(n^\delta)$.

Theorem 12 (Multivariate large powers theorem). *Consider a power series $B(z_1, \dots, z_d)$ with nonnegative coefficients, full rank (Definition 3), aperiodic (Definition 5) and convergent on a neighborhood Ω of 0. Let $K(\boldsymbol{t}) = \log\left(B(e^{\boldsymbol{t}})\right)$ denote its cumulant generating function. Let $\boldsymbol{\lambda}$ denote a vector of positive rational numbers such that the function $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ does not tend to its infimum on the boundary of $\Psi = \{\boldsymbol{t} \in \mathbb{R}^d,\ e^{\boldsymbol{t}} \in \Omega\}$. Let $\boldsymbol{\tau}$ denote the unique point of Ψ where $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ reaches its minimum and $\mathcal{H}$ denote the Hessian matrix of $K(\boldsymbol{t})$ at $\boldsymbol{\tau}$. Set $\boldsymbol{\zeta} = e^{\boldsymbol{\tau}}$. Consider a power series $A(\boldsymbol{z})$ analytic on the torus of radius $\boldsymbol{\zeta}$ and satisfying $A(\boldsymbol{\zeta}) \neq 0$. Then, considering large values of n such that $\boldsymbol{\lambda} n$ is a vector of integers,*

$$[\boldsymbol{z}^{\boldsymbol{\lambda} n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n \sim \frac{A(\boldsymbol{\zeta})}{\sqrt{(2\pi n)^d \det(\mathcal{H})}} \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\lambda} n}}.$$

Proof. Lemma 6 ensures that $K(\boldsymbol{t})$ is strictly convex, so $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ is strictly convex as well. Since it does not reach its minimum on the boundary of Ψ, there exists a unique point $\boldsymbol{\tau}$ in Ψ where $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ reaches its minimum. We write the coefficient extraction as a Cauchy integral.

$$[\boldsymbol{z}^{\boldsymbol{\lambda} n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n = \frac{1}{(2i\pi)^d} \oint A(\boldsymbol{z}) B(\boldsymbol{z})^n \frac{d\boldsymbol{z}}{\boldsymbol{z}^{\boldsymbol{\lambda} n + \mathbf{1}}}.$$

The analyticity of $B(\boldsymbol{z})$ at $\boldsymbol{\zeta}$ and its nonnegative coefficients ensure that $B(\boldsymbol{z})$ is analytic on the torus of radius $\boldsymbol{\zeta}$. We choose for the integration domain the torus of radius $\boldsymbol{\zeta}$

$$[\boldsymbol{z}^{\boldsymbol{\lambda} n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\theta} \in [-\pi, \pi]^d} A(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}}) B(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}})^n \frac{d\boldsymbol{\theta}}{\boldsymbol{\zeta}^{\boldsymbol{\lambda} n} e^{in \boldsymbol{\lambda}^\intercal \boldsymbol{\theta}}}.$$

Introducing

$$\phi(\boldsymbol{\theta}) := i\boldsymbol{\lambda}^\intercal \boldsymbol{\theta} - \log\left(\frac{B(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}})}{B(\boldsymbol{\zeta})}\right),$$

we have

$$[\boldsymbol{z}^{\boldsymbol{\lambda} n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\lambda} n}} \frac{1}{(2\pi)^d} \int_{\boldsymbol{\theta} \in [-\pi, \pi]^d} A(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta}.$$

Since $|B(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}})|$ reaches its maximal value $B(\boldsymbol{\zeta})$ only at $\boldsymbol{\theta} = 0$, the real part of $\phi(\boldsymbol{\theta})$ is positive except at $\boldsymbol{\theta} = \mathbf{0}$. The Hessian matrix of $\phi(\boldsymbol{\theta})$ at $\boldsymbol{\theta} = \mathbf{0}$ is equal to the Hessian matrix $\mathcal{H}$ of $K(\boldsymbol{t})$ at $\boldsymbol{t} = \log(\boldsymbol{\zeta})$, and is nonsingular according to Lemma 6. The result of the theorem follows by application of the Laplace method, recalled in Theorem 11. □

Applications often require more flexibility than provided by our previous result. Instead of extracting the coefficient of index $\boldsymbol{\lambda} n$ for some fixed vector $\boldsymbol{\lambda}$ of rational values, we might consider an index $\boldsymbol{\kappa}_n \sim \boldsymbol{\lambda} n$, which allows us to cover the case where $\boldsymbol{\lambda}$ contains irrational values. To achieve this result, we follow and modify slightly the proof of Theorem 11.

Theorem 13. *Consider $\delta < 1/2$ and a sequence of vectors of real values*

$$\boldsymbol{\kappa}_n = \boldsymbol{\lambda} n + O(n^\delta).$$

Under the assumptions of Theorem 12, with $\boldsymbol{\tau}$ defined as the minimum of $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ and $\boldsymbol{\zeta} = e^{\boldsymbol{\tau}}$, we have

$$[\boldsymbol{z}^{\boldsymbol{\kappa}_n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n \sim \frac{A(\boldsymbol{\zeta})}{\sqrt{(2\pi n)^d \det(\mathcal{H})}} \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}.$$

Proof. We write again the coefficient extraction as a Cauchy integral on the torus of radius $\boldsymbol{\zeta}$

$$[\boldsymbol{z}^{\boldsymbol{\kappa}_n}] A(\boldsymbol{z}) B(\boldsymbol{z})^n = \frac{1}{(2\pi)^d} \int_{(-\pi, \pi]^d} A(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}}) B(\boldsymbol{\zeta} e^{i\boldsymbol{\theta}})^n \frac{d\boldsymbol{\theta}}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n} e^{i\boldsymbol{\kappa}_n^\intercal \boldsymbol{\theta}}}$$

and introduce

$$\phi(\boldsymbol{\theta}) := i\boldsymbol{\lambda}^\intercal\boldsymbol{\theta} - \log\left(\frac{B(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})}{B(\boldsymbol{\zeta})}\right) \tag{13}$$

to rewrite it as

$$[\boldsymbol{z}^{\boldsymbol{\kappa}_n}]A(\boldsymbol{z})B(\boldsymbol{z})^n = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})e^{-n\phi(\boldsymbol{\theta})}\frac{d\boldsymbol{\theta}}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal\boldsymbol{\theta}}}. \tag{14}$$

Central part. We introduce a positive sequence ε_n converging to 0 and specified later. The central part of the integral is defined as

$$C_n := \frac{1}{(2\pi)^d}\int_{(-\varepsilon_n,\varepsilon_n)^d} A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})e^{-n\phi(\boldsymbol{\theta})}\frac{d\boldsymbol{\theta}}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal\boldsymbol{\theta}}}.$$

Since $\phi(\mathbf{0}) = \mathbf{0}$ and, by definition of $\boldsymbol{\zeta}$, the gradient of $\phi(\boldsymbol{\theta})$ also vanishes at $\mathbf{0}$, the Taylor expansion of $\phi(\boldsymbol{\theta})$ is

$$\phi(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^\intercal\mathcal{H}\boldsymbol{\theta} + O(\|\boldsymbol{\theta}\|^3). \tag{15}$$

For $\boldsymbol{\theta} \in (-\varepsilon_n, \varepsilon_n)^d$, we have

$$n\phi(\boldsymbol{\theta}) = \frac{n}{2}\boldsymbol{\theta}^\intercal\mathcal{H}\boldsymbol{\theta} + O(n{\varepsilon_n}^3).$$

We will choose ε_n to ensure $n{\varepsilon_n}^3 \to 0$. We will also ensure $n^\delta\varepsilon_n \to 0$, so $e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal\boldsymbol{\theta}} \sim 1$ uniformly for $\theta \in (-\varepsilon_n, \varepsilon_n)$. The change of variable $\boldsymbol{t} = \sqrt{n}\boldsymbol{\theta}$ is applied

$$C_n = \frac{n^{-d/2}}{(2\pi)^d}\int_{(-\sqrt{n}\varepsilon_n,\sqrt{n}\varepsilon_n)^d} A(\boldsymbol{\zeta}e^{in^{-1/2}\boldsymbol{t}})e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2+O(n{\varepsilon_n}^3)}d\boldsymbol{t}.$$

We have $A(\boldsymbol{\zeta}e^{in^{-1/2}\boldsymbol{t}}) \to A(\boldsymbol{\zeta})$ and $n{\varepsilon_n}^3 \to 0$, so

$$C_n \sim A(\boldsymbol{\zeta})\frac{n^{-d/2}}{(2\pi)^d}\int_{(-\sqrt{n}\varepsilon_n,\sqrt{n}\varepsilon_n)^d} e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t}.$$

We recognise a Cauchy integral and choose ε_n such that $\sqrt{n}\varepsilon_n \to +\infty$, so

$$C_n \sim A(\boldsymbol{\zeta})\frac{n^{-d/2}}{(2\pi)^d}\int_{(-\infty,+\infty)^d} e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t} \sim \frac{A(\boldsymbol{\zeta})}{\sqrt{(2\pi n)^d\det(\mathcal{H})}}.$$

The three constraints on ε_n are satisfied by choosing, for example, $\varepsilon_n = n^{-\beta}$ with $\beta \in (\max(\delta, 1/3), 1/2)$.

Tail. The tail corresponds to $(-\pi,\pi]^d \setminus (-\varepsilon_n,\varepsilon_n)^d$. The tail part of the integral (14) is bounded by

$$\int_{(-\pi,\pi]^d\setminus(-\varepsilon_n,\varepsilon_n)^d}\left|A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})e^{-n\phi(\boldsymbol{\theta})}\right|d\boldsymbol{\theta} \le \left(\sup_{\boldsymbol{\theta}\in(-\pi,\pi)^d}|A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})|\right)\int_{(-\pi,\pi]^d\setminus(-\varepsilon_n,\varepsilon_n)^d} e^{-n\,\mathrm{Re}(\phi(\boldsymbol{\theta}))}d\boldsymbol{\theta}.$$

We will prove in the next paragraph that for all large enough n, the minimum of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ on the tail is reached at some points from $[-\varepsilon_n,\varepsilon_n]^d \setminus (-\varepsilon_n,\varepsilon_n)^d$. Thus, the tail is bounded by

$$O(1)\sup_{\boldsymbol{\theta}\in[-\varepsilon_n,\varepsilon_n]^d\setminus(-\varepsilon_n,\varepsilon_n)^d} e^{-n\,\mathrm{Re}(\phi(\boldsymbol{\theta}))}.$$

We replace $\phi(\boldsymbol{\theta})$ with its Taylor expansion

$$-n\,\mathrm{Re}(\phi(\boldsymbol{\theta})) = -\frac{n}{2}\boldsymbol{\theta}^\intercal\mathcal{H}\boldsymbol{\theta} + O(n{\varepsilon_n}^3).$$

Our choice of ε_n ensures the O tends to 0. We divide $\boldsymbol{\theta}$ by ε_n to bound the tail by

$$O(1)\sup_{\boldsymbol{\theta}\in[-\varepsilon_n,\varepsilon_n]^d\setminus(-\varepsilon_n,\varepsilon_n)^d}\exp\left(-\frac{n{\varepsilon_n}^2}{2}({\varepsilon_n}^{-1}\boldsymbol{\theta})^\intercal\mathcal{H}({\varepsilon_n}^{-1}\boldsymbol{\theta})\right).$$

The vector ${\varepsilon_n}^{-1}\boldsymbol{\theta}$ has at least one coefficient equal to -1 or 1, so it stays bounded away from $\mathbf{0}$. Since $\mathcal{H}$ is positive definite, there exists a positive constant c such that $({\varepsilon_n}^{-1}\boldsymbol{\theta})^{\intercal}\mathcal{H}({\varepsilon_n}^{-1}\boldsymbol{\theta}) \geq c$, so the tail is bounded by

$$O(1)\exp\left(-c\frac{n{\varepsilon_n}^2}{2}\right).$$

Since $n{\varepsilon_n}^2 \to +\infty$, this converges exponentially fast to 0, so the tail is negligible compared to the central part of the integral.

Locating the minimum of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ *on the tail.* By definition of $\phi(\boldsymbol{\theta})$, we have

$$\begin{aligned}\mathrm{Re}(\phi(\boldsymbol{\theta})) &= \mathrm{Re}\left(i\boldsymbol{\lambda}^{\intercal}\boldsymbol{\theta} - \log\left(\frac{B(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})}{B(\boldsymbol{\zeta})}\right)\right)\\ &= \log(B(\boldsymbol{\zeta})) - \log\left(\left|B(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})\right|\right).\end{aligned}$$

Thus, the minima of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ correspond to maxima of $|B(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})|$. By assumption, on the polydisc of radius $\boldsymbol{\zeta}$, the maximum of $|B(\boldsymbol{z})|$ is reached only at $\boldsymbol{z} = \boldsymbol{\zeta}$, so the minimum of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ is reached only at $\boldsymbol{\theta} = \mathbf{0}$. As all points of the central part tend to $\boldsymbol{\zeta}$ with n, this minimum tends to $\mathrm{Re}(\phi(\mathbf{0})) = 0$. By contradiction, let us assume that there is an infinite set of indices n for which there exists a point in $(-\pi,\pi)^d \setminus [-\varepsilon_n,\varepsilon_n]^d$ where $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ reaches its minimum on the tail $(-\pi,\pi)^d \setminus (-\varepsilon_n,\varepsilon_n)^d$. We extract a converging subsequence $\boldsymbol{u}_n$. The limit $\boldsymbol{v}$ must be $\mathbf{0}$, otherwise $\lim_n \mathrm{Re}(\phi(\boldsymbol{u}_n)) = \mathrm{Re}(\phi(\boldsymbol{v})) > 0$, which contradicts the fact that the minimum on the tail tends to 0. The gradient of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ vanishes at each $\boldsymbol{\theta} = \boldsymbol{u}_n$. From Equation (15), we deduce that the gradient of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ at $\boldsymbol{\theta} = \boldsymbol{u}_n$ is proportional to $\mathcal{H}\boldsymbol{u}_n + O(\|\boldsymbol{u}_n\|^2)$. Let q_n denote the minimum of the absolute value of all coefficients of $\boldsymbol{u}_n$, then for all n

$$\mathcal{H}\frac{\boldsymbol{u}_n}{q_n} + O(\|\boldsymbol{u}_n\|) = \mathbf{0}.$$

Left-multiplying by $\mathcal{H}^{-1}$, we deduce that $\boldsymbol{u}_n/q_n$ converges to $\mathbf{0}$, which is impossible as at least one of its coefficients is equal to 1. This concludes the proof that for all large enough n, the minima of $\mathrm{Re}(\phi(\boldsymbol{\theta}))$ on $(-\pi,\pi]^d \setminus (-\varepsilon_n,\varepsilon_n)^d$ belongs to $[-\varepsilon_n,\varepsilon_n]^d \setminus (-\varepsilon_n,\varepsilon_n)^d$. □

A.3 Combining Large Powers and Singularity Analysis

The following result generalizes Theorem 13 to the case where the saddle point meets singularities. It is the analytic result behind Theorems 3, 6 and 8. This is a multivariate combination of the large powers theorem (Flajolet and Sedgewick 2009, Theorem VIII.8) and singularity analysis (Flajolet and Sedgewick 2009, Section VI). The presence of large powers simplifies the singularity analysis part and we avoid the complications related to multivariate singularity analysis (Pemantle et al. 2024). We will derive a more precise version of it (asymptotic expansion) in Appendix A.4.

Theorem 14. *Consider a power series $B(z_1,\dots,z_d)$ with nonnegative coefficients, full rank, aperiodic and convergent on a neighborhood Ω of 0. Let $K(\boldsymbol{t}) = \log(B(e^{\boldsymbol{t}}))$ denote its cumulant generating function. Let $\boldsymbol{\lambda}$ denote a vector of positive real values such that the function $K(\boldsymbol{t}) - \boldsymbol{\lambda}^{\intercal}\boldsymbol{t}$ does not tend to its infimum on the boundary of $\Psi = \{\boldsymbol{t} \in \mathbb{R}^d,\ e^{\boldsymbol{t}} \in \Omega\}$. Let $\boldsymbol{\tau}$ denote the unique point of Ψ where $K(\boldsymbol{t}) - \boldsymbol{\lambda}^{\intercal}\boldsymbol{t}$ reaches its minimum. Set $\boldsymbol{\zeta} = e^{\boldsymbol{\tau}}$.*

We assume $\zeta_j \leq 1$ for all j and denote by $\mathcal{S}$ the set of indices such that $\zeta_j < 1$, and by $\mathcal{C}$ the corresponding set for $\zeta_j = 1$. Note that by assumption, we have $\mathcal{S} \cup \mathcal{C} = \{1,\dots,d\}$. Let $\mathcal{H}$ denote the Hessian matrix of $K(\boldsymbol{t})$ at $\boldsymbol{\tau}$ and $\mathcal{M}$ the submatrix of $\mathcal{H}^{-1}$ corresponding to the rows and column in $\mathcal{C}$. Consider a power series $A(\boldsymbol{z})$ analytic on the torus of radius $\boldsymbol{\zeta}$ and satisfying $A(\boldsymbol{\zeta}) \neq 0$. Consider a real value $\delta < 1/2$ and a sequence of vectors of real values

$$\boldsymbol{\kappa}_n = \boldsymbol{\lambda}n + O(n^{\delta}).$$

Then

$$[z^{\boldsymbol{\kappa}_n}]\frac{A(z)B(z)^n}{\prod_{j\in\mathcal{S}\cup C}(1-z_j)} \sim \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{1}{\sqrt{(2\pi)^d n^{|\mathcal{S}|}\det(\mathcal{H})}}\int_{(0,+\infty)^{|C|}} e^{-\boldsymbol{u}^\intercal \mathcal{M}\boldsymbol{u}/2}d\boldsymbol{u}.$$

Proof. We write the coefficient extraction as a Cauchy integral and apply again the change of variable $\boldsymbol{z} = \boldsymbol{\zeta}e^{i\boldsymbol{\theta}}$. However, for $j \in C$, as $\zeta_j = 1$ and $z_j = 1$ is a singularity of the integrand, we cannot let θ_j go through 0. So in that case, fixing $\varepsilon_n = n^{-\beta}$ with $\beta \in (\max(\delta, 1/3), 1/2)$, as in the proof of Theorem 13, we define the path of integration for θ_j as $(-\pi, -\varepsilon_n)$, then a half-circle centered at 0, starting at $-\varepsilon_n$, going through $i\varepsilon_n$ and stopping at ε_n, then $(\varepsilon_n, \pi]$. For $j \in \mathcal{S}$, we have $\zeta_j < 1$ so we choose the integration path $\theta_j \in (-\pi, \pi]$. The function $\phi(\boldsymbol{t})$ is defined as in (13) and Equation (14) becomes

$$[z^{\boldsymbol{\kappa}_n}]\frac{A(z)B(z)^n}{\prod_{j\in\mathcal{S}\cup C}(1-z_j)} = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{1}{(2\pi)^d}\int \frac{A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})e^{-n\phi(\boldsymbol{\theta})}}{\prod_{j\in\mathcal{S}\cup C}(1-\zeta_j e^{i\theta_j})}\frac{d\boldsymbol{\theta}}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal\boldsymbol{\theta}}}.$$

Central part. It corresponds to the following part of the domain of integration, denoted by γ_n. For $j \in C$, we choose θ_j on the half-circle described above. For $j \in \mathcal{S}$, we choose $\theta_j \in (-\varepsilon_n, \varepsilon_n)$. On this domain, we apply a Taylor expansion of the integrand. Note that for $j \in C$, as $\zeta_j = 1$, we have $1 - \zeta_j e^{i\theta_j} \sim -i\theta_j$, while for $j \in \mathcal{S}$, we have $1 - \zeta_j e^{i\theta_j} \sim 1 - \zeta_j$.

$$\frac{1}{(2\pi)^d}\int_{\gamma_n} \frac{A(\boldsymbol{\zeta}e^{i\boldsymbol{\theta}})e^{-n\phi(\boldsymbol{\theta})}}{\prod_{j\in\mathcal{S}\cup C}(1-\zeta_j e^{i\theta_j})}\frac{d\boldsymbol{\theta}}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal\boldsymbol{\theta}}} \sim \frac{1}{(2\pi)^d}\int_{\gamma_n}\frac{A(\boldsymbol{\zeta})e^{-n\boldsymbol{\theta}^\intercal\mathcal{H}\boldsymbol{\theta}/2+O(n\|\boldsymbol{\theta}\|^3)}}{\prod_{j\in C}(-i\theta_j)\prod_{j\in\mathcal{S}}(1-\zeta_j)}d\boldsymbol{\theta}. \tag{16}$$

Note that our choice of ε_n ensures that $O(n\|\boldsymbol{\theta}\|^3)$ tends to 0. The change of variable $\boldsymbol{t} = \sqrt{n}\boldsymbol{\theta}$ is applied. The integral is now equivalent with

$$\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{i^{|C|}}{(2\pi)^d n^{|\mathcal{S}|/2}}\int\frac{e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}}{\prod_{j\in C}t_j}d\boldsymbol{t},$$

where the domain of integration is the following. For $j \in C$, t_j is on the line from $-\sqrt{n}\varepsilon_n$ to i, then on the line from i to $\sqrt{n}\varepsilon_n$. For $j \in \mathcal{S}$, $t_j \in (-\sqrt{n}\varepsilon_n, \sqrt{n}\varepsilon_n)$. The tails of this integral are exponentially small, so we add them to simplify the expression. The central asymptotics becomes

$$\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{i^{|C|}}{(2\pi)^d n^{|\mathcal{S}|/2}}\int_{(i-\infty,i+\infty)^{|C|}\times(-\infty,+\infty)^{|\mathcal{S}|}}\frac{e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}}{\prod_{j\in C}t_j}d\boldsymbol{t}.$$

To further simplify the expression, we inject the following identity, valid for any t with positive imaginary part

$$\frac{1}{t} = -i\int_0^{+\infty} e^{itu}du.$$

Applying this identity for $t = t_j$ and $j \in C$ yields

$$\frac{A(\boldsymbol{\zeta})}{(2\pi)^d n^{|\mathcal{S}|/2}\prod_{j\in\mathcal{S}}(1-\zeta_j)}\int_{(i-\infty,i+\infty)^{|C|}\times(-\infty,+\infty)^{|\mathcal{S}|}}\int_{(0,+\infty)^{|C|}}\exp\left(-\frac{1}{2}\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t} + i\sum_{j\in C}u_j t_j\right)d\boldsymbol{u}d\boldsymbol{t}.$$

Let $(\boldsymbol{u}_C, \boldsymbol{0}_{\mathcal{S}})$ denote the vector of dimension d with u_j at any index $j \in C$ and 0 at any index $j \in \mathcal{S}$. The matrix $\mathcal{H}$ is positive definite, so its inverse and square root are well defined. We interchange the integrals and apply the

change of variable $\boldsymbol{s} = \sqrt{\mathcal{H}}\boldsymbol{t}$

$$\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{1}{(2\pi)^d\sqrt{n^{|\mathcal{S}|}\det(\mathcal{H})}} \times \int_{(0,+\infty)^{|\mathcal{C}|}}\int_{(i-\infty,i+\infty)^{|\mathcal{C}|}\times(-\infty,+\infty)^{|\mathcal{S}|}} \exp\left(-\frac{1}{2}\boldsymbol{s}^\intercal\boldsymbol{s} + i(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})^\intercal\mathcal{H}^{-1/2}\boldsymbol{s}\right)d\boldsymbol{s}d\boldsymbol{u}.$$

The argument of the exponential is factorized

$$\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{1}{(2\pi)^d\sqrt{n^{|\mathcal{S}|}\det(\mathcal{H})}}\iint \exp\left(-\frac{1}{2}\left(\boldsymbol{s}-i\mathcal{H}^{-1/2}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})\right)^\intercal\left(\boldsymbol{s}-i\mathcal{H}^{-1/2}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})\right) - \frac{1}{2}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})^\intercal\mathcal{H}^{-1}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})\right)d\boldsymbol{s}d\boldsymbol{u}.$$

The integrals with respect to each s_j are now separated, each a Gaussian integral with value $\sqrt{2\pi}$. We deduce that the central part has asymptotics

$$\frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{1}{\sqrt{(2\pi)^d n^{|\mathcal{S}|}\det(\mathcal{H})}}\int_{(0,+\infty)^{|\mathcal{C}|}}\exp\left(-\frac{1}{2}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})^\intercal\mathcal{H}^{-1}(\boldsymbol{u}_{\mathcal{C}},\boldsymbol{0}_{\mathcal{S}})\right)d\boldsymbol{u}.$$

Tail. The proof that the tail is exponentially small compared to the central part is the same as in the proof of Theorem 13. □

A.4 Asymptotic Expansion

We now explain how to obtain more precise asymptotic results, applied in Section 8. As mentioned before, our treatment of the subcritical case, in Theorem 13, is an application of a technique known as *large powers*, relying on the *Laplace method*. It is well known (see *e.g.,* (Pemantle et al. 2024)) that the corresponding asymptotic expansion consists of terms in increasing powers of $1/n$. The critical and mixed cases from Theorem 14 combine properties from large powers and *singularity analysis*. In univariate singularity analysis (Flajolet and Sedgewick 2009, Section VI), the asymptotic expansion is also found to be in powers of $1/n$. Somewhat surprisingly, our next result shows that in the critical and mixed cases, the asymptotic expansion is in powers of $1/\sqrt{n}$ instead.

Theorem 15. *We keep the notations of Theorem 14 and replace our assumption $\boldsymbol{\kappa}_n = \boldsymbol{\lambda}n + O(n^\delta)$ for some $\delta < 1/2$ with*

$$\boldsymbol{\kappa}_n = \boldsymbol{\lambda}n + O(1).$$

Then for any $r \geq 1$, there exist functions $a_{k,n}$ of $\boldsymbol{\kappa}_n - \boldsymbol{\lambda}n$ (which are bounded with respect to n) such that

$$[\boldsymbol{z}^{\boldsymbol{\kappa}_n}]\frac{A(\boldsymbol{z})B(\boldsymbol{z})^n}{\prod_{j\in\mathcal{S}\cup\mathcal{C}}(1-z_j)} = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{1}{n^{|\mathcal{S}|/2}}\left(\sum_{k=0}^{r-1}a_{k,n}n^{-k/2} + O(n^{-r/2})\right). \tag{17}$$

An algorithm computing them is provided in the proof. In particular, $a_0 := a_{0,n}$ is independent of n and equal to

$$a_0 = \frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)}\frac{1}{\sqrt{(2\pi)^d\det(\mathcal{H})}}\int_{(0,+\infty)^{|\mathcal{C}|}}e^{-\boldsymbol{u}^\intercal\mathcal{M}\boldsymbol{u}/2}d\boldsymbol{u}$$

and we will express $a_{1,n}$ in Lemma 8. Furthermore, in the particular case $\mathcal{C} = \emptyset$, then $a_{k,n} = 0$ for all odd k.

Proof. We fix some $r \geq 1$. Let us say that a sequence u_n *tends superpolynomially fast* to 0 if for any integer k, the sequence $n^k u_n$ tends to 0. For example, $e^{-\sqrt{n}}$ tends superpolynomially fast to 0. We proved that the tails tend

superpolynomially fast to 0, so its contribution to Equation (17) is negligible. We apply the change of variable $\boldsymbol{t} = \sqrt{n}\boldsymbol{\theta}$ in the left-hand side of Equation (16) and deduce

$$[z^{\boldsymbol{\kappa}_n}]\frac{A(z)B(z)^n}{\prod_{j\in\mathcal{S}\cup C}(1-z_j)} = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{1}{(2\pi)^d n^{d/2}}\int \frac{A(\boldsymbol{\zeta}e^{i\boldsymbol{t}/\sqrt{n}})e^{-n\phi(\boldsymbol{t}/\sqrt{n})}}{\prod_{j\in\mathcal{S}\cup C}(1-\zeta_j e^{it_j/\sqrt{n}})}\frac{d\boldsymbol{t}}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal \boldsymbol{t}/\sqrt{n}}}\left(1+O(n^{-r/2})\right)$$

where the domain of integration is the following. As in the proof of Theorem 14, we set $\varepsilon_n = n^{-\beta}$ for some $\beta \in (1/3, 1/2)$. This ensures $\sqrt{n}\varepsilon_n \to +\infty$. For $j \in C$, t_j is on the line from $-\sqrt{n}\varepsilon_n$ to i, then on the line from i to $\sqrt{n}\varepsilon_n$. For $j \in \mathcal{S}$, $t_j \in (-\sqrt{n}\varepsilon_n, \sqrt{n}\varepsilon_n)$. Let us set

$$F(\boldsymbol{t}, y) := \frac{A(\boldsymbol{\zeta}e^{it\boldsymbol{y}})\exp\left(\frac{1}{2}\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t} - \phi(\boldsymbol{t}y)y^{-2}\right)}{\left(\prod_{j\in C}\frac{1-e^{it_j y}}{-it_j y}\right)\prod_{j\in\mathcal{S}}(1-\zeta_j e^{it_j y})}\frac{1}{e^{i(\boldsymbol{\kappa}_n-\boldsymbol{\lambda}n)^\intercal \boldsymbol{t}y}}. \tag{18}$$

Recalling that $\zeta_j = 1$ for $j \in C$ and injecting $F(\boldsymbol{t}, y)$ in our previous expression leads to

$$[z^{\boldsymbol{\kappa}_n}]\frac{A(z)B(z)^n}{\prod_{j\in\mathcal{S}\cup C}(1-z_j)} = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{i^{|C|}}{(2\pi)^d n^{|\mathcal{S}|/2}}\int \frac{F(\boldsymbol{t}, n^{-1/2})}{\prod_{j\in C} t_j}e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t}\left(1+O(n^{-r/2})\right). \tag{19}$$

The term $\frac{1-e^{it_j y}}{-it_j y}$ has a Taylor expansion at $y = 0$. Observe also that the valuation of $\frac{1}{2}\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t} - \phi(\boldsymbol{t}y)y^{-2}$ in y is at least 1, and that in the series expansion at $(\boldsymbol{t}, y) = (\boldsymbol{0}, 0)$, each monomial where the sum of the exponents of the t_j is p has y at the power $p-2$. This implies that $F(\boldsymbol{t}, y)$ has a Taylor expansion at $y = 0$, denoted by

$$F(\boldsymbol{t}, y) = \sum_{k=0}^{r-1} F_k(\boldsymbol{t})y^k + O\left((t_1\cdots t_d)^{\beta_r}y^r\right)$$

where β_r is some sequence depending on r (we could prove $\beta_r \le 3r$ but will not need it) and each $F_k(\boldsymbol{t})$ is a polynomial in $\boldsymbol{t}$ and in $\boldsymbol{\kappa}_n - \boldsymbol{\lambda}n$, which is a bounded sequence in n. With our domain of integration, we have for any β_r

$$\int \frac{O\left((t_1\cdots t_d)^{\beta_r}n^{-r/2}\right)}{\prod_{j\in C} t_j}e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t} = O(n^{-r/2})$$

so, injecting in Equation (19) the Taylor expansion of $F(\boldsymbol{t}, \boldsymbol{y})$ at $\boldsymbol{y} = n^{-1/2}$, we obtain

$$[z^{\boldsymbol{\kappa}_n}]\frac{A(z)B(z)^n}{\prod_{j\in\mathcal{S}\cup C}(1-z_j)} = \frac{B(\boldsymbol{\zeta})^n}{\boldsymbol{\zeta}^{\boldsymbol{\kappa}_n}}\frac{1}{n^{|\mathcal{S}|/2}}\left(\sum_{k=0}^{r-1}\left[\frac{i^{|C|}}{(2\pi)^d}\int\frac{F_k(\boldsymbol{t})}{\prod_{j\in C} t_j}e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t}\right]n^{-k/2} + O(n^{-r/2})\right).$$

The tails of the integrals are exponentially small, so we add them to simplify the expression. After deformation, the domain of integration becomes $(i-\infty, i+\infty)^{|C|}\times(-\infty, +\infty)^{|\mathcal{S}|}$. Defining

$$a_{k,n} := \frac{i^{|C|}}{(2\pi)^d}\int_{(i-\infty,i+\infty)^{|C|}\times(-\infty,+\infty)^{|\mathcal{S}|}}\frac{F_k(\boldsymbol{t})}{\prod_{j\in C} t_j}e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2}d\boldsymbol{t},$$

we obtain Equation (17).

The expression of a_0 was obtained and simplified in the proof of Theorem 14. Let us now show how to apply the same ideas to the other coefficients. Write for any complex t with positive imaginary part

$$\frac{i}{t} = \int_0^{+\infty} e^{itu}du,$$

then

$$a_{k,n} = \frac{1}{(2\pi)^d}\iint_{(0,+\infty)^{|C|}} F_k(\boldsymbol{t})e^{-\boldsymbol{t}^\intercal\mathcal{H}\boldsymbol{t}/2+i(\boldsymbol{u}_C,\boldsymbol{0}_S)^\intercal\boldsymbol{t}}d\boldsymbol{u}d\boldsymbol{t}.$$

We apply the change of variable $\boldsymbol{s} = \sqrt{\mathcal{H}}\boldsymbol{t}$ and factorize the argument of the exponential as in the proof of Theorem 14

$$\begin{aligned} a_{k,n} &= \frac{1}{(2\pi)^d\sqrt{\det(\mathcal{H})}} \iint_{(0,+\infty)^{|C|}} F_k(\mathcal{H}^{-1/2}\boldsymbol{s}) e^{-\boldsymbol{s}^\intercal \boldsymbol{s}/2 + i(\boldsymbol{u}_C,\boldsymbol{0}_S)^\intercal \mathcal{H}^{-1/2}\boldsymbol{s}} d\boldsymbol{u}d\boldsymbol{s} \\ &= \frac{1}{(2\pi)^d\sqrt{\det(\mathcal{H})}} \iint_{(0,+\infty)^{|C|}} F_k(\mathcal{H}^{-1/2}\boldsymbol{s}) e^{-(\boldsymbol{s}-i\mathcal{H}^{-1/2}(\boldsymbol{u}_C,\boldsymbol{0}_S))^\intercal(\boldsymbol{s}-i\mathcal{H}^{-1/2}(\boldsymbol{u}_C,\boldsymbol{0}_S))/2 - \boldsymbol{u}^\intercal\mathcal{M}\boldsymbol{u}/2} d\boldsymbol{u}d\boldsymbol{s}. \end{aligned}$$

We interchange the integrals and shift the vector $\boldsymbol{s}$ by $i\mathcal{H}^{-1/2}(\boldsymbol{u},\boldsymbol{0})$. Recalling that $\mathcal{M}$ is the submatrix of $\mathcal{H}^{-1}$ indexed by C, we then have

$$a_{k,n} = \frac{1}{(2\pi)^d\sqrt{\det(\mathcal{H})}} \int_{(0,+\infty)^{|C|}} \int F_k\left(\mathcal{H}^{-1/2}\boldsymbol{s} + i\mathcal{H}^{-1}(\boldsymbol{u}_C,\boldsymbol{0}_S)\right) e^{-\boldsymbol{s}^\intercal\boldsymbol{s}/2 - \boldsymbol{u}^\intercal\mathcal{M}\boldsymbol{u}/2} d\boldsymbol{s}d\boldsymbol{u}.$$

Recall that $F_k(\boldsymbol{t})$ is a polynomial, so there exist polynomials $G_{k,\boldsymbol{p}}(\boldsymbol{u})$ (which are also polynomials in $\boldsymbol{\kappa}_n - \boldsymbol{\lambda}n$) and a finite set $S_k \subset \mathbb{Z}^d_{\geq 0}$ such that

$$F_k\left(\mathcal{H}^{-1/2}\boldsymbol{s} + i\mathcal{H}^{-1}(\boldsymbol{u}_C,\boldsymbol{0}_S)\right) = \sum_{\boldsymbol{p}\in S_k} \boldsymbol{s}^{\boldsymbol{p}} G_{k,\boldsymbol{p}}(\boldsymbol{u}).$$

We deduce

$$a_{k,n} = \frac{1}{(2\pi)^d\sqrt{\det(\mathcal{H})}} \sum_{\boldsymbol{p}\in S_k} \int_{(0,+\infty)^{|C|}} \int \boldsymbol{s}^{\boldsymbol{p}} G_{k,\boldsymbol{p}}(\boldsymbol{u}) e^{-\boldsymbol{s}^\intercal\boldsymbol{s}/2 - \boldsymbol{u}^\intercal\mathcal{M}\boldsymbol{u}/2} d\boldsymbol{s}d\boldsymbol{u}.$$

Each integral with respect to s_j, for $j \in \mathcal{S}\cup C$, is simplified using the formula

$$\int_{(i-\infty,i+\infty)} s^p e^{-s^2/2} ds = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \sqrt{2\pi}\frac{(2q)!}{2^q q!} & \text{if } p = 2q. \end{cases}$$

We observe that in the expression of $a_{k,n}$, only vectors $\boldsymbol{p}$ where all coefficients are even may contribute. Let us define

$$S_k^\star = \{\boldsymbol{q},\ 2\boldsymbol{q} \in S_k\},$$

then

$$a_{k,n} = \frac{1}{\sqrt{(2\pi)^d \det(\mathcal{H})}} \sum_{\boldsymbol{q}\in S_k^\star} \left(\prod_{j\in\mathcal{S}\cup C} \frac{(2q_j)!}{2^{q_j} q_j!}\right) \int_{(0,+\infty)^{|C|}} G_{k,2\boldsymbol{q}}(\boldsymbol{u}) e^{-\boldsymbol{u}^\intercal\mathcal{M}\boldsymbol{u}/2} d\boldsymbol{u}. \tag{20}$$

In the particular case $C = \emptyset$, the function F_k is simply composed with $\mathcal{H}^{-1/2}\boldsymbol{s}$. If k is odd, by construction, the sum of the exponents in each monomial of $F_k(\boldsymbol{t})$ is odd. This implies that each monomial of $F_k(\mathcal{H}^{-1/2}\boldsymbol{s})$ contains some s_j at an odd power, so $a_{k,n} = 0$.

An alternative approach to compute the coefficients $a_{k,n}$ is to apply the change of variable $-\boldsymbol{t}^\intercal\boldsymbol{t}/2 = \phi(\boldsymbol{\theta})$ in the left-hand side of Equation (16). We refer the interested reader to (Pemantle et al. 2024, Chapter 5). □

Our next result provides an explicit expression for the term $a_{1,n}$ of the asymptotic expansion from Theorem 15.

Lemma 8. *With ζ defined as in Theorem 15, define the polynomial*

$$\phi_3(\boldsymbol{t}) := [y^3] \log\left(\frac{B(\zeta e^{\boldsymbol{t}y})}{B(\zeta)}\right).$$

Observe that the sum of the exponents of each of its monomials is 3. *Let* $\phi_3^\star(\boldsymbol{u})$ *denote the polynomial obtained by replacing each monomial* $t_j t_k t_\ell$ *(with* j, k, ℓ *in* $\mathcal{S} \cup C$ *not necessarily distinct) with*

$$\begin{aligned} &- \boldsymbol{e}_j^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \boldsymbol{e}_k^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \\ &+ \left((\mathcal{H}^{-1})_{k,\ell} \boldsymbol{e}_j + (\mathcal{H}^{-1})_{j,\ell} \boldsymbol{e}_k + (\mathcal{H}^{-1})_{j,k} \boldsymbol{e}_\ell \right)^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}). \end{aligned} \tag{21}$$

Let $\boldsymbol{b}$ *denote the vector of size* d *with* $b_j = -1/2$ *for* $j \in C$ *and* $b_j = \frac{\zeta_j}{1-\zeta_j}$ *for* $j \in \mathcal{S}$. *Then the term* $a_{1,n}$ *in the asymptotic expansion from Theorem 15 is equal to*

$$\begin{aligned} a_{1,n} =& \frac{A(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)} \frac{1}{\sqrt{(2\pi)^d \det(\mathcal{H})}} \\ &\times \int_{(0,+\infty)^{|C|}} \left(\phi_3^\star(\boldsymbol{u}) - \left[\frac{\nabla_A(\boldsymbol{\zeta})}{A(\boldsymbol{\zeta})} + \boldsymbol{b} - (\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n) \right]^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \right) e^{-\boldsymbol{u}^\intercal \mathcal{M} \boldsymbol{u}/2} d\boldsymbol{u}. \end{aligned}$$

Proof. Our first step is to compute $F_1(\boldsymbol{t}) = [y^1]F(\boldsymbol{t}, y)$, where $F(\boldsymbol{t}, y)$ is defined in Equation (18). Using the expression of $\phi_3(\boldsymbol{t})$ from the lemma, we have the following Taylor expansions

$$\begin{aligned} A(\boldsymbol{\zeta} e^{i\boldsymbol{t}y}) &= A(\boldsymbol{\zeta}) + i\nabla_A(\boldsymbol{\zeta})\boldsymbol{t}y + O(y^2), \\ \exp\left(\frac{1}{2}\boldsymbol{t}^\intercal \mathcal{H} \boldsymbol{t} - \phi(\boldsymbol{t}y) y^{-2} \right) &= 1 - i\phi_3(\boldsymbol{t})y + O(y^2), \\ \frac{1}{\frac{1-e^{it_j y}}{-it_j y}} &= 1 - \frac{i}{2} t_j y + O(y^2), \\ \frac{1}{1-\zeta_j e^{it_j y}} &= \frac{1}{1-\zeta_j} + \frac{\zeta_j}{(1-\zeta_j)^2} i t_j y + O(y^2), \\ \frac{1}{e^{i(\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n)^\intercal \boldsymbol{t} y}} &= 1 - i(\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n)^\intercal \boldsymbol{t} y + O(y^2). \end{aligned}$$

Using the vector $\boldsymbol{b}$ from the current lemma, we obtain the following Taylor expansion for the products

$$\frac{1}{\prod_{j\in C} \frac{1-e^{it_j y}}{-it_j y}} \frac{1}{\prod_{j\in\mathcal{S}}(1-\zeta_j e^{it_j y})} = 1 + i\boldsymbol{b}^\intercal \boldsymbol{t} y + O(y^2).$$

We deduce

$$F_1(\boldsymbol{t}) = \frac{-iA(\boldsymbol{\zeta})}{\prod_{j\in\mathcal{S}}(1-\zeta_j)} \left(\phi_3(\boldsymbol{t}) - \left[\frac{\nabla_A(\boldsymbol{\zeta})}{A(\boldsymbol{\zeta})} + \boldsymbol{b} - (\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n) \right]^\intercal \boldsymbol{t} \right).$$

Following the proof of Theorem 15, our next step is to compute

$$\sum_{\boldsymbol{q}\in S_1^\star} \left(\prod_{j\in\mathcal{S}\cup C} \frac{(2q_j)!}{2^{q_j} q_j!} \right) G_{1,2\boldsymbol{q}}(\boldsymbol{u}).$$

It is obtained by the following transformation of $F_1(\boldsymbol{t})$. Replace $\boldsymbol{t}$ with $\mathcal{H}^{-1/2}\boldsymbol{s} + i\mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S})$, then for each j and k, replace s_j^k with 0 if k is odd, and with $\frac{(2q)!}{2^q q!}$ if $k = 2q$. Clearly, any t_j at the power 1 will be transformed into $i\mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S})$. By construction, $\phi_3(\boldsymbol{t})$ is a polynomial where the sum of exponents in each monomial is 3. Consider three integers j, k, ℓ in $\mathcal{S} \cup C$. The monomial $t_j t_k t_\ell$ will be transformed into

$$\left(\boldsymbol{e}_j^\intercal \mathcal{H}^{-1/2}\boldsymbol{s} + i\boldsymbol{e}_j^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \right) \left(\boldsymbol{e}_k^\intercal \mathcal{H}^{-1/2}\boldsymbol{s} + i\boldsymbol{e}_k^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \right) \left(\boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1/2}\boldsymbol{s} + i\boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \boldsymbol{0}_\mathcal{S}) \right)$$

where each s_h at an odd power is replaced by 0 and each s_h^2 is replaced by 1, for all $h \in \mathcal{S} \cup C$. The expression becomes

$$\begin{aligned} -i\boldsymbol{e}_j^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S})\boldsymbol{e}_k^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S})\boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S}) + i\boldsymbol{e}_j^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S}) \sum_{h=1}^{d} \boldsymbol{e}_k^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \\ + i\boldsymbol{e}_k^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S}) \sum_{h=1}^{d} \boldsymbol{e}_j^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \\ + i\boldsymbol{e}_\ell^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S}) \sum_{h=1}^{d} \boldsymbol{e}_j^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \boldsymbol{e}_k^\intercal \mathcal{H}^{-1/2}\boldsymbol{e}_h \end{aligned}$$

which, given the symmetry of $\mathcal{H}^{-1/2}$, is equal to (21). With $\phi_3^\star(\boldsymbol{u})$ defined as in the lemma, we deduce that after evaluating $F_1(\boldsymbol{t})$ at $\boldsymbol{t} = \mathcal{H}^{-1/2}\boldsymbol{s} + i\mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S})$, then replacing for each j any odd power of s_j with 0 and s_j^2 with 1, we obtain

$$\sum_{\boldsymbol{q} \in S_1^\star} \left(\prod_{j \in \mathcal{S} \cup C} \frac{(2q_j)!}{2^{q_j} q_j!} \right) G_{1,2\boldsymbol{q}}(\boldsymbol{u}) = \frac{A(\boldsymbol{\zeta})}{\prod_{j \in \mathcal{S}}(1-\zeta_j)} \left(\phi_3^\star(\boldsymbol{u}) - \left[\frac{\nabla_A(\boldsymbol{\zeta})}{A(\boldsymbol{\zeta})} + \boldsymbol{b} - (\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n) \right]^\intercal \mathcal{H}^{-1}(\boldsymbol{u}_C, \mathbf{0}_\mathcal{S}) \right).$$

Injecting this in Equation (20) for $k = 1$ yields the expression of $a_{1,n}$ from the current lemma. □

B Properties and Proofs for the General Independent Culture (GIC)

B.1 Satisfying the Conditions of the Asymptotic Theorems in Appendix A

In this paper, the characteristic polynomial $P(\boldsymbol{x})$ encodes a probability distribution over the preference rankings of a voter. This guarantees its convergence on a neighborhood Ω of 0. Since we assume that the distribution of voter preferences over rankings is generic, *i.e.,* every ranking r has a positive probability, it follows that all coefficients of the characteristic polynomial $P(\boldsymbol{x})$ from Definition 1 are positive. The following result establishes that this assumption ensures that the remaining conditions required for the application of our asymptotic theorems (Theorems 13 to 15) are satisfied.

Lemma 9. *Consider a polynomial $P(x_1, \ldots, x_d)$ where for any $\mathcal{X} \subseteq \{1, \ldots, d\}$, the monomial $\prod_{j \in \mathcal{X}} x_j$ has a positive coefficient, while any other monomial has a zero coefficient. As usual, define its cumulant generating function as*

$$K : \boldsymbol{t} \mapsto \log(P(e^{\boldsymbol{t}})).$$

Then $P(\boldsymbol{x})$ has full rank (Definition 3) and is aperiodic (Definition 5). Furthermore, for any $\boldsymbol{\lambda} \in (0,1)^d$ and $\boldsymbol{\ell}$ on the boundary of $[-\infty, +\infty]^d$, the function $t \mapsto K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ tends to infinity as $\boldsymbol{t}$ tends to $\boldsymbol{\ell}$. Thus, there exists a unique vector $\boldsymbol{\tau} \in \mathbb{R}^d$ where the gradient of K vanishes.

Proof. By assumption, the support of $P(\boldsymbol{x})$ contains $\mathbf{0}$ and $\boldsymbol{e}_j$ for each $j \in \{1, \ldots, d\}$. Proposition 1 then implies that $P(\boldsymbol{x})$ has full rank and is aperiodic.

Now, consider $\boldsymbol{t}$ converging to $\boldsymbol{\ell}$. Let $\mathcal{X}$ denote the set of indices j such that $\boldsymbol{\ell}_j = +\infty$. Let $p_\mathcal{X}$ denote the coefficient of $P(\boldsymbol{x})$ corresponding to the monomial of exponents in $\mathcal{X}$. Let $\mathbf{1}_\mathcal{X}$ denote the vector of dimension d with value 1 at indices from $\mathcal{X}$, and 0 otherwise. Then

$$P(e^{\boldsymbol{t}}) \geq p_\mathcal{X} \exp\left(\mathbf{1}_\mathcal{X}^\intercal \boldsymbol{t}\right),$$

so

$$K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t} \geq \log(p_\mathcal{X}) + \mathbf{1}_\mathcal{X}^\intercal \boldsymbol{t} - \boldsymbol{\lambda}^\intercal \boldsymbol{t} \geq \log(p_\mathcal{X}) + (\mathbf{1}_\mathcal{X} - \boldsymbol{\lambda})^\intercal \boldsymbol{t}.$$

The vector $\mathbf{1}_{\mathcal{X}} - \boldsymbol{\lambda}$ has positive coefficients at the indices where $\boldsymbol{t}$ tends to $+\infty$, and negative coefficients at the indices where $\boldsymbol{t}$ tends to a constant or $-\infty$. Since $\boldsymbol{\ell}$ is on the boundary of $[-\infty, +\infty]^d$, it has at least one element equal to $-\infty$ or $+\infty$, so $(\mathbf{1}_{\mathcal{X}} - \boldsymbol{\lambda})^\intercal \boldsymbol{t}$ tends to infinity. This implies that $K(\boldsymbol{t}) - \boldsymbol{\lambda}^\intercal \boldsymbol{t}$ converges to $+\infty$ as well.

According to Lemma 6, K is strictly convex. If its minimum is not reached on the boundary of its domain of definition, it must be reached in its interior and is unique. Thus, there exists a unique point $\boldsymbol{\tau}$ where the gradient of K vanishes. □

B.2 Connection Between the Hessian Matrix and the Correlation Matrix

This section demonstrates the connection discussed in Section 5 between our Hessian matrix $\mathcal{H}_K(\boldsymbol{\tau})$ and the correlation matrix R_m of Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005).

For each $j \in \mathcal{A}$, let X_j denote the random variable taking value 1 if candidate j is ranked higher than candidate m, and 0 otherwise. Set $\boldsymbol{Y} = 2\boldsymbol{X} - \mathbf{1}$. In this paper, we consider the covariance matrix $\mathcal{H}_K(\boldsymbol{\tau})$ of $\boldsymbol{X}$, whose coefficient (j,k) is

$$\mathbb{E}(X_j X_k) - \mathbb{E}(X_j)\mathbb{E}(X_k),$$

while Niemi and Weisberg (1968) and Krishnamoorthy and Raghavachari (2005) consider the correlation matrix R_m of $\boldsymbol{Y}$, whose coefficient (j,k) is

$$\frac{\mathbb{E}(Y_j Y_k) - \mathbb{E}(Y_j)\mathbb{E}(Y_k)}{\sqrt{(1-\mathbb{E}(Y_j)^2)(1-\mathbb{E}(Y_k)^2)}}.$$

Observe that

$$\begin{aligned}\mathbb{E}(Y_j Y_k) - \mathbb{E}(Y_j)\mathbb{E}(Y_k) &= \mathbb{E}((2X_j-1)(2X_k-1)) - \mathbb{E}(2X_j-1)\mathbb{E}(2X_k-1)\\ &= 4\left(\mathbb{E}(X_j X_k) - \mathbb{E}(X_j)\mathbb{E}(X_k)\right).\end{aligned}$$

Let D denote the invertible diagonal matrix whose jth element is $(1-\mathbb{E}(Y_j)^2)^{-1/2}$, then

$$R_m = 4D\mathcal{H}_K(\boldsymbol{\tau})D.$$

Thus, the change of variable $\boldsymbol{u} = 2D\boldsymbol{v}$ links the two formulas

$$\frac{1}{\sqrt{\det(R_m)}}\int_{(0,+\infty)^{m-1}} e^{-\boldsymbol{v}^\intercal {R_m}^{-1}\boldsymbol{v}/2} d\boldsymbol{v} = \frac{1}{\sqrt{\det(\mathcal{H}_K(\boldsymbol{\tau}))}}\int_{(0,+\infty)^{m-1}} e^{-\boldsymbol{u}^\intercal \mathcal{H}_K(\boldsymbol{\tau})^{-1}\boldsymbol{u}/2} d\boldsymbol{u}.$$

B.3 Inversion Lemma for a Saddle Point Coordinate

This section consists in proving Lemma 2, rewritten below.

Lemma. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two disjoint sets of adversaries. Let $\tilde{\boldsymbol{\zeta}}$ be the saddle point associated with $\mathbb{P}\left(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\right)$. Let $j \in \mathcal{X}$, $\mathcal{X}' = \mathcal{X} \setminus \{j\}$, and $\mathcal{Y}' = \mathcal{Y} \cup \{j\}$. Then the saddle point associated with $\mathbb{P}\left(m \succ_{\boldsymbol{\alpha}} \mathcal{X}' \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}'\right)$ has its k-th coordinate equal to $\tilde{\zeta}_k$ if $k \neq j$ and $\frac{1}{\tilde{\zeta}_j}$ if $k = j$.*

Proof. Transferring a candidate from $\mathcal{X}$ to $\mathcal{Y}$ corresponds to the following algebraic operation:

$$P^{\mathcal{X}'}_{\mathcal{Y}'}\left(\boldsymbol{x}_{\mathcal{X}'}, \boldsymbol{y}_{\mathcal{Y}'}\right) = y_j P^{\mathcal{X}}_{\mathcal{Y}}\left(\boldsymbol{x}_{\mathcal{X}'}, \frac{1}{y_j}, \boldsymbol{y}_{\mathcal{Y}}\right), \tag{22}$$

where y_j encodes when m is preferred to j, whereas x_j encoded the opposite.

Now, Let us denote $(\boldsymbol{\beta}, \boldsymbol{\alpha})$ the threshold vector associated with $\mathbb{P}\left(m \succ_{\boldsymbol{\alpha}} \mathcal{X} \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}\right)$, and $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ the threshold vector associated with $\mathbb{P}\left(m \succ_{\boldsymbol{\alpha}} \mathcal{X}' \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}'\right)$. Remark that $\hat{\boldsymbol{\beta}}$ is the vector $\boldsymbol{\beta}$ where the coordinate β_j has been

removed, and $\hat{\boldsymbol{\alpha}}$ is the vector $\boldsymbol{\alpha}$ where the coordinate α_j has been added. Finally, let us denote $\hat{\boldsymbol{\zeta}}$ the saddle point associated with $\mathbb{P}\big(m \succ_{\boldsymbol{\alpha}} \mathcal{X}' \wedge m \preccurlyeq_{\boldsymbol{\alpha}} \mathcal{Y}'\big)$. By definition of $\hat{\boldsymbol{\zeta}}$, extending the remark in Section 3.3, we have

$$\hat{\boldsymbol{\zeta}} = \underset{(\mathbb{R}_{>0})^{|\mathcal{X}'\cup\mathcal{Y}'|}}{\arg\min} \frac{P^{\mathcal{X}'}_{\mathcal{Y}'}\big(\boldsymbol{x}_{\mathcal{X}'}, \boldsymbol{y}_{\mathcal{Y}'}\big)}{\boldsymbol{x}_{\mathcal{X}'}\hat{\boldsymbol{\beta}}\boldsymbol{y}_{\mathcal{Y}'}{}^{\hat{\boldsymbol{\alpha}}}}.$$

So injecting Equation (22) we get

$$\hat{\boldsymbol{\zeta}} = \underset{(\mathbb{R}_{>0})^{|\mathcal{X}'\cup\mathcal{Y}'|}}{\arg\min} \frac{P^{\mathcal{X}}_{\mathcal{Y}}\big(\boldsymbol{x}_{\mathcal{X}'}, \frac{1}{y_j}, \boldsymbol{y}_{\mathcal{Y}'}\big)}{\boldsymbol{x}_{\mathcal{X}'}\hat{\boldsymbol{\beta}}y_j{}^{-\beta_j}\boldsymbol{y}_{\mathcal{Y}}{}^{\boldsymbol{\alpha}}}.$$

Since the minimum of a function over a positive orthant remains unchanged under the inversion of a coordinate, setting $x_j = \frac{1}{y_j}$ yields

$$\min_{(\mathbb{R}_{>0})^{|\mathcal{X}'\cup\mathcal{Y}'|}} \frac{P^{\mathcal{X}}_{\mathcal{Y}}\big(\boldsymbol{x}_{\mathcal{X}'}, \frac{1}{y_j}, \boldsymbol{y}_{\mathcal{Y}'}\big)}{\boldsymbol{x}_{\mathcal{X}'}\hat{\boldsymbol{\beta}}y_j{}^{-\beta_j}\boldsymbol{y}_{\mathcal{Y}}{}^{\boldsymbol{\alpha}}} = \min_{(\mathbb{R}_{>0})^{|\mathcal{X}\cup\mathcal{Y}|}} \frac{P^{\mathcal{X}}_{\mathcal{Y}}\big(\boldsymbol{x}_{\mathcal{X}}, \boldsymbol{y}_{\mathcal{Y}}\big)}{\boldsymbol{x}_{\mathcal{X}}{}^{\boldsymbol{\beta}}\boldsymbol{y}_{\mathcal{Y}}{}^{\boldsymbol{\alpha}}},$$

and more importantly, the transformation $x_j = \frac{1}{y_j}$ yields

$$\hat{\boldsymbol{\zeta}} = (\tilde{\zeta}_1, \dots, \tilde{\zeta}_{j-1}, \frac{1}{\tilde{\zeta}_j}, \tilde{\zeta}_{j+1}, \dots, \tilde{\zeta}_{m-1}).$$

□

C Properties and Proofs for Particular Cultures

C.1 Saddle Point, Hessian Matrix and Terms of the Asymptotic Expansion in Impartial Culture

We provide the proofs of Theorems 4 and 10, both considering Impartial Culture on m candidates. For each $j \in \mathcal{A}$, let X_j denote the random variable taking value 1 if candidate j is ranked higher than candidate m, and 0 otherwise. The probability generating function of the vector $\boldsymbol{X}$ is $P(\boldsymbol{x})$ and its cumulant generating function $\boldsymbol{t} \mapsto \mathbb{E}(e^{\boldsymbol{t}^\intercal \boldsymbol{X}})$ is $K : \boldsymbol{t} \mapsto \log(P(e^{\boldsymbol{t}}))$. We have $\mathbb{E}(\boldsymbol{X}) = \mathbf{1}/2$. For our analysis, we need to compute the second and third central moments of $\boldsymbol{X}$, which correspond to the terms of order 2 and 3 in the Taylor expansion of K.

Second central moment. The Hessian of K at $\mathbf{0}$ is the covariance matrix of $\boldsymbol{X}$, whose coefficient (j,k) is equal to

$$\mathcal{H}_K(\mathbf{0})_{j,k} = \mathbb{E}((X_j - 1/2)(X_k - 1/2)) = \mathbb{E}(X_j X_k) - \frac{1}{4}.$$

For $j = k$, we deduce $\mathcal{H}_K(\mathbf{0})_{j,k} = 1/4$. In a random ranking r, the probability that each candidate from a given subset $\mathcal{Y}$ of adversaries beats candidate m is $\frac{1}{|\mathcal{Y}|+1}$. Indeed, the ranking induced by r on $\mathcal{Y} \cup \{m\}$ is uniformly distributed, so the probability that m is in the last position is $\frac{1}{|\mathcal{Y}|+1}$. This implies for $j \neq k$

$$\mathcal{H}_K(\mathbf{0})_{j,k} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Letting I denote the identity matrix of dimension $m-1$ and J the matrix containing only 1s, we deduce

$$\mathcal{H}_K(\mathbf{0}) = \frac{1}{12}J + \frac{1}{6}I.$$

The expressions of its inverse and determinant from Equation (7) follow.

Third central moment. A useful property of the cumulant generating function is that the third term of its Taylor expansion $\phi_3 := \boldsymbol{t} \mapsto [y^3]\log(P(e^{\boldsymbol{t}\,y}))$ (using the notation from Lemma 8) is equal to the third central moment of $\boldsymbol{X}$. Thus

$$\phi_3(\boldsymbol{t}) = \sum_{j,k,\ell \in \{1,\dots,m-1\}} \mathbb{E}((X_j - 1/2)(X_k - 1/2)(X_\ell - 1/2))t_j t_k t_\ell.$$

We develop

$$\mathbb{E}((X_j - 1/2)(X_k - 1/2)(X_\ell - 1/2)) = \mathbb{E}(X_j X_k X_\ell) - \frac{1}{2}(\mathbb{E}(X_j X_k) + \mathbb{E}(X_j X_\ell) + \mathbb{E}(X_k X_\ell)) + \frac{3}{4}\mathbb{E}(X_j) - \frac{1}{8}.$$

As we saw earlier, we have $\mathbb{E}(X_j X_k X_\ell) = \frac{1}{4}$. In all cases, whether j, k and ℓ are distinct or some of them are equal, we find the third central moment to be 0 and deduce $\phi_3(\boldsymbol{t}) = 0$.

Proof of Theorem 4. The number of permutations of $\{1, \dots, m\}$ where the set of elements above m is exactly $\mathcal{X}$ is $|\mathcal{X}|!(m-1-|\mathcal{X}|)!$. Thus, for the Impartial Culture, the characteristic polynomial from Definition 1 is

$$P(\boldsymbol{x}) = \sum_{\mathcal{X} \subseteq \mathcal{A}} \frac{|\mathcal{X}|!(m-1-|\mathcal{X}|)!}{m!} \prod_{j \in \mathcal{X}} x_j.$$

The saddle point and log saddle point are $\boldsymbol{\zeta} = \mathbf{1}$ and $\boldsymbol{\tau} = \mathbf{0}$, and we have $\boldsymbol{\beta} = \mathbf{1}/2$. Theorem 4 is obtained by application of Theorem 3 with those parameters.

Proof of Theorem 10. The term a_0 is computed in Theorem 4. The term $a_{1,n}$ is obtained by application of Lemma 8 with $c = d = m-1$, $\boldsymbol{b} = -\mathbf{1}/2$, $A(\boldsymbol{z}) = 1$ so $\nabla_A(\boldsymbol{\zeta}) = \mathbf{0}$, $\mathcal{H} = \mathcal{H}_K(\mathbf{0})$, $\phi_3^\star(\boldsymbol{u}) = 0$ because $\phi_3(\boldsymbol{t}) = 0$, and $\boldsymbol{\kappa}_n - \boldsymbol{\lambda} n = (\lceil n/2 \rceil - 1 - n/2)\mathbf{1}$.

C.2 Saddle Points in the Mallows Models $\mathcal{M}_{m\,\text{last}}$ and $\mathcal{M}_{m\,\text{first}}$

This section is dedicated to proving Lemma 1 and Lemma 3. Since the proof follows the exact same steps in both cases, we present it only for the culture $\mathcal{M}_{m\,\text{last}}$.

Lemma. *Under the Mallows culture $\mathcal{M}_{m\,\text{last}}$, the log saddle point $\boldsymbol{\tau}$ is given by*

$$\boldsymbol{\tau} = \left(\frac{-m\rho}{2}, \frac{(-m+2)\rho}{2}, \dots, \frac{(m-4)\rho}{2}\right),$$

where ρ is the concentration parameter of the culture, defined in Section 2.1.

Proof. To prove this result, we leverage the probabilistic interpretation of the saddle point, namely that in the distribution induced by the log saddle point $\boldsymbol{\tau}$, candidate m is, in expectation, precisely at the threshold for being an $\boldsymbol{\alpha}$-winner.

In the general case, this interpretation translates to the following equation for each ζ_j

$$\sum_{r,\ j \succ m} p_r \Big(\prod_{k \succ m \text{ in } r} \zeta_k \Big) = \sum_{r,\ j \prec m} p_r \Big(\prod_{k \succ m \text{ in } r} \zeta_k \Big).$$

For the Mallows culture $\mathcal{M}_{m\,\text{last}}$, the equilibrium condition can be formulated in terms of ranking pairs. Specifically, due to the structure of the Mallows distribution, a ranking of the form (σ, m, σ') balances with the ranking obtained by swapping σ and σ', *i.e.,* (σ', m, σ).

For instance, when $m = 5$, the ranking $(1, 3, 2, 5, 4)$ must be in equilibrium with $(4, 5, 1, 3, 2)$, yielding

$$p_{(1,3,2,5,4)}\zeta_1\zeta_2\zeta_3 = p_{(4,5,1,3,2)}\zeta_4.$$

Therefore, for an arbitrary m, the saddle point satisfies $m!/2$ equilibrium equations. Indeed, for each suitable pair of rankings r and r', the saddle point satisfies

$$p_r \prod_{k \succ m \text{ in } r} \zeta_k = p_{r'} \prod_{k \succ m \text{ in } r'} \zeta_k.$$

By considering the equation

$$p_{\{j\}}\zeta_j = p_{\mathcal{A}\setminus\{j\}} \prod_{k \neq j} \zeta_k$$

for each adversary j, we obtain a system of $m-1$ independent equations with $m-1$ unknowns (except when $m=3$, where the system reduces to a single equation). Solving this system, while ensuring that all coordinates of the saddle point are nonzero, yields the desired result. In the case of $m=3$, the system simplifies to $p_{\{1\}}\zeta_1 = p_{\{2\}}\zeta_2$. Adding the equation $p_{\{1,2\}}\zeta_1\zeta_2 = p_\emptyset$ provides two independent equations, allowing for a unique solution.

□

C.3 Asymptotic Behavior in the Mallows Model $\mathcal{M}_{3\text{ first}}$

This section outlines the main steps leading to the asymptotic formula

$$1 - \mathbb{P}(3 \text{ is CW}) \underset{n\to+\infty}{\sim} \sqrt{\frac{2}{\pi n}} \frac{e^{-\rho\lceil n/2\rceil}}{1-e^{-\rho}} \left(\frac{2}{1+e^{-\rho}}\right)^n.$$

First, it is needed to iteratively apply Equation (22) on the supercritical coordinates, until every term is subcritical, which gives the following equality

$$1 - \mathbb{P}(3 \text{ is CW}) = \left(\mathbb{P}(3 \preccurlyeq_{\frac{1}{2}} \{1\}) + \mathbb{P}(3 \preccurlyeq_{\frac{1}{2}} \{2\})\right) - \mathbb{P}(3 \preccurlyeq_{\frac{1}{2}} \{1,2\}).$$

Subcriticality. Note that the characteristic polynomial associated with $\mathbb{P}(3 \preccurlyeq_{\frac{1}{2}} \{1,2\})$, namely $P^{\emptyset}_{\{1,2\}}(y_1y_2)$, is actually the characteristic polynomial of the probability $\mathbb{P}(3 \text{ is CW})$ under the Mallows model $\mathcal{M}_{3\text{ last}}$. Therefore, we already know that the saddle point for this term is subcritical.

The characteristic polynomials $P^{\{2\}}_{\{1\}}(y_1,1)$ and $P^{\{1\}}_{\{2\}}(1,y_2)$ associated respectively with $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{1\})$ and $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\})$ are both univariate and of the form $\gamma(q+py)$. Denoting ξ a saddle point associated with such a polynomial, a straightforward calculation yields $\xi = \frac{q}{p}$. The verification of subcriticality follows directly, by simply writing explicitly

$$P^{\{2\}}_{\{1\}}(y_1,1) = \gamma\left((2e^{-2\rho} + e^{-3\rho}) + (1+2e^{-\rho})y_1\right),$$

and

$$P^{\{1\}}_{\{2\}}(1,y_2) = \gamma\left((e^{-\rho} + e^{-2\rho} + e^{-3\rho}) + (1+e^{-\rho}+e^{-2\rho})y_2\right).$$

Dominating term. The first two terms can be associated with a Mallows culture with two candidates and a concentration parameter $\tilde{\rho} = \log(\frac{q}{p})$. In the case of $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\})$, $\tilde{\rho} = \rho$. In the case of $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{1\})$, $\tilde{\rho}$ is equal to ρ multiplied by a quantity strictly less than 1. Therefore, this Mallows is less concentrated than the first one. Thus, asymptotically, $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\})$ will dominate $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{1\})$.

In particular, as $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{1,2\}) \leq \mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{1\})$, the third term of the sum is also dominated by $\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\})$. Thus,

$$1 - \mathbb{P}(3 \text{ is CW}) \underset{n\to+\infty}{\sim} \mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\}).$$

Now, using Theorem 8, we get

$$\mathbb{P}(3 \preccurlyeq_{\boldsymbol{\alpha}} \{2\}) \underset{n\to+\infty}{\sim} \frac{P_{\{2\}}^{\{1\}}(1,\xi)^n}{(1-\xi)\xi^{\lfloor n/2\rfloor}} \frac{1}{\sqrt{2\pi n^{\frac{1}{4}}}},$$

where $\xi = e^{-\rho}$, thus $P_{\{2\}}^{\{1\}}(1,\xi) = 2\gamma e^{-\rho}(1 + e^{-\rho} + e^{-2\rho})$, with $\gamma = \frac{1}{1+2e^{-\rho}+2e^{-2\rho}+e^{-3\rho}}$.

Straightforward calculations yield the final formula.

Received 30 June 2025; accepted 21 September 2026