Abduction in Łukasiewicz Logic With Rational Interval Terms

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We explore abductive reasoning in contexts involving fuzzy statements (statements with truth degrees) such as 'the lift is heavily loaded' (meaning, e.g., that the lift is carrying more than 70% of its maximal load), 'the symptoms are severe', 'it is cold outside', etc. Such statements are both used to express *observations* and to *explain* observed events. That is, both the observed event and our explanation of it may have truth degrees.

To formalise these contexts, we use infinitely-valued Łukasiewicz fuzzy logic Ł with semantics defined over the real-valued interval $[0, 1]$. Here, 0 and 1 are interpreted as 'absolutely false' and 'absolutely true', respectively, and the remaining values as degrees of truth. We consider two entailment relations for Ł: the 'truth-preserving' one (if the premise is absolutely true, then the conclusion is absolutely true) and the 'degree-preserving' (the truth degree of the premise should be at most as high as the truth degree of the conclusion).

For each of these entailment relations, we define and motivate the notions of abduction problems and explanations in the language of Ł expanded with 'interval literals' of the form $p{\geq}\overline{c}$, $p{\leq}\overline{c}$, and their negations that express the sets of values a variable can have. We analyse the complexity of standard abductive reasoning tasks (solution recognition, solution existence, and relevance / necessity of hypotheses) in Ł for the case of the full language and for the case of theories containing only disjunctive clauses and show that, in contrast to classical propositional logic, the abduction in the clausal fragment has lower complexity than in the general case.

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1 Introduction

Deduction, induction, and abduction are three main forms of reasoning (Flach and Kakas 2000). Abduction (finding explanations) has multiple applications in artificial intelligence, such as diagnosis (El Ayeb et al. 1993; J. R. Josephson and S. G. Josephson 2009; Koitz-Hristov and Wotawa 2018), commonsense reasoning (Bhagavatula et al. 2020; Paul 1993), formalisation of scientific reasoning (Magnani 2011), and machine learning (Dai et al. 2019). In *logic-based abduction* (Eiter and Gottlob 1995), the reasoning task is to find an explanation for an observation χ from a theory Γ, i.e., a formula ϕ such that $\Gamma, \phi \models \chi$ but $\Gamma, \phi \not\models \bot$ (i.e., Γ and ϕ should *consistently entail* χ). Thus, abduction is also connected to reasoning with incomplete information. If a theory Γ does not entail an observed event χ nor its negation, it is *incomplete*. In this case, we assume an additional statement ϕ consistent with Γ that makes χ follow from Γ expanded with ϕ. In other words, ϕ explains χ *relative to* Γ and makes it complete w.r.t. χ.

Observe, however, that in many applications, one needs not only to state *whether* a statement is true but also to specify *to which degree* it holds. E.g., a lift may be safe to use only when loaded to at least $\frac{1}{4}$ of its maximal capacity (otherwise, its software will register it as empty) and to no more than $\frac{2}{3}$ of the capacity. Then, we need to

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know not only whether the lift is loaded but also how much it is loaded (i.e., the *proportion* between its actual load and its maximal load). Moreover, one may need to specify an *interval* of values which make some condition true. For example, the cruising speed of a car can be defined as $60 \ldots 90$ kmh (with the maximal speed being 150 kmh). As classical logic has only two values, it is not well-suited to reason about contexts involving truth degrees and ranges of values. On the other hand, *fuzzy logics* evaluate formulas in the real-valued interval $[0, 1]$ and thus are much better suited to formalising such contexts than classical logic. In particular, to formalise the examples above, we can set $v(l) \in [\frac{1}{4}, \frac{2}{3}]$ and $v(s) \in [0.4, 0.6]$ where the truth degrees of l and s denote, respectively, the load of the lift and the speed of the car w.r.t. its maximal speed.

Fuzzy Logic. Originally, fuzzy logics were introduced by Zadeh (1975, 1965) to reason about imprecise statements such as 'it is cold outside', 'the symptoms are severe', etc. In fuzzy logics, the formulas are evaluated over the interval $[0, 1]$, and their values are interpreted as *degrees of truth* from 0 (absolutely false) to 1 (absolutely true). Fuzzy logic has also been applied to reasoning about uncertainty (Baldi et al. 2020; Hájek and Tulipani 2001) and beliefs (Rodriguez et al. 2022). In knowledge representation and reasoning, fuzzy logics have found multiple applications in representing graded and fuzzy ontologies (Straccia 2016). In such ontologies, concept assertions and terminological axioms have degrees of truth. Moreover, fuzzy versions of description logics and their computational properties have been extensively investigated (Borgwardt 2014; Borgwardt, Distel, et al. 2014; Borgwardt and Peñaloza 2017, 2012).

Additionally, fuzzy logic has found numerous applications in artificial intelligence. Recent work in machine learning integrates deep learning-based perception with symbolic knowledge representation. Neurosymbolic frameworks such as (Badreddine et al. 2022; Diligenti et al. 2017) adopt semantics of fuzzy logic to support learning and reasoning in real-world domains. Krieken et al. (2022) analyse how different fuzzy logic semantics affect the behaviour of learning. Fuzzy logic has also been used in reasoning problems with knowledge graphs (Chen et al. 2022) and MaxSAT (Haniková et al. 2023). Furthermore, t-norms (functions used to interpret conjunctions in fuzzy logic) are applied to autonomous driving with requirements (Stoian et al. 2023).

Abduction in Fuzzy Logic. Abduction in different systems of fuzzy logic has long attracted attention. To the best of our knowledge, it was first presented by Yamada and Mukaidono (1995). There, the authors formalised abduction problems in the infinitely-valued Łukasiewicz fuzzy logic Ł and proposed explanations in the form of *fuzzy sets*, i.e., assignments of values from $[0, 1]$ to propositional variables. This approach was further expanded by Vojtáš (1999) to Gödel and Product fuzzy logics. Solutions to abduction problems in multiple fuzzy logics were further systematised by Allonnes et al. (2007) and Chakraborty et al. (2013). Abduction in fuzzy logic has found multiple applications, in particular, diagnosis (Miyata et al. 1998), machine learning (Bergadano et al. 2000), fuzzy logic programming (Ebrahim 2001; Vojtáš 2001), decision-making and learning in the presence of incomplete information (Mellouli and Bouchon-Meunier 2003; Tsypyschev 2017), and robot perception (Shanahan 2005).

Still, the complexity of fuzzy abduction largely remains unexplored. To the best of our knowledge, the only discussion is given by Vojtáš (1999). There, the author explores fuzzy abduction in definite logic programs. A fuzzy definite logic program is a set of rules of the form $\langle B \leftarrow A_1, \ldots, A_n, x\rangle$ where A_i's and B are atoms; x belongs to the real-valued interval $[0, 1]$ and denotes the truth degree of the rule. In particular, the author claims that

> [a]s linear programming is lying in NP complexity class (even much lower) as prolog does, to find minimal solutions for a definite [fuzzy logic programming abduction problem] ... does not increase the complexity and remains in NP — (Vojtáš 1999, §4.4)

Thus, there seems to be no formal study of the complexity of abduction in fuzzy logic.

Contributions. In this paper, we make a step towards a systematic study of the complexity of abduction in fuzzy logic. We concentrate on the infinite-valued Łukasiewicz logic Ł as it is one of the most expressive ones. In particular, it can express rational numbers and continuous linear functions over $[0, 1]$ (cf. (McNaughton 1951) for details). In addition, it still retains many classical relations between its conjunction, disjunction, implication, and negation. We formalise abduction in Ł and explore its computational properties. Our contribution is twofold. First, we propose and motivate a new form of hypotheses — *interval literals* — the formulas of the form $p \diamond \overline{c}$ with $\diamond \in \{\leq, \geq, >, <\}$ and c being a rational number in $[0, 1]$. Such literals express intervals of values a variable is permitted to have. The solutions to abduction problems are then represented as conjunctions of interval literals (*interval terms*). Thus, our solutions generalise the fuzzy-set solutions to abduction problems in fuzzy logics. Given a 'traditional' fuzzy set-membership function $\mathbf{m}$, a statement $\mathbf{m}(p) = c$ can be expressed via (a conjunction of) two interval literals: $p{\geq}\overline{c}$ and $p{\leq}\overline{c}$. To identify informative solutions, we will typically impose a properness condition, requiring that a solution does not entail the observation by itself (i.e., it must be combined with the theory to explain the observation). Moreover, we consider two kinds of solution minimality based on Ł-entailment. This gives rise to the notions of $\models_{Ł}$-minimal solutions and theory-minimal solutions.

Second, we study the complexity of standard reasoning problems (solution recognition, solution existence, relevance and necessity of hypotheses). We consider the cases of theories containing arbitrary formulas and those comprised of disjunctive clauses only. Additionally, we examine abductive reasoning w.r.t. two different entailment relations: the standard one (*truth-preserving*) — 'if the premise has value 1, then so does the conclusion', and the *degree-preserving* entailment (Font et al. 2006) — 'the truth degree of the premise is at least as high as that of the conclusion'.

We demonstrate that the complexity of abduction in Łukasiewicz logic with truth-preserving entailment coincides with that of classical propositional abductive reasoning in the general case (cf. Table 1 for a summary of results). We then consider several disjunctive-clause fragments of Łukasiewicz logic. In particular, we investigate the complexity of abduction in the *simple clause fragment* (where clauses are disjunctions of variables and their negations) and of various *interval clause* fragments (where clauses are disjunctions of *interval literals*). We establish that the complexity of abductive reasoning in simple clause theories coincides with that of classical Horn abduction. This result (Theorems 5.7–5.10) exhibits a contrast with classical logic, where abductive reasoning with clause theories has the same complexity as abduction with arbitrary theories.

Moreover, we show that Łukasiewicz abduction in interval-clause theories is reducible to classical abduction in clause theories (Theorem 5.14). Importantly, this allows us to apply classical reasoning methods to generate solutions to abduction problems with interval-clause theories. We then use this result to establish the complexity of abductive reasoning in various subclasses of interval-clause theories (cf. Tables 2 and 3). In particular, we consider abduction interval-clause theories that correspond to classical (definite) Horn and IHS-B^{-} (Creignou and Zanuttini 2006; Khanna et al. 2002; Nordh and Zanuttini 2008) theories.

Finally, we expand our complexity evaluations onto Łukasiewicz logic with degree-preserving entailment. We prove that the complexity of abduction in the general case (Theorems 6.7–6.11) as well as in the case of interval-clause theories (Corollary 6.12) coincides with that of the truth-preserving entailment. We also obtain upper bounds on the complexity of abduction in simple clause theories under the degree-preserving entailment and show that it is at most as hard as simple clause abduction under the truth-preserving entailment (Theorems 6.15 and 6.16).

This paper is an extended version of an earlier conference submission (Inoue and Kozhemiachenko 2025). The main novel contribution is the study of the complexity of abduction in Łukasiewicz logic with *degree-preserving entailment* (Section 6). In addition, we include the proofs omitted from the conference version of the paper, consider the complexity of Ł-abduction in additional disjunctive-clause fragments, and expand the discussion.

Plan of the Paper. The paper is structured as follows. In Section 2, we present the Łukasiewicz logic and its expansion with interval literals. We also recall its important semantical and computational properties. In Section 3, we formulate the notions of abduction problems and solutions and outline a procedure for solution generation to Łukasiewicz logic abduction problems. Sections 4 and 5 are dedicated to the complexity of abductive reasoning in

Łukasiewicz logic and its clause fragments. In Section 6, we consider abductive reasoning in Łukasiewicz logic with degree-preserving entailment. Finally, in Section 7, we summarise results of the paper and outline the plan for future research.

2 Łukasiewicz Logic With Rational Interval Literals

In this section, we present Łukasiewicz logic with interval literals ($Ł^{\mathbb{Q}}$). We will also discuss its semantics and complexity. Let us now provide the syntax and semantics of $Ł^{\mathbb{Q}}$.

Definition 2.1. We let Pr to be a fixed set of propositional variables, $\diamond \in \{\leq, <, \geq, >\}$, and $c \in [0, 1]$ be a *rational number*. The language $\mathcal{L}^{\mathbb{Q}}_{Ł}$ is defined via the following grammar

$$\phi \coloneqq p \in \mathsf{Pr} \mid p \diamond \overline{\mathsf{c}} \mid \neg\phi \mid (\phi \odot \phi) \mid (\phi \oplus \phi) \mid (\phi \rightarrow \phi)$$

In the grammar above, $\overline{\mathsf{c}}$ is a *rational constant*, and $p \diamond \overline{\mathsf{c}}$ is a *(rational) interval literal*.

We define $\mathcal{L}_{Ł}$ to be the fragment of $\mathcal{L}^{\mathbb{Q}}_{Ł}$ *without* interval literals. Similarly, we will use Ł to denote the fragment of $Ł^{\mathbb{Q}}$ without interval literals.

Before proceeding to the semantics of $Ł^{\mathbb{Q}}$, let us introduce the notation we will utilise in this paper.

Convention 1 (Notation). We use the following shorthands:

$$\top \coloneqq p \oplus \neg p \qquad \bot \coloneqq p \odot \neg p \qquad \phi \leftrightarrow \chi \coloneqq (\phi \rightarrow \chi) \odot (\chi \rightarrow \phi)$$

Additionally, given an interval literal $p \diamond \overline{\mathsf{c}}$, we use $p \blacklozenge \overline{\mathsf{c}}$ to denote $\neg(p \diamond \overline{\mathsf{c}})$.

For a set of formulas Γ and a formula ϕ, we write $\mathsf{Pr}(\phi)$ and $\mathsf{Pr}[\Gamma]$ to denote the set of all variables occurring in ϕ and Γ, respectively. Additionally, $\mathsf{lgth}(\phi)$ and $\mathsf{lgth}[\Gamma]$ denote the *lengths* of ϕ and Γ, respectively, i.e., the number of symbols needed to write them down. Given a set X, $|X|$ denotes the number of elements in it.

We use $\mathbb{R}$ and $\mathbb{Q}$ to denote the sets of real and rational numbers, respectively. When dealing with intervals, square brackets mean that the endpoint is included in the interval, and round brackets that it is excluded. Lower index $_{\mathbb{Q}}$ means that the interval contains rational numbers only. E.g.:

$$[1/2, 2/3] = \{x \mid x \in \mathbb{R}, x \geq 1/2, x \leq 2/3\} \qquad (1/2, 2/3]_{\mathbb{Q}} = \{x \mid x \in \mathbb{Q}, x > 1/2, x \leq 2/3\}$$

Let us now present the semantics of $Ł^{\mathbb{Q}}$.

Definition 2.2 (Semantics of Łukasiewicz logic). An Ł-*valuation* is a function $v : \mathsf{Pr} \rightarrow [0, 1]$ extended to the complex formulas as follows:

$$v(\neg\phi) = 1 - v(\phi) \qquad v(p \diamond \overline{\mathsf{c}}) = \begin{cases} 1 & \text{if } v(p) \diamond c \\ 0 & \text{otherwise} \end{cases}$$

$$v(\phi \odot \chi) = \max(0, v(\phi) + v(\chi) - 1) \quad v(\phi \oplus \chi) = \min(1, v(\phi) + v(\chi)) \quad v(\phi \rightarrow \chi) = \min(1, 1 - v(\phi) + v(\chi))$$

For a rational interval literal $p \diamond \overline{\mathsf{c}}$, we call $\overline{\mathsf{c}}$ its *boundary value*. The set of *permitted values* of $p \diamond \overline{\mathsf{c}}$ ($\mathsf{V}(p \diamond \overline{\mathsf{c}})$) is defined as follows.

$$\mathsf{V}(p \diamond \overline{\mathsf{c}}) = \{v(p) \mid v(p \diamond \overline{\mathsf{c}}) = 1\}$$

We say that $\phi \in \mathcal{L}^{\mathbb{Q}}_{Ł}$ is Ł-*valid* ($Ł \models \phi$) if $v(\phi) = 1$ for every Ł-valuation v; ϕ is Ł-*satisfiable* if $v(\phi) = 1$ for some Ł-valuation v.

We define two notions of equivalence — *strong equivalence* ($\phi \equiv_{Ł} \chi$) and *weak equivalence* ($\phi \simeq_{Ł} \chi$):

$$\phi \equiv_{Ł} \chi \text{ iff } \forall v : v(\phi) = v(\chi) \qquad \phi \simeq_{Ł} \chi \text{ iff } \forall v : v(\phi) = 1 \leftrightarrows v(\chi) = 1$$

Given a finite $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{Ł}^{\mathbb{Q}}$, we say that Γ *entails* χ *in* Ł ($\Gamma \models_{Ł} \chi$) iff $v(\chi) = 1$ in every v such that $v(\phi) = 1$ for all $\phi \in \Gamma$, and that Γ *consistently entails* χ *in* Ł ($\Gamma \models_{Ł}^{\mathsf{cons}} \chi$) iff $\Gamma \not\models_{Ł} \bot$ and $\Gamma \models_{Ł} \chi$. We will call $\models_{Ł}$ the *truth-preserving* entailment.

Let us note important semantical properties of $Ł^{\mathbb{Q}}$.

Remark 1 (Semantical properties of Łukasiewicz logic). First, we observe that every connective *behaves classically on* $\{0, 1\}$: in particular, $\oplus$ behaves like disjunction and $\odot$ like conjunction. Thus, we will call $\oplus$ *strong disjunction* and $\odot$ *strong conjunction.*

Second, *deduction theorem does not hold for* $\models_{Ł}$: indeed, while $p \models_{Ł} p \odot p$, it is easy to show that $Ł \not\models p \rightarrow (p \odot p)$ by setting $v(p) = \frac{1}{2}$. Third, $\odot$ and $\oplus$ *are not idempotent*: setting $v(p) = \frac{1}{2}$, we have $v(p \odot p) = 0$ and $v(p \oplus p) = 1$.

Still, $\neg$, $\odot$, $\oplus$, and $\rightarrow$ interact in an expected manner. In particular, it is easy to check that $v(\top) = 1$ and $v(\bot) = 0$ for every valuation and that the following pairs of formulas are indeed *strongly equivalent*:

$$\begin{aligned} \neg(\phi \odot \chi) &\equiv_{Ł} \neg\phi \oplus \neg\chi & \neg(\phi \oplus \chi) &\equiv_{Ł} \neg\phi \odot \neg\chi \\ \neg(\phi \rightarrow \chi) &\equiv_{Ł} \phi \odot \neg\chi & \neg\phi \oplus \chi &\equiv_{Ł} \phi \rightarrow \chi \\ \neg\neg\phi &\equiv_{Ł} \phi & (\phi \odot \chi) \rightarrow \psi &\equiv_{Ł} \phi \rightarrow (\chi \rightarrow \psi) \end{aligned} \tag{1}$$

Furthermore, one can use Definition 2.2 to see that $\phi \leftrightarrow \chi$ corresponds to 'the values of ϕ and χ are not too different':

$$v(\phi \leftrightarrow \chi) = 1 - |v(\phi) - v(\chi)| \tag{2}$$

It is also important to observe that *weak* conjunction ($\wedge$) and disjunction ($\vee$) are definable as follows:

$$\phi \vee \chi \coloneqq (\phi \rightarrow \chi) \rightarrow \chi \qquad\qquad \phi \wedge \chi \coloneqq \neg(\neg\phi \vee \neg\chi) \tag{3}$$

Using Definition 2.2 and (3), one can derive their semantics:

$$v(\phi \wedge \chi) = \min(v(\phi), v(\chi)) \qquad\qquad v(\phi \vee \chi) = \max(v(\phi), v(\chi))$$

As one can see, $\wedge$ and $\vee$ satisfy De Morgan laws: $\neg(\phi \wedge \chi) \equiv_{Ł} \neg\phi \vee \neg\chi$ and $\neg(\phi \vee \chi) \equiv_{Ł} \neg\phi \wedge \neg\chi$. Note, however, that $\neg\phi \vee \chi$ cannot be treated as an implication because modus ponens fails: $\neg\phi \vee \chi, \phi \not\models_{Ł} \chi$ if $v(\phi) = v(\chi) = \frac{1}{2}$.

In what follows, we will write $\Gamma, \phi \models_{Ł} \chi$ as a shorthand for $\Gamma \cup \{\phi\} \models_{Ł} \chi$. Similarly, if the set of premises is given explicitly, we omit brackets. E.g., $\phi, \chi \models_{Ł} \psi$ stands for $\{\phi, \chi\} \models_{Ł} \psi$. Note, furthermore, that the comma in the set of premises can be equivalently interpreted as $\wedge$ and $\odot$, respectively, because $(\phi \wedge \chi) \simeq_{Ł} (\phi \odot \chi)$. Hence, for $\Gamma = \{\phi_1, \ldots, \phi_n\}$, we have

$$\Gamma \models_{Ł} \chi \text{ iff } \bigodot_{i=1}^{n} \phi_i \models_{Ł} \chi \text{ iff } \bigwedge_{i=1}^{n} \phi_i \models_{Ł} \chi \tag{4}$$

We finish the section with a discussion of interval literals. The idea of interval literals comes from a logic first introduced by Pavelka (1979a,b,c) as an extension of Ł with constants for every *real number*. It turns out, however (cf. (Hájek 1998, §3.3) for details) that if one adds constants only for *rational* numbers, the resulting logic ('Rational Pavelka logic' or RPL) will have the same expressivity as the original Pavelka's logic and as Łukasiewicz logic. In particular, for every set of RPL-formulas $\Gamma \cup \{\chi\}$, there is $\Gamma' \cup \{\chi'\} \subseteq \mathcal{L}_{Ł}$ such that $\Gamma \models_{Ł} \chi$ iff $\Gamma' \models_{Ł} \chi'$ (Hájek 1998, Lemmas 3.3.11 and 3.3.13). Note, however, that in the general case, the length of Γ' is not polynomially bounded.

To simulate the two-valued behaviour of interval terms, one also needs to introduce the 'Delta operator' $\triangle$ proposed by Baaz (1996) with the following semantics: $v(\triangle\phi) = 1$ if $v(\phi) = 1$ and $v(\triangle\phi) = 0$, otherwise.[1] Now,

[1]Note furthermore that given an Ł-valuation v, $v(\triangle\phi) = \mathbf{H}(v(\phi) - 1)$ where $\mathbf{H}$ is the Heaviside step function.

using the following equivalences (as expected, $v(\overline{c}) = c$):

$$p \leq \overline{c} \equiv_{Ł} \triangle(p \to \overline{c}) \qquad p \geq \overline{c} \equiv_{Ł} \triangle(\overline{c} \to p) \qquad p < \overline{c} \equiv_{Ł} \neg\triangle(\overline{c} \to p) \qquad p > \overline{c} \equiv_{Ł} \neg\triangle(p \to \overline{c}) \tag{5}$$

and expanding the constraint tableaux by Hähnle (1999) with rules for $\triangle$, we can obtain the complexity of $Ł^{\mathbb{Q}}$.[2]

Proposition 2.3.

(1) Ł*-satisfiability of* $\mathcal{L}_{Ł}^{\mathbb{Q}}$*-formulas is* NP*-complete.*

(2) Entailment in Ł *of* $\mathcal{L}_{Ł}^{\mathbb{Q}}$*-formulas is* coNP*-complete.*

Proof sketch. It is known from (Mundici 1987, Theorem 3.4) and (Haniková 2011, Corollary 4.1.3) that satisfiability and entailment of $\mathcal{L}_{Ł}$*-formulas* (i.e., formulas that do not contain interval literals) are NP- and coNP-complete, respectively. Thus, we obtain hardness.

For NP and coNP membership, we expand the constraint tableaux from (Hähnle 1999) with additional rules for $\triangle$ (cf., e.g., (Bílková et al. 2025)) as shown below (vertical bars stand for the branching, P is a linear polynomial with integer coefficients and variables ranging over $[0,1]$, and y is a fresh variable ranging over $[0,1]$).[3]

$$\neg_{\leq}: \frac{\neg\phi \leq P}{\phi \geq 1 - P} \qquad \to_{\leq}: \frac{\phi \to \chi \leq P}{P \geq 1 \;\Bigg|\; \begin{matrix}\phi \geq 1 - P + y\\ \chi \leq y\end{matrix}} \qquad \triangle_{\geq}: \frac{\triangle\phi \geq P}{P \leq 0 \;\Bigg|\; \begin{matrix}\phi \geq y\\ y \geq 1\end{matrix}}$$

$$\neg_{\geq}: \frac{\neg\phi \geq P}{\phi \leq 1 - P} \qquad \to_{\geq}: \frac{\phi \to \chi \geq P}{\begin{matrix}\phi \leq 1 - P + y\\ \chi \geq y\end{matrix}} \qquad \triangle_{\leq}: \frac{\triangle\phi \leq P}{P \geq 1 \;\Bigg|\; \begin{matrix}\phi \leq y\\ y < 1\end{matrix}}$$

As expected, a *constraint tableau* is a downward branching tree containing constraints of the form $\psi \leq P$, $\psi \geq P$, $P \leq P'$, and $P < P'$ with ψ being a formula and P and P' being linear polynomials. Each branch can be extended by one of the rules above.

Now, given $\phi \in \mathcal{L}_{Ł}^{\mathbb{Q}}$, we first eliminate interval literals by applying equivalences from (5). Then, using equivalences from (1), we replace subformulas $\chi \oplus \psi$ and $\chi \odot \psi$ with $\neg\chi \to \psi$ and $\neg(\chi \to \neg\psi)$, respectively. This produces a formula $\phi^{\triangle}$ that is strongly equivalent to ϕ such that $\mathsf{lgth}(\phi^{\triangle}) = O(\mathsf{lgth}(\phi))$. Then we construct a constraint tableau starting with $\{\phi^{\triangle} \geq 1\}$. Each branch of the tableau gives rise to a system of linear inequalities if we treat occurrences of formulas as variables and replace every rational constant $\overline{c}$ with its corresponding rational number c. We call a branch *open* if its corresponding system of inequalities has a solution over $[0,1]$ and *closed* if it does not. A branch is *complete* if for each premise of a rule that appears on it, at least one conclusion of that rule also belongs to the branch. One can easily check that $\phi \in \mathcal{L}_{Ł}^{\mathbb{Q}}$ is Ł-satisfiable iff a tableau beginning with $\{\phi^{\triangle} \geq 1\}$ has at least one complete open branch. Indeed, observe that a solution over $[0,1]$ to the system of inequalities corresponding to a branch induces a satisfying Ł-valuation for ϕ.

Furthermore, every rule decomposes the labelled formula $\chi \leq P$ ($\chi \geq P$) in the premise into its immediate subformulas, and expands the polynomial P with at most one new variable and one new constant. Thus, the length of the conclusion of a rule is larger than the length of the premise only by a constant number of symbols. As we can apply a tableau rule to a given labelled formula only once, it follows that the number of applications

[2]To the best of our knowledge, the next statement has not been explicitly proven in the literature. Note, however, that it is a straightforward combination of the NP- and coNP-completeness of RPL and the expansion of Ł with $\triangle$ (Běhounek et al. 2011, §4.4). We present a proof to keep the paper self-contained.

[3]We note briefly that it is possible to expand the original constraint tableaux calculus with derived rules for $\odot$ and $\oplus$, as well as rules for interval literals instead of $\triangle$. We are choosing a more compact presentation here since we only need tableaux to demonstrate the (expected) complexity result.

of rules to $\chi \geq P$ or $\chi \leq P$ is bounded by the number of subformulas in χ. Hence, the size of each branch is polynomial in the length of χ.

Thus, to check that ϕ is Ł-satisfiable, we guess a complete branch of the tableau starting with $\{\phi^{\triangle}{\geq}1\}$ and solve its system of inequalities over $[0, 1]$. This gives us the required NP procedure. For the coNP-membership, we proceed as follows. Given Γ and χ, we eliminate interval literals, $\odot$, and $\oplus$ and then construct a tableau for $\{\phi^{\triangle}{\geq}1 \mid \phi{\in}\Gamma\} \cup \{\chi^{\triangle}{\leq}y, y{<}1\}$ with y being a fresh variable. Note that since our rules are given for constraints with $\leq$ and $\geq$, $\chi^{\triangle} < 1$ cannot appear in the tableau. Thus, we need to introduce a new variable. It is clear that *all branches* of this tableau are *closed* iff $\Gamma \models_{Ł} \chi$. □

3 Abduction in Łukasiewicz Logic

In this section, we present our framework for abduction in Ł. We begin by considering solutions to abduction problems (explanations). A solution τ is often somehow restricted because not every formula can be intuitively accepted as an explanation. Traditional restrictions are usually as follows (cf. discussion in (Marquis 2000, §4.2) or (Aliseda 2006, §3.3)). First, τ can be represented as a conjunction of literals (*term*) formed from a pre-defined set of permitted hypotheses (abducibles). This can be interpreted as collecting abducibles that jointly explain the observation. Furthermore, it can be required that τ does not entail χ by itself. In other words, χ can only follow from Γ *together with* τ. Aliseda (2006) calls such τ *explanatory*; in other presentations (Bienvenu et al. 2024, 2026; Inoue and Kozhemiachenko 2025) and in this paper, such explanations are called *proper*. Another usual restriction is that τ should only contain atoms occurring in $\Gamma \cup \{\tau\}$. Observe, however, that some representations of abduction (e.g., *creative abduction* (Schurz 2008) and *meta-level abduction* (Inoue 2016)) allow for solutions that use new atoms. Additionally, one can demand that τ should constitute the weakest possible (or minimal) explanation. If solutions are represented as terms, the weakest solutions w.r.t. entailment in classical logic are the subset-minimal ones.

3.1 Interval Terms

In our framework, we will consider explanations in the form of conjunctions of hypotheses. In Łukasiewicz logic, one can introduce the analogues of classical propositional logic (CPL) terms and clauses (*disjunctions* of literals). The difference with the classical notion is that we use $\odot$ and $\oplus$ instead of $\wedge$ and $\vee$.

Definition 3.1 (Simple literals, clauses, and terms).

- A *simple literal* is a propositonal variable or its negation.
- A *simple clause* is a *strong disjunction* of simple literals, i.e., a formula of the form $\bigoplus_{i=1}^{n} l_i$ for some $n \in \mathbb{N}$.
- A *simple term* is a *strong conjunction* of simple literals, i.e., a formula of the form $\bigodot_{i=1}^{n} l_i$ for some $n \in \mathbb{N}$.

One can observe, however, that *simple terms* from Definition 3.1 are too restrictive if we want to use them as solutions for abduction problems. Indeed, a simple term τ has value 1 iff all its literals have value 1. But in a context with fuzzy propositions, we might have statements such as 'p has value $\frac{2}{3}$' or 'q has value at least $\frac{1}{4}$'. Furthermore, sometimes, we might express a *proportion* or a *difference* between values: e.g., 'the value of p is twice as large as the value of q' or 'the values of p and q differ by $\frac{1}{3}$'. In these cases, it might not be enough to assume that some statements are (absolutely) true or false to explain an observation. The following examples illustrate such contexts.

Example 3.2. There is a lift with a weight sensor that controls two indicators: green and blue. The green indicator is on when the weight sensor detects the load of *at least* $\frac{1}{4}$ of its maximal capacity. Blue light is on when the lift is loaded to *at most* $\frac{2}{3}$ of its maximal capacity. We see that both indicators are lit.

Let us now formalise this problem in Ł. We interpret the value of c as the percentage of capacity to which the lift is loaded and translate the condition 'the lift is loaded to at least $1/4$ of its capacity' as $c \oplus c \oplus c \oplus c$. Observe that

$v(c \oplus c \oplus c \oplus c) = 1$ iff $v(c) \geq 1/4$. To represent the other condition 'the lift is loaded to at most $2/3$ of its capacity', we write $\neg c \oplus \neg c \oplus \neg c$. We have that $v(\neg c \oplus \neg c \oplus \neg c) = 1$ iff $v(\neg c) \geq 1/3$, i.e., iff $v(c) \leq 2/3$. Finally, we use g and b to represent that the green and blue lights are on. We obtain the following theory Γ_{lift} and observation χ_{lift}:

$$\Gamma_{\mathsf{lift}} = \{(c \oplus c \oplus c \oplus c) \rightarrow g, (\neg c \oplus \neg c \oplus \neg c) \rightarrow b\} \qquad \chi_{\mathsf{lift}} = g \odot b$$

To explain why both indicators are on, we need to present a formula ϕ such that $\Gamma_{\mathsf{lift}}, \phi \models^{\mathsf{cons}}_{\text{Ł}} g \odot b$. For this, we need that $v(\neg c \oplus \neg c \oplus \neg c) = 1$ and $v(c \oplus c \oplus c \oplus c) = 1$ in every valuation that makes Γ_{lift} true. As we noted above, this requires that $v(c) \in [\frac{1}{4}, \frac{2}{3}]$. On the other hand, a simple term τ has value 1 iff all its literals have value 1 (i.e., all variables in τ should have value 0 or 1). Thus, there is no simple term τ containing c or $\neg c$ such that $\Gamma_{\mathsf{lift}}, \tau \models^{\mathsf{cons}}_{\text{Ł}} g \odot b$. Hence ϕ cannot be a simple term.

Example 3.3. There is a ship with two cargo compartments (port and starboard) being loaded. The cargo should be balanced evenly between two compartments (the load difference between compartments should be at most 20%), loaded to at least 50% of total capacity, but not overloaded (at most 75% of the total capacity). If both conditions are fulfilled, the ship is loaded correctly and is clear to sail. Upon inspection, it is concluded that the ship is indeed loaded correctly.

This context can be formalised in Ł as follows. The values of p and s are the loads of the port and starboard compartments, respectively. The value of d corresponds to the *imbalance* (load difference) between the compartments, and the value of l to the total load. Using (2) and Definition 2.2, we write $\neg(p \leftrightarrow s) \leftrightarrow d$ to express that the imbalance is the difference between the loads and $(p \oplus s) \leftrightarrow l$ to express that the total load is the sum of the loads of both compartments. Indeed, one can see that $v(\neg(p \leftrightarrow s) \leftrightarrow d) = 1$ iff $v(d) = |v(p) - v(s)|$ and $v((p \oplus s) \leftrightarrow l) = 1$ iff $v(p) + v(s) = v(l)$ or $\min(v(p) + v(s), v(l)) \geq 1$. We will use c to denote that the ship was cleared to sail.

Now, it remains to express that $v(d) \leq \frac{1}{5}$ and $\frac{1}{2} \leq v(l) \leq \frac{3}{4}$. To do that, we can encode rational numbers with Łukasiewicz logic formulas using the method from (Kozhemiachenko and Sedlár 2025, Proposition 3).[4] The encoding works as follows. Given $\frac{m}{n} \in (0,1)_{\mathbb{Q}}$, we proceed in two stages. First, we introduce a fresh variable q and write a formula $\psi_{1/n}(q)$ s.t. $v(\psi_{1/n}) = 1$ iff $v(q) = \frac{1}{n}$. Second, (if $m > 1$) we introduce another fresh variable q' and write a formula $\psi_{m/n}(q, q')$ s.t. $v(\psi_{m/n}) = 1$ iff $v(q') = m \cdot v(q')$.

$$\psi_{1/n} \coloneqq \neg q \leftrightarrow (\underbrace{q \oplus \ldots \oplus q}_{n-1 \text{ times}}) \qquad \psi_{m/n} = q' \leftrightarrow (\underbrace{q \oplus \ldots \oplus q}_{m \text{ times}}) \tag{6}$$

Now note from (6) and Definition 2.2 that $v(\psi_{1/n} \odot \psi_{m/n}) = 1$ iff $v(q) = \frac{1}{n}$ and $v(q') = \frac{m}{n}$.

Thus, to formalise our context, we need auxiliary formulas $\psi_{1/5}$, $\psi_{1/2}$, $\psi_{1/4}$, and $\psi_{3/4}$:

$$\psi_{1/5} = \neg d' \leftrightarrow (d' \oplus d' \oplus d' \oplus d') \quad \psi_{1/2} = l' \leftrightarrow \neg l' \quad \psi_{1/4} = (\neg l^* \leftrightarrow (l^* \oplus l^* \oplus l^*)) \quad \psi_{3/4} = (l'' \leftrightarrow (l^* \oplus l^* \oplus l^*))$$

This gives us the following theory Γ_{ship} and observation χ_{ship}:

$$\Gamma_{\mathsf{ship}} = \{\neg(p \leftrightarrow s) \leftrightarrow d, (p \oplus s) \leftrightarrow l, \psi_{1/5}, \psi_{1/2}, \psi_{1/4}, \psi_{3/4}, ((d \rightarrow d') \odot (l \rightarrow l'') \odot (l' \rightarrow l)) \rightarrow c\}$$
$$\chi_{\mathsf{ship}} = c$$

To make the decision, the inspector should have checked the load in both compartments. Formally, we need a formula ϕ such that $\mathrm{Pr}(\phi) = \{p, s\}$ and $\Gamma_{\mathsf{ship}}, \phi \models^{\mathsf{cons}}_{\text{Ł}} c$. For this we need that $|v(p) - v(s)| \leq \frac{1}{5}$ and $\frac{1}{2} \leq v(p) + v(s) \leq \frac{3}{4}$. As neither p nor s can have value 0 or 1, no simple term over p and s can be an explanation of c. On the other hand, the pair of values $v(p) = \frac{3}{8}$ and $v(s) = \frac{2}{8}$ suits our purpose.

[4]One can also use an alternative (though a bit less concise) encoding by Hájek (1998, Lemma 3.3.11). Note, furthermore, that one can encode rational numbers using interval literals as shown in Section 5.2. In these examples, however, we show that even if the theory *does not* contain interval literals, it may still happen that we cannot explain the observation using only *simple literals*.

One way to circumvent the fact that simple terms are not sufficiently expressive to solve abduction problems formulated in Ł is to adopt the proposal of Yamada and Mukaidono (1995) and Vojtáš (1999) and define solutions to fuzzy abduction problems as fuzzy sets, i.e., assignments of values to propositional variables. This approach, however, has a drawback. In this setting, one can only express exact values of variables but not *intervals* of their values. Note, however, that we cannot compare two solutions σ and τ w.r.t. Łukasiewicz logic entailment unless one uses more hypotheses than the other. Now, observe from Example 3.2 that any assignment of a value from $[\frac{1}{4}, \frac{2}{3}]$ to c explains χ_{lift} relative to Γ_{lift}. Similarly (cf. Example 3.3), any assignment of a pair of values from $[\frac{1}{4}; \frac{3}{8}]$ to p and s explains χ_{ship} relative to Γ_{ship}.

In this section, we propose an alternative. For that, we define terms that allow us to express both exact values of variables and their intervals and compare different solutions w.r.t. entailment in Ł. In other words, our solutions can be interpreted as *interval-valued* fuzzy sets (Bouchet et al. 2023; Couso and Dubois 2014).[5] Moreover, as we will see, every abduction problem will have only finitely many solutions in our setting, and we will be able to meaningfully select the weakest possible solutions w.r.t. entailment in Ł.

Definition 3.4 (Interval terms and clauses).

- A *(rational) interval term* has the form $\bigodot_{i=1}^{n} \lambda_i$ with each λ_i being an interval literal.
- A *(rational) interval clause* has the form $\bigoplus_{i=1}^{n} \lambda_i$ with each λ_i being an interval literal.

We observe that it follows from (1) that interval clauses and terms are dual in the following sense:

$$\neg \bigodot_{i=1}^{n} (p_i \diamond \overline{\mathsf{c}}_i) \equiv_{\text{Ł}} \bigoplus_{i=1}^{n} (p_i \blacklozenge \overline{\mathsf{c}}_i) \qquad\qquad \neg \bigoplus_{i=1}^{n} (p_i \diamond \overline{\mathsf{c}}_i) \equiv_{\text{Ł}} \bigodot_{i=1}^{n} (p_i \blacklozenge \overline{\mathsf{c}}_i) \tag{7}$$

Remark 2. From Definition 3.4, it is clear that interval terms generalise simple terms in the following sense: for every simple term τ, there is an interval term τ' such that $\tau \simeq_{\text{Ł}} \tau'$. Namely, given $\tau = \bigodot_{i=1}^{m} p_i \odot \bigodot_{j=1}^{n} \neg q_j$, we can define $\tau' = \bigodot_{i=1}^{m} (p_i \geq \overline{1}) \odot \bigodot_{j=1}^{n} (q_j \leq \overline{0})$. Moreover, entailment between interval literals behaves as expected: given two interval literals $p \diamond \overline{\mathsf{c}}$ and $p \diamond \overline{\mathsf{d}}$, $p \diamond \overline{\mathsf{c}} \models_{\text{Ł}} p \diamond \overline{\mathsf{d}}$ iff $\mathsf{V}(p \diamond \overline{\mathsf{c}}) \subseteq \mathsf{V}(p \diamond \overline{\mathsf{d}})$ (recall Definition 2.2). We note briefly that interval terms allow us to express not only the values of variables but also of arbitrary formulas (cf. (Flaminio 2007) for an alternative approach). For $\phi \in \mathcal{L}_{\text{Ł}}$ and $p \in \mathsf{Pr}$ such that $p \notin \mathsf{Pr}(\phi)$, $c \in [0, 1]_{\mathbb{Q}}$, and $\diamond \in \{\leq, <, \geq, >\}$, we have $v(\phi) \diamond c$ iff $v((\phi \leftrightarrow p) \odot (p \diamond \overline{\mathsf{c}})) = 1$.

The following statement establishes the complexity of the entailment of interval terms.

Proposition 3.5. *Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\text{Ł}}^{\mathbb{Q}}$ be finite and τ and τ' be interval terms. Then the following statements hold.*

(1) It takes polynomial time to decide whether $\tau \models_{\text{Ł}} \tau'$.
(2) It is coNP*-complete to decide whether $\tau \models_{\text{Ł}} \chi$.*
(3) It is coNP*-complete to decide $\Gamma, \tau \models_{\text{Ł}} \tau'$.*

Proof. We begin with Item 1. For an an interval term σ and $p \in \mathsf{Pr}(\sigma)$, set $\mathsf{V}(p, \sigma) = \bigcap_{p \diamond \overline{\mathsf{c}} \in \sigma} \mathsf{V}(p \diamond \overline{\mathsf{c}})$ (recall Definition 2.2 for $\mathsf{V}(p \diamond \overline{\mathsf{c}})$). One can straightforwardly verify that σ is Ł-satisfiable iff $\mathsf{V}(p, \sigma) \neq \varnothing$ for every $p \in \mathsf{Pr}(\sigma)$. Moreover, if σ is not satisfiable, then $v(\sigma) = 0$ for every v. One can see that it takes polynomial time to verify whether an interval term is Ł-satisfiable.

Now let $\sigma = \bigodot_{i=1}^{m} p_i \diamond \overline{\mathsf{c}}_i$ and $\tau = \bigodot_{i=1}^{n} q_i \diamond \overline{\mathsf{d}}_i$ be two interval terms. By Definition 2.2, we have that $\sigma \models_{\text{Ł}} \tau$ iff $\sigma \models_{\text{Ł}} q_i \diamond \overline{\mathsf{d}}_i$ for all $i \in \{1, \ldots, n\}$. In turn, $\sigma \models_{\text{Ł}} q_i \diamond \overline{\mathsf{d}}_i$ iff $\sigma, q_i \blacklozenge \overline{\mathsf{d}}_i \models_{\text{Ł}} \bot$, i.e., iff $\sigma \odot q_i \blacklozenge \overline{\mathsf{d}}_i$ is Ł-unsatisfiable.

[5]The difference between 'standard' and interval-valued fuzzy sets is (as the name suggests) that in the first case statements $x \in X$ are evaluated as *numbers* from $[0, 1]$, and in the second case as *closed intervals* in $[0, 1]$. Note, however, that interval literals can also represent *open* intervals in $[0, 1]$.

As we have shown in the previous paragraph, it can be decided in polynomial time whether an interval term is Ł-satisfiable. It follows that Ł-entailment between terms is also polynomially decidable.

For Items 2 and 3, the membership is evident from Proposition 2.3. Hardness in Item 2 can be obtained by the following reduction from Ł-validity, which is coNP-complete by Proposition 2.3. Namely, $\chi \in \mathcal{L}_Ł^{\mathbb{Q}}$ is Ł-valid (i.e., is entailed by the empty set of formulas) iff $p{\geq}\overline{1} \models_Ł \chi$ for some $p \notin \mathsf{Pr}(\chi)$. The hardness part of Item 3 is obtained by a reduction from Ł-unsatisfiability. Let $\Gamma = \{\phi\}$. It is clear that ϕ is Ł-unsatisfiable iff $\phi, p{\geq}\overline{1} \models q{\geq}\overline{1}$ for some $p, q \notin \mathsf{Pr}(\phi)$. □

Finally, we observe from Proposition 3.5 that (as expected) the strength of an interval term w.r.t. Łukasiewicz logic is, in a sense, *inversely proportional* to the size of intervals of permitted values of its variables. That is, the larger the intervals of permitted values are, the weaker the term is. The next statement formalises this intuition.

Corollary 3.6. *Let σ and τ be two interval terms. Then $\sigma \models_Ł \tau$ iff $\mathsf{V}(p, \sigma) \subseteq \mathsf{V}(p, \tau)$ for every $p \in \mathsf{Pr}(\tau)$.*

3.2 Abduction Problems

We are now ready to present abduction in Ł. Our idea is to use interval terms as solutions to problems of the form $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$. Here, Γ is a theory that encodes our background knowledge, χ is the observation we want to explain, and H is the set of interval literals (hypotheses) that one can use to build explanations (solutions). One can restrict H in two ways. First, one may allow *arbitrary* interval literals over a given *finite* set of variables. Second, one can explicitly define a finite set of interval literals. We choose the second option, as the first one leads to *infinite* sets of solutions. For instance, in Example 3.2, allowing *any* interval literal over c will result in $\Gamma_{\mathsf{lift}}, (c \geq \overline{\mathrm{d}}) \odot (c \leq \overline{\mathrm{d}}') \models_Ł^{\mathsf{cons}} \chi_{\mathsf{lift}}$ whenever $d, d' \in [\frac{1}{4}, \frac{2}{3}]_{\mathbb{Q}}$ and $d \leq d'$.

Definition 3.7 (Ł-abduction problems and solutions).

- An Ł-*abduction problem* is a tuple $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ with $\Gamma \cup \{\chi\}$ a finite set of $\mathcal{L}_Ł^{\mathbb{Q}}$-formulas, and H a finite set of *interval literals* such that $\mathsf{Pr}[\mathsf{H}] \subseteq \mathsf{Pr}[\Gamma \cup \{\chi\}]$. We call Γ a *theory*, χ an *observation*, and members of H *hypotheses*.
- An Ł-*solution* of $\mathbb{P}$ is an *interval term* τ composed of hypotheses such that $\Gamma, \tau \models_Ł^{\mathsf{cons}} \chi$.
- A solution τ is *proper* if $\tau \not\models_Ł \chi$.
- A proper solution τ is $\models_Ł$*-minimal (entailment-minimal)* if there is no proper solution σ such that $\tau \models_Ł \sigma$ and $\sigma \not\models_Ł \tau$.
- A proper solution τ is *theory-minimal* if there is no proper solution σ such that $\Gamma, \sigma \not\models_Ł \tau$ and $\Gamma, \tau \models_Ł \sigma$.

We will also use the notation from Convention 1 when dealing with abduction problems. Recall that we write $\mathsf{Pr}[\mathbb{P}]$ and $\mathsf{lgth}[\mathbb{P}]$ to denote the set of propositional variables that occur in $\mathbb{P}$ and the number of symbols in $\mathbb{P}$.

The next property of solutions follows immediately from Corollary 3.6 and Definition 3.7.

Proposition 3.8. *Let σ be a solution to an Ł-abduction problem $\mathbb{P}$. Then any interval term τ such that $\mathsf{V}(p, \tau) \subseteq \mathsf{V}(p, \sigma)$ and $\mathsf{V}(p, \tau) \neq \varnothing$ for all $p \in \mathsf{Pr}(\tau)$ is a solution. In particular, entailment-minimal solutions are those with the largest possible intervals of permitted values for every variable.*

Convention 2. Given an abduction problem $\mathbb{P}$, we will use $\mathcal{S}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$, $\mathcal{S}^{\min}(\mathbb{P})$, and $\mathcal{S}^{\mathsf{Th}}(\mathbb{P})$ to denote the sets of all solutions, all proper solutions, all $\models_Ł$-minimal solutions, and all theory-minimal solutions of $\mathbb{P}$, respectively.

From Definition 3.7, it is evident that there are finitely many (at most exponentially many in the number of hypotheses in H) solutions for each abduction problem. We also present two notions of minimal solutions: entailment-minimality and theory-minimality. Entailment-minimality is an analogue of *subset-minimality* from (Eiter and Gottlob 1995) (cf. (Aliseda 2006) for a similar notion) in the setting of Łukasiewicz logic. Theory-minimal solutions correspond to *least specific* solutions in the terminology of Sakama and Inoue (1995) and Stickel (1990) or *least* *presumptive* solutions in the terminology of Poole (1989) (cf. (Mayer and Pirri 1996) for a similar notion of the *explanans w.r.t. a criterion*). Note also (cf., e.g., (Inoue 2002; Marquis 2000; Reiter and Kleer 1987)) that (classical) entailment-minimal solutions can be interpreted as duals of *prime implicates*. Theory-minimal solutions in classical logic can also be seen as duals of *theory prime implicates* from (Marquis 1995).

Observe, furthermore, that even though a theory-minimal solution is entailment-minimal, the converse is not always the case. Indeed, let $\mathbb{P} = \langle\{p \rightarrow q, q \rightarrow r\}, r\rangle$. Then there are two entailment-minimal solutions: $p{\geq}\overline{1}$ and $q{\geq}\overline{1}$. Note, however, that $p \rightarrow q, p{\geq}\overline{1} \models_{\text{Ł}} q{\geq}\overline{1}$. Thus, $p{\geq}\overline{1}$ *is not theory-minimal*.

Let us now see how we can solve abduction problems using interval terms. Recall Examples 3.2 and 3.3.

Example 3.9. We continue Example 3.2. We need to formulate an abduction problem $\mathbb{P}_{\text{lift}} = \langle\Gamma_{\text{lift}}, \chi_{\text{lift}}, \mathsf{H}_{\text{lift}}\rangle$. Γ_{lift} and χ_{lift} are already given in Example 3.2. It remains to form the set of hypotheses we are allowed to use. Assume for simplicity that we can measure the load of our lift in twelfths of its capacity. Thus, we can set

$$\mathsf{H}_{\text{lift}} = \{c \diamond \overline{\tfrac{i}{12}} \mid \diamond \in \{\leq, \geq, <, >\} \text{ and } i \in \{0, \ldots, 12\}\}$$

It is now easy to check that $(c{\geq}\overline{\frac{3}{12}}) \odot (c{\leq}\overline{\frac{8}{12}})$ is indeed the only theory-minimal solution of $\mathbb{P}_{\text{lift}}$. Moreover, as expected,

$$\mathcal{S}(\mathbb{P}_{\text{lift}}) = \left\{(c{\triangleright}\overline{\tfrac{i}{12}}) \odot (c{\triangleleft}\overline{\tfrac{i'}{12}}) \;\middle|\; \begin{array}{c}\triangleleft \in \{\leq, <\},\ \triangleright \in \{\geq, >\}\\ i \geq 3,\ i' \leq 8,\ i < i'\end{array}\right\} \cup \{(c{\leq}\overline{\tfrac{i}{12}}) \odot (c{\geq}\overline{\tfrac{i}{12}}) \mid 3 \leq i \leq 8\}$$

That is, given any interval $\mathcal{D} \subseteq \left[\frac{1}{4}, \frac{2}{3}\right]$, every interval term τ such that $v(\tau) = 1$ iff $v(c) \in \mathcal{D}$ is a solution to $\mathbb{P}_{\text{lift}}$.

Example 3.10. Recall Example 3.3. Let us formulate $\mathbb{P}_{\text{ship}} = \langle\Gamma_{\text{ship}}, \chi_{\text{ship}}, \mathsf{H}_{\text{ship}}\rangle$. The theory and the observation are already given in Example 3.3, so we only need to specify H_{ship}. As the imbalance between the two compartments should be at most $\frac{1}{5}$ of the total load, and the total load in the two compartments should be at most $\frac{3}{4}$, we set

$$\mathsf{H}_{\text{ship}} = \left\{p \diamond \overline{\tfrac{i}{40}} \mid \diamond \in \{\leq, \geq, <, >\} \text{ and } i \in \{0, \ldots, 40\}\right\} \cup \left\{s \diamond \overline{\tfrac{i}{40}} \mid \diamond \in \{\leq, \geq, <, >\} \text{ and } i \in \{0, \ldots, 40\}\right\}$$

It is now easy to check that every interval term $(p \geq \overline{e}_1) \odot (p \leq \overline{e}_1) \odot (s \geq \overline{e}_2) \odot (s \leq \overline{e}_2)$ such that $\frac{1}{2} \leq e_1 + e_2 \leq \frac{3}{4}$ and $|e_1 - e_2| \leq \frac{1}{5}$ is a solution to $\mathbb{P}_{\text{lift}}$. Moreover, in contrast to the problem in Example 3.9, there are several theory-minimal solutions, for example:

$$\sigma_1 = (p{\geq}\overline{\tfrac{10}{40}}) \odot (p{\leq}\overline{\tfrac{15}{40}}) \odot (s{\geq}\overline{\tfrac{10}{40}}) \odot (s{\leq}\overline{\tfrac{15}{40}}) \qquad \sigma_2 = (p{\geq}\overline{\tfrac{8}{40}}) \odot (p{\leq}\overline{\tfrac{15}{40}}) \odot (s{\geq}\overline{\tfrac{12}{40}}) \odot (s{\leq}\overline{\tfrac{15}{40}})$$

Indeed, one can check that if we expand permitted values of p or s in σ_1 and σ_2, the resulting terms will not be solutions to $\mathbb{P}_{\text{ship}}$. In particular, if we let $\sigma_1' = (p{\geq}\overline{\frac{9}{40}}) \odot (p{\leq}\overline{\frac{15}{40}}) \odot (s{\geq}\overline{\frac{10}{40}}) \odot (s{\leq}\overline{\frac{15}{40}})$ and set $v(p) = \frac{9}{40}$ and $v(s) = \frac{10}{40}$, then $v((p \oplus s)) = \frac{19}{40}$ which means that $v(l) = \frac{19}{40} < \frac{1}{2}$ (i.e., the ship is *underloaded*). Thus, v can be extended to falsify $\Gamma_{\text{ship}}, \sigma_1' \models_{\text{Ł}} c$, for example, as follows: $v(d) = \frac{1}{40}$, $v(l) = \frac{19}{40}$, $v(c) = \frac{38}{40}$. Using Definition 2.2 and (2), one can check that indeed, $v(\phi) = 1$ for every $\phi \in \Gamma_{\text{ship}}$, but $v(c) < 1$. Furthermore, $\Gamma_{\text{ship}}, \sigma_1 \not\models_{\text{Ł}} \sigma_2$ and $\Gamma_{\text{ship}}, \sigma_2 \not\models_{\text{Ł}} \sigma_1$, as required.

It is also important to observe that the set of solutions depends not only on the variables in H but on their permitted values as well. Indeed, if in Example 3.9, we supposed that we can measure the load in *tenths* of the maximal capacity (i.e., if $\mathsf{H} = \{c \diamond \overline{\frac{i}{10}} \mid \diamond \in \{\leq, \geq, <, >\} \text{ and } i \in \{0, \ldots, 10\}\}$), the theory-minimal solution would be $(c{\geq}\overline{\frac{3}{10}}) \odot (c{\leq}\overline{\frac{6}{10}})$. Hence, to test a solution τ for minimality, we need to calculate the permitted value that is the nearest to the boundary value for every interval literal in τ.

3.3 Generating Solutions to Ł-abduction Problems

In practice, one is usually interested in *generating* solutions to a given abduction problem. In this section, we will discuss solution generation to abduction problems in Łukasiewicz logic. First, we recall the notion of the classical abduction problems and solutions. We adapt the definitions from (Eiter and Gottlob 1995) and (Creignou and Zanuttini 2006) for our notation. We use terminology and notation for CPL analogous to those introduced for Ł, e.g., speaking about CPL-validity and using $\models_{\mathsf{CPL}}$ for the classical entailment relation.

Definition 3.11. Let $\mathcal{L}_{\mathsf{CPL}}$ be the propositional language over $\{\neg,\wedge,\vee,\rightarrow\}$. We define

- an $\mathcal{L}_{\mathsf{CPL}}$-*term* to be a formula of the form $\bigwedge_{i=1}^{n} l_i$ where each l is a simple literal;
- an $\mathcal{L}_{\mathsf{CPL}}$-*clause* to be a formula of the form $\bigvee_{i=1}^{n} l_i$ where each l is a simple literal.

Definition 3.12. A *classical abduction problem* is a tuple $\mathbb{P}=\langle\Gamma,\chi,\mathsf{H}\rangle$ such that $\Gamma\cup\{\chi\}\subseteq\mathcal{L}_{\mathsf{CPL}}$ and H is a set of simple literals such that $\mathsf{Pr}[\mathsf{H}]\subseteq\mathsf{Pr}[\Gamma\cup\{\chi\}]$.

- A *solution* of $\mathbb{P}$ is an $\mathcal{L}_{\mathsf{CPL}}$-term τ composed of literals from H such that $\Gamma,\tau\models^{\mathsf{cons}}_{\mathsf{CPL}}\psi$.
- A solution τ is *proper* if $\tau\not\models_{\mathsf{CPL}}\psi$.
- A proper solution τ is $\models_{\mathsf{CPL}}$-*minimal*[6] if there is no proper solution ϕ such that $\tau\models_{\mathsf{CPL}}\phi$ and $\phi\not\models_{\mathsf{CPL}}\tau$.
- A proper solution τ is *theory-minimal* if there is no proper solution ϕ such that $\Gamma,\tau\models_{\mathsf{CPL}}\phi$ and $\Gamma,\phi\not\models_{\mathsf{CPL}}\tau$.

Among the best-known techniques applied to solution generation to classical abduction problems are those based on *consequence finding* via resolution, cf. (Inoue 2002, 1992; Marquis 2000, 1995; Val 1999, 2000). The general idea of such algorithms is as follows. Given $\mathbb{P}=\langle\Gamma,\chi,\mathsf{H}\rangle$, one aims to produce $\mathcal{L}_{\mathsf{CPL}}$-clauses composed from negated literals in H that are entailed by $\Gamma\cup\{\neg\chi\}$. The *negations* of these clauses are then used as candidates for solutions. Unfortunately, such approaches would not be straightforward to implement because they rely on the soundness of contraposition: $\Gamma,\tau\models_{\mathsf{CPL}}\chi$ iff $\Gamma,\neg\chi\models_{\mathsf{CPL}}\neg\tau$. In Łukasiewicz logic, however, contraposition is not sound: $\neg((p\rightarrow\neg p)\rightarrow\neg p)\models_{Ł} q$ but $\neg q\not\models_{Ł}(p\rightarrow\neg p)\rightarrow\neg p$. Another issue is that resolution techniques for fuzzy logics usually consider theories represented as sets of formulas of the form $\bigoplus_{i=1}^{n} l_i$ or $\bigvee_{i=1}^{n} l_i$ (recall (3) for the weak disjunction $\vee$) with l's being simple literals (Habiballa 2012; Mundici and Olivetti 1998). Unfortunately, such representation is insufficient in the general case as not every $\mathcal{L}_{Ł}$-formula is equivalent to a (weak or strong) conjunction of $\bigoplus_{i=1}^{n} l_i$'s or $\bigvee_{i=1}^{n} l_i$'s (Bofill et al. 2019; Nola and Lettieri 2004).

Still, it is not necessary to represent abductive reasoning in terms of consequence finding. In particular, Mayer and Pirri (1996) treat abduction independently from deductive reasoning and generate solutions by a special calculus. Furthermore, new algorithms for solution generation that do not rely on consequence finding have been proposed recently (Lagerkvist et al. 2025). The general idea of these algorithms is to test all subsets of $\Delta\subseteq\mathsf{H}$ whether $\Gamma\cup\Delta$ is CPL-satisfiable and $\Gamma\cup\Delta\cup\{\neg\chi\}$ is not. The (relative) efficiency is achieved by beginning with $\Delta=\mathsf{H}$ and then by removing literals one-by-one, thus constructing the 'next weakest' candidate and keeping track of tested candidates to avoid repetitions.

In Łukasiewicz logic, however, it is not sufficient to remove a literal from an interval term to obtain a 'next weakest' term. Indeed, if $\tau=(p\geq\overline{\tfrac{2}{3}})\odot(q\geq\overline{\tfrac{3}{4}})$, and the set of hypotheses H contains $\{q\geq\overline{\tfrac{i}{4}}\mid i\in\{0,\ldots,4\}\}$, then the next weakest term is not $p\geq\overline{\tfrac{2}{3}}$ but $(p\geq\overline{\tfrac{2}{3}})\odot(q\geq\overline{\tfrac{2}{4}})$. Moreover, if H also contains $\{p\geq\overline{\tfrac{i}{3}}\mid i\in\{0,\ldots,3\}\}$, then $(p\geq\overline{\tfrac{1}{3}})\odot(q\geq\overline{\tfrac{3}{4}})$ is another next weakest term. Namely (recall Corollary 3.6), we need to *expand* intervals of permitted values to construct next weakest terms.[7] Furthermore, sometimes we cannot obtain a solution by just expanding the interval of permitted values. For example, given a theory $\{(p\leftrightarrow\neg p)\rightarrow q\}$, the observation q has only one explanation: $p\geq\overline{\tfrac{1}{2}}\odot p\leq\overline{\tfrac{1}{2}}$. Hence, if we want to construct it, we might need to *shrink* the interval of

[6] $\subseteq$-minimal in the terminology of Eiter and Gottlob (1995).

[7] Note that the removal of a literal $p\geq\overline{c}$ ($p\leq\overline{c}$) corresponds to its expansion to $p\geq\overline{0}$ ($p\leq\overline{1}$).

permitted values. This means that we need to test *next strongest* terms. Thus, constructing the 'next' solution candidate involves search and may be inherently more complicated than in classical logic.

The following definitions formalise the notions of the next weakest and the next strongest interval terms w.r.t. a given set of hypotheses.

Definition 3.13 (Next weakest and strongest interval terms). For a finite set of interval literals H, $\lambda \in \mathsf{H}$, and $c \in [0,1]_{\mathbb{Q}}$, we define

$$\overline{\mathrm{c}}^{\uparrow}(p) = \min\left\{c' \in \mathbb{Q} \mid c' > c \text{ and } p{\leq}\overline{\mathrm{c}}' \in \mathsf{H};\ \text{or } p{<}\overline{\mathrm{c}}' \in \mathsf{H}\right\}$$
$$\overline{\mathrm{c}}^{\downarrow}(p) = \max\left\{c' \in \mathbb{Q} \mid c' < c \text{ and } p{\geq}\overline{\mathrm{c}}' \in \mathsf{H};\ \text{or } p{>}\overline{\mathrm{c}}' \in \mathsf{H}\right\}$$

Now, let $\overline{\mathrm{c}}^{\downarrow}(p)$ and $\overline{\mathrm{d}}^{\uparrow}(p)$ exist in H for p and c. Given an interval literal λ, we define *the next weakest* ($\lambda^{\flat}_{\mathsf{H}}$) and *the next strongest* ($\lambda^{\pitchfork}_{\mathsf{H}}$) interval literals w.r.t. H:

$$(p \geq \overline{\mathrm{c}})^{\flat}_{\mathsf{H}} = \begin{cases} p > \overline{\mathrm{c}}^{\downarrow} & \text{if } p{>}\overline{\mathrm{c}}^{\downarrow} \in \mathsf{H} \\ p \geq \overline{\mathrm{c}}^{\downarrow} & \text{otherwise} \end{cases} \qquad (p > \overline{\mathrm{c}})^{\flat}_{\mathsf{H}} = \begin{cases} p \geq \overline{\mathrm{c}} & \text{if } p{\geq}\overline{\mathrm{c}} \in \mathsf{H} \\ p > \overline{\mathrm{c}}^{\downarrow} & \text{if } p{>}\overline{\mathrm{c}}^{\downarrow} \in \mathsf{H} \\ p \geq \overline{\mathrm{c}}^{\downarrow} & \text{otherwise} \end{cases}$$

$$(p \leq \overline{\mathrm{d}})^{\flat}_{\mathsf{H}} = \begin{cases} p < \overline{\mathrm{d}}^{\uparrow} & \text{if } p{<}\overline{\mathrm{d}}^{\uparrow} \in \mathsf{H} \\ p \leq \overline{\mathrm{d}}^{\uparrow} & \text{otherwise} \end{cases} \qquad (p < \overline{\mathrm{d}})^{\flat}_{\mathsf{H}} = \begin{cases} p \leq \overline{\mathrm{d}} & \text{if } p{\leq}\overline{\mathrm{d}} \in \mathsf{H} \\ p < \overline{\mathrm{d}}^{\uparrow} & \text{if } p{<}\overline{\mathrm{d}}^{\downarrow} \in \mathsf{H} \\ p \leq \overline{\mathrm{d}}^{\uparrow} & \text{otherwise} \end{cases}$$

$$(p \leq \overline{\mathrm{c}})^{\pitchfork}_{\mathsf{H}} = \begin{cases} p < \overline{\mathrm{c}} & \text{if } p{<}\overline{\mathrm{c}} \in \mathsf{H} \\ p \leq \overline{\mathrm{c}}^{\downarrow} & \text{if } p{\leq}\overline{\mathrm{c}}^{\downarrow} \in \mathsf{H} \\ p < \overline{\mathrm{c}}^{\downarrow} & \text{otherwise} \end{cases} \qquad (p < \overline{\mathrm{c}})^{\pitchfork}_{\mathsf{H}} = \begin{cases} p \leq \overline{\mathrm{c}}^{\downarrow} & \text{if } p{\leq}\overline{\mathrm{c}} \in \mathsf{H} \\ p < \overline{\mathrm{c}}^{\downarrow} & \text{otherwise} \end{cases}$$

$$(p \geq \overline{\mathrm{d}})^{\pitchfork}_{\mathsf{H}} = \begin{cases} p > \overline{\mathrm{d}} & \text{if } p{>}\overline{\mathrm{d}} \in \mathsf{H} \\ p \geq \overline{\mathrm{d}}^{\uparrow} & \text{if } p{\geq}\overline{\mathrm{d}} \in \mathsf{H} \\ p > \overline{\mathrm{d}}^{\uparrow} & \text{otherwise} \end{cases} \qquad (p > \overline{\mathrm{d}})^{\pitchfork}_{\mathsf{H}} = \begin{cases} p \geq \overline{\mathrm{d}}^{\uparrow} & \text{if } p{\geq}\overline{\mathrm{d}} \in \mathsf{H} \\ p > \overline{\mathrm{d}}^{\uparrow} & \text{otherwise} \end{cases}$$

Finally, let $\mathsf{L}' \subseteq \mathsf{H}$ be a finite set of interval literals and $\tau = \lambda \odot \bigodot_{\lambda' \in \mathsf{L}'} \lambda'$ be an interval term. We define the *next weakest interval term* w.r.t. λ and H as follows:

$$\tau^{\flat}_{\lambda} = \begin{cases} \lambda^{\flat}_{\mathsf{H}} \odot \bigodot\limits_{\lambda' \in \mathsf{L}'} \lambda' & \text{if } \lambda^{\flat}_{\mathsf{H}} \text{ is defined} \\ \bigodot\limits_{\lambda' \in \mathsf{L}'} \lambda' & \text{otherwise} \end{cases}$$

Similarly, if $\lambda^{\pitchfork}_{\mathsf{H}}$ is defined and $\lambda^{\pitchfork}_{\mathsf{H}} \odot \bigodot_{\lambda' \in \mathsf{L}'} \lambda$ is Ł-satisfiable, we define the *next strongest interval term* w.r.t. λ and H as follows: $\tau^{\pitchfork}_{\lambda} = \lambda^{\pitchfork}_{\mathsf{H}} \odot \bigodot_{\lambda' \in \mathsf{L}'} \lambda$.

The following statement shows that, given an interval term τ and a set of interval literals H, $\tau^{\flat}_{\lambda}$ is indeed a 'next weakest term' w.r.t. H. That is, any interval term τ' such that $\tau \models_{Ł} \tau'$ and $\tau' \models_{Ł} \tau^{\flat}_{\lambda}$ is (strongly) equivalent to either τ or $\tau^{\flat}_{\lambda}$. Dually, $\tau^{\pitchfork}_{\lambda}$ is a 'next strongest' term w.r.t. H.

Proposition 3.14. *Let H and L be finite sets of interval terms such that $\mathsf{L} \subseteq \mathsf{H}$, and $\tau = \lambda \odot \bigodot_{\lambda^* \in \mathsf{L}} \lambda^*$. Then the following statements hold.*

(1) For every $\mathsf{L}' \subseteq \mathsf{H}$ and $\tau' = \bigodot_{\lambda' \in \mathsf{L}'} \lambda'$ such that $\tau \models_{Ł} \tau'$ and $\tau' \models_{Ł} \tau^{\flat}_{\lambda}$, it follows that $\tau \equiv_{Ł} \tau'$ or $\tau' \equiv_{Ł} \tau^{\flat}_{\lambda}$.

(2) For every $\mathsf{L}''\subseteq\mathsf{H}$ *and* $\tau''=\bigodot_{\lambda''\in\mathsf{L}''}\lambda''$ *such that* $\tau^{\pitchfork}_{\lambda}\models_{Ł}\tau''$ *and* $\tau''\models_{Ł}\tau$*, it follows that* $\tau\equiv_{Ł}\tau''$ *or* $\tau''\equiv_{Ł}\tau^{\pitchfork}_{\lambda}$*.*

Proof. Note, first of all, that $(\cdot)^{\flat}_{\mathsf{H}}$ and $(\cdot)^{\pitchfork}_{\mathsf{H}}$ are indeed dual transformations for any given H. Indeed, by Definition 3.13, we have $(\lambda^{\flat}_{\mathsf{H}})^{\pitchfork}_{\mathsf{H}}=\lambda$ and $\lambda=(\lambda^{\pitchfork}_{\mathsf{H}})^{\flat}_{\mathsf{H}}$ (provided that $\lambda^{\flat}_{\mathsf{H}}$ and $\lambda^{\pitchfork}_{\mathsf{H}}$ are defined). Thus, it suffices to show only one of the two items in the statement.

We show the first item. Observe that $\tau\models_{Ł}\tau^{\flat}_{\lambda}$ and $\tau^{\flat}_{\lambda}\not\models_{Ł}\tau$. Indeed, using Definition 2.2, one can check that for any given H, $\lambda\models_{Ł}\lambda^{\flat}_{\mathsf{H}}$ and $\lambda^{\flat}_{\mathsf{H}}\not\models_{Ł}\lambda$ when $\lambda^{\flat}_{\mathsf{H}}$ is defined. As $\tau^{\flat}_{\lambda}$ differs from τ only in λ (recall Definition 3.13), our observation holds.

Assume now for contradiction that there are some H, $\tau=\lambda\odot\bigodot_{\lambda^*\in\mathsf{L}}\lambda^*$, $\tau^{\flat}_{\lambda}$, and τ' such that $\tau\models_{Ł}\tau'$, $\tau'\models_{Ł}\tau^{\flat}_{\lambda}$, $\tau'\not\models_{Ł}\tau$, and $\tau^{\flat}_{\lambda}\not\models_{Ł}\tau'$. By Corollary 3.6, given two interval terms π and σ, it holds that $\pi\models_{Ł}\sigma$ iff $\mathsf{V}(p,\pi)\subseteq\mathsf{V}(p,\sigma)$ for every $p\in\mathsf{Pr}(\sigma)$ (if $p\notin\mathsf{Pr}(\pi)$, we set $\mathsf{V}(p,\pi)=[0,1]$).

Now, we have four cases, depending on the type of λ: (i) $\lambda=p{\geq}\overline{\mathsf{c}}$, (ii) $\lambda=p{>}\overline{\mathsf{c}}$, (iii) $\lambda=p{\leq}\overline{\mathsf{c}}$, and (iv) $\lambda=p{<}\overline{\mathsf{c}}$. Consider case (i). By Definition 3.13, we have that (a) $\lambda^{\flat}_{\mathsf{H}}=p{>}\overline{\mathsf{c}}^{\downarrow}$, (b) $\lambda^{\flat}_{\mathsf{H}}=p{\geq}\overline{\mathsf{c}}^{\downarrow}$, or (c) $\lambda^{\flat}_{\mathsf{H}}$ is undefined. In the cases (a) and (b), observe that there is no $\lambda''\in\mathsf{H}$ such that $\lambda\neq\lambda''$, $\lambda''\neq\lambda^{\flat}_{\mathsf{H}}$, $\lambda\models_{Ł}\lambda''$, and $\lambda''\models_{Ł}\lambda^{\flat}$. In the case (c), there is no $\lambda''\in\mathsf{H}$ such that $\lambda\models_{Ł}\lambda''$ and $\lambda\neq\lambda''$. Thus, $\tau\models_{Ł}\tau'$ implies that either (1) $\lambda\in\tau$, (2) $\lambda^{\dagger}\in\tau$ for some $\lambda^{\dagger}$ such that $\lambda\models_{Ł}\lambda^{\dagger}$ and $\lambda^{\dagger}\not\models_{Ł}\lambda$, or (3) τ does not contain an interval literal $\lambda^{\ddagger}$ s.t. $\lambda\models_{Ł}\lambda^{\ddagger}$.

In the first case, recall that $\tau'\models_{Ł}\tau^{\flat}_{\lambda}$. But since $\tau=\lambda\odot\bigodot_{\lambda^*\in\mathsf{L}}\lambda^*$ and $\tau^{\flat}_{\lambda}=\lambda^{\flat}_{\mathsf{H}}\odot\bigodot_{\lambda^*\in\mathsf{L}}\lambda^*$ or $\tau^{\flat}_{\lambda}=\bigodot_{\lambda^*\in\mathsf{L}}\lambda^*$, it follows that $\tau'=\lambda\odot\bigodot_{\lambda^*\in\mathsf{L}}\lambda^*$ (otherwise, it will either not entail $\tau^{\flat}_{\lambda}$ or not be entailed by τ). This contradicts the assumption that $\tau'\models_{Ł}\tau^{\flat}_{\lambda}$. In the second case, either $\lambda^{\dagger}\equiv_{Ł}\lambda^{\flat}_{\mathsf{H}}$ (whence $\tau'\equiv_{Ł}\tau^{\flat}_{\lambda}$, contradicting the assumption that $\tau^{\flat}_{\lambda}\not\models_{Ł}\tau'$) or $\lambda^{\flat}_{\mathsf{H}}\models_{Ł}\lambda^{\dagger}$ and $\lambda^{\dagger}\not\models_{Ł}\lambda^{\flat}_{\mathsf{H}}$ (whence $\tau'\not\models_{Ł}\tau^{\flat}_{\lambda}$, contradicting the assumption that $\tau'\models_{Ł}\tau^{\flat}_{\lambda}$). In the third case, it follows that $\tau^{\flat}_{\lambda}\models_{Ł}\tau'$, contradicting the assumption that $\tau^{\flat}_{\lambda}\not\models_{Ł}\tau'$.

Cases (ii)–(iv) can be dealt with similarly. □

We can now sketch a (naive) procedure for solution generation to Ł-abduction problems. We assume for simplicity that the set of hypotheses H contains tautological literals $p{\geq}\overline{0}$ and $p{\leq}\overline{1}$ for every $p\in\mathsf{Pr}[\mathsf{H}]$. The procedure runs as follows. Given $\mathbb{P}=\langle\Gamma,\chi,\mathsf{H}\rangle$, we begin with the largest possible intervals of permitted values and then shrink or shift them incrementally to find a solution. For each constructed candidate τ, we verify whether $\Gamma,\tau\not\models_{Ł}\bot$ and $\Gamma,\tau\models_{Ł}\chi$. For that, we can use a reduction from Łukasiewicz logic to mixed-integer programming (cf., e.g., (Hähnle 1999)) or a constraint tableaux calculus as shown in Proposition 2.3 and an MIP solver. Note, furthermore, that as the semantics of Łukasiewicz logic can be equivalently defined over $[0,1]_{\mathbb{Q}}$ (Esteva et al. 2002), it is possible to utilise *rational MIP solvers* such as the one proposed by Cook et al. (2013). To illustrate the procedure, we consider the following example.

Example 3.15. Let $\mathbb{P}=\langle\{(p\leftrightarrow\neg p)\rightarrow q\},q,\{p\underline{\diamond}\overline{c}\mid c\in\{0,\frac{1}{2},1\},\underline{\diamond}\in\{\leq,\geq\}\}\rangle$. Our first solution candidate is $\tau_0=(p{\geq}\overline{0})\odot(p{\leq}\overline{1})$. Clearly, τ_0 does not solve $\mathbb{P}$. Therefore, we need to shrink the interval of permitted values of p. This gives us the following terms $(\tau_0)^{\pitchfork}_{\lambda}$ with $\lambda\in\tau_0$:

$$(\tau_0)^{\pitchfork}_{p\geq\overline{0}}=(p{\geq}\overline{\tfrac{1}{2}})\odot(p{\leq}\overline{1})\qquad\qquad(\tau_0)^{\pitchfork}_{p\leq\overline{1}}=(p{\geq}\overline{0})\odot(p{\leq}\overline{\tfrac{1}{2}})$$

Again, neither of these terms is a solution to $\mathbb{P}$. We shrink the intervals once again and rename $(\tau_0)^{\pitchfork}_{p\geq\overline{0}}$ as $\tau^{\geq}_1$ and $(\tau_0)^{\pitchfork}_{p\leq\overline{1}}$ as $\tau^{\leq}_1$. This produces the following terms (note that some of them are repeated):

$$(\tau^{\geq}_1)^{\pitchfork}_{p\geq\overline{1/2}}=(p{\geq}\overline{1})\odot(p{\leq}\overline{1})\quad(\tau^{\geq}_1)^{\pitchfork}_{p\leq\overline{1}}=(p{\geq}\overline{\tfrac{1}{2}})\odot(p{\leq}\overline{\tfrac{1}{2}})\quad(\tau^{\leq}_1)^{\pitchfork}_{p\geq\overline{0}}=(p{\geq}\overline{\tfrac{1}{2}})\odot(p{\leq}\overline{\tfrac{1}{2}})\quad(\tau^{\leq}_1)^{\pitchfork}_{p\leq\overline{1/2}}=(p{\geq}\overline{0})\odot(p{\leq}\overline{0})$$

Now, one can see that $(p{\geq}\overline{\tfrac{1}{2}})\odot(p{\leq}\overline{\tfrac{1}{2}})$ is indeed a solution to $\mathbb{P}$ (and no other term is a solution). Note, furthermore, that we cycled through all intervals that we could express using our hypotheses since we began with the largest possible one and reduced it to a single point. Thus, we do not need to produce 'next strongest' terms anymore (they will all be Ł-unsatisfiable) and can stop the procedure.

Finally, note that if a problem has *non-minimal w.r.t.* Ł*-entailment* solutions, the algorithm will produce an entailment-minimal solution first. To see that, let $\mathbb{P}' = \langle\{(\neg p \rightarrow p) \rightarrow q\}, q, \{p\underline{\diamond}\overline{c} \mid c \in \{0, \frac{1}{2}, 1\}, \underline{\diamond} \in \{\leq, \geq\}\}\rangle$. Now, as one can see, there are three solutions of $\mathbb{P}'$: $p\geq\overline{\frac{1}{2}}$, $(p\geq\overline{\frac{1}{2}}) \odot (p\leq\overline{\frac{1}{2}})$, and $(p\geq\overline{1})$. Furthermore, $p\geq\overline{\frac{1}{2}}$ is (the) entailment-minimal solution. Thus, as we see from the previous part of the example, $p\geq\overline{\frac{1}{2}}$ (represented as $(p\geq\overline{\frac{1}{2}}) \odot (p\leq\overline{1})$) will be generated at the second stage of the procedure, while the remaining solutions at the third.

Note from Example 3.15 that since we are beginning with the largest possible intervals, it follows by Corollary 3.6 that the first solution constructed by this procedure will be *entailment-minimal*. Indeed, as $(\cdot)^{\pitchfork}_{\mathsf{H}}$ and $(\cdot)^{\flat}_{\mathsf{H}}$ are dual to one another, if τ is *the first* solution to $\mathbb{P}$ constructed by the procedure in Example 3.15, the *previous* solution candidates were, in fact, $\tau^{\flat}_{\lambda}$ for some $\lambda \in \tau$ (next weakest w.r.t. τ). And since none of them was a valid solution, it follows that τ is indeed entailment-minimal.

4 Complexity of Abductive Reasoning

In this section, we will establish the complexity of abductive reasoning. We will consider three standard tasks:

- *solution recognition* — given a problem $\mathbb{P}$ and an interval term τ, determine whether τ is a (proper, entailment-minimal, or theory-minimal) solution;
- *solution existence* — given a problem $\mathbb{P}$, determine whether $\mathcal{S}(\mathbb{P}) = \varnothing$;
- *relevance and necessity of hypotheses* — given a problem $\mathbb{P}$ and a hypothesis λ, determine whether there is a solution where it occurs and whether it occurs in all solutions.

Let us now introduce some formula transformations that we will employ in the proofs.

Convention 3.

- Let $\phi \in \mathcal{L}_{\mathsf{CPL}}$. We define $\phi^{Ł} \in \mathcal{L}^{\mathbb{Q}}_{Ł}$ as follows:

  $$p^{Ł} = p \qquad (\neg\phi)^{Ł} = \neg\phi^{Ł} \qquad (\phi \wedge \chi)^{Ł} = \phi^{Ł} \odot \chi^{Ł} \qquad (\phi \vee \chi)^{Ł} = \phi^{Ł} \oplus \chi^{Ł} \qquad (\phi \rightarrow \chi)^{Ł} = \phi^{Ł} \rightarrow \chi^{Ł}$$

  Given $\Gamma \subseteq \mathcal{L}_{\mathsf{CPL}}$, we set $\Gamma^{Ł} = \{\phi^{Ł} \mid \phi \in \Gamma\}$.
- Let $\phi \in \mathcal{L}_{Ł}$. We define $\phi_{\mathsf{cl}} \in \mathcal{L}_{\mathsf{CPL}}$ as follows:

  $$p_{\mathsf{cl}} = p \qquad (\neg\phi)_{\mathsf{cl}} = \neg\phi_{\mathsf{cl}} \qquad (\phi \odot \chi)_{\mathsf{cl}} = \phi_{\mathsf{cl}} \wedge \chi_{\mathsf{cl}} \qquad (\phi \oplus \chi)_{\mathsf{cl}} = \phi_{\mathsf{cl}} \vee \chi_{\mathsf{cl}} \qquad (\phi \rightarrow \chi)_{\mathsf{cl}} = \phi_{\mathsf{cl}} \rightarrow \chi_{\mathsf{cl}}$$

  Given $\Gamma \subseteq \mathcal{L}_{Ł}$, we set $\Gamma_{\mathsf{cl}} = \{\phi_{\mathsf{cl}} \mid \phi \in \Gamma\}$.
- Let $\tau \in \mathcal{L}_{\mathsf{CPL}}$ and $\mathsf{H} \subseteq \mathcal{L}_{\mathsf{CPL}}$ be as follows: $\tau = \bigwedge\limits_{i=1}^{m} p_i \wedge \bigwedge\limits_{j=1}^{n} \neg q_i$, and $\mathsf{H} = \{r_1, \ldots, r_k, \neg s_1, \ldots, \neg s_l\}$. We define

  $$\tau^{\odot} = \bigodot_{i=1}^{m}(p_i\geq\overline{1}) \odot \bigodot_{j=1}^{n}(q_j\leq\overline{0}) \qquad \mathsf{H}^{\odot} = \{r_i\geq\overline{1} \mid i \in \{1, \ldots, k\}\} \cup \{s_i\leq\overline{0} \mid i \in \{1, \ldots, l\}\}$$

- Let $\sigma = \bigodot\limits_{i=1}^{m}(p_i\geq\overline{1}) \odot \bigodot\limits_{j=1}^{n}(q_j\leq\overline{0})$ and $\mathsf{H} = \{r_i\geq\overline{1} \mid i \in \{1, \ldots, k\}\} \cup \{s_i\leq\overline{0} \mid i \in \{1, \ldots, l\}\}$. We define

  $$\sigma_{\mathsf{cl}} = \bigwedge_{i=1}^{m} p_i \wedge \bigwedge_{j=1}^{n} \neg q_i \qquad \mathsf{H}_{\mathsf{cl}} = \{r_1, \ldots, r_k, \neg s_1, \ldots, \neg s_l\}$$

- Let $\Gamma \subseteq \mathcal{L}_{\mathsf{CPL}}$ and $\mathcal{X} \subseteq \mathsf{Pr}$ be finite. We define $\Gamma^{\sharp}_{\mathcal{X}} = \Gamma^{Ł} \cup \{p \vee \neg p \mid p \in \mathsf{Pr}[\Gamma] \cup \mathcal{X}\}$.

Table 1. Complexity of abductive reasoning problems. Notation for different types of solution sets (e.g. $\mathcal{S}(\mathbb{P})$) can be found in Convention 2. We use 'X-c.' to abbreviate X-complete.

Recognition and existence	Ł		CPL	
$\tau \in \mathcal{S}(\mathbb{P})$? / $\tau \in \mathcal{S}^{\mathsf{p}}(\mathbb{P})$?	DP-c.	Th. 4.2 and 4.3	DP-c.	(Eiter and Gottlob 1995, §4, Remark)
$\tau \in \mathcal{S}^{\min}(\mathbb{P})$?	DP-c.	Th. 4.5	DP-c.	(Marquis 2000, Prop. 115)
$\tau \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$?	in Π_2^{P}	Th. 4.6	in Π_2^{P}	by (Marquis 2000, Prop. 115)
$\mathcal{S}(\mathbb{P}) \neq \varnothing$? / $\mathcal{S}^{\mathsf{p}}(\mathbb{P}) \neq \varnothing$?	Σ_2^{P}-c.	Th. 4.7	Σ_2^{P}-c.	(Eiter and Gottlob 1995, Th. 4.2.)

Relevance	Ł		CPL	
w.r.t. $\mathcal{S}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$, $\mathcal{S}^{\min}(\mathbb{P})$	Σ_2^{P}-c.	Th. 4.8	Σ_2^{P}-c.	(Eiter and Gottlob 1995, Th. 4.1.1., 4.2.1.)
w.r.t. $\mathcal{S}^{\mathsf{Th}}(\mathbb{P})$	in Σ_3^{P}	Th. 4.9	in Σ_3^{P}	by (Eiter and Gottlob 1995, Th. 4.2.1.)

Necessity	Ł		CPL	
w.r.t. $\mathcal{S}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$, $\mathcal{S}^{\min}(\mathbb{P})$	Π_2^{P}-c.	Th. 4.8	Π_2^{P}-c.	(Eiter and Gottlob 1995, Th. 4.1.1., 4.2.1.)
w.r.t. $\mathcal{S}^{\mathsf{Th}}(\mathbb{P})$	in Π_3^{P}	Th. 4.9	in Π_3^{P}	by (Eiter and Gottlob 1995, Th. 4.2.1.)

Before proceeding to the complexity of abductive reasoning in Łukasiewicz logic, let us provide some technical remarks. First, as Łukasiewicz connectives behave classically on $\{0,1\}$, it follows that if ϕ is CPL-satisfiable, then $\phi^{\text{Ł}}$ is Ł-satisfiable. Thus, for $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathsf{CPL}}$, if a (classical) valuation v witnesses $\Gamma \not\models_{\mathsf{CPL}} \chi$, then v witnesses $\Gamma^{\text{Ł}} \not\models_{\text{Ł}} \chi^{\text{Ł}}$. Hence, given $\Delta \cup \{\psi\} \subseteq \mathcal{L}_{\text{Ł}}$, $\Delta \models_{\text{Ł}} \psi$ implies $\Delta_{\mathsf{cl}} \models_{\mathsf{CPL}} \psi_{\mathsf{cl}}$. Furthermore, one can straightforwardly embed Łukasiewicz entailment into classical entailment, as the next statement shows.

Lemma 4.1. *Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathsf{CPL}}$ and $\mathsf{Pr}[\Gamma \cup \{\chi\}] \subseteq \mathcal{X}$. Then $\Gamma \models_{\mathsf{CPL}} \chi$ iff $\Gamma^{\sharp}_{\chi} \models_{\text{Ł}} \chi^{\text{Ł}}$.*

Proof. Observe that given an Ł-valuation v, $v(p \vee \neg p) = 1$ iff $v(p) \in \{0,1\}$. Furthermore, on $\{0,1\}$ $\odot$ works the same as $\wedge$ and $\oplus$ as $\vee$ (cf. Definition 2.2 and (3)). Now let v be a *classical valuation* witnessing $\Gamma \not\models_{\mathsf{CPL}} \chi$. It is clear that v also witnesses $\Gamma^{\text{Ł}}, \{p \vee \neg p \mid p \in \mathcal{X}\} \not\models_{\text{Ł}} \chi$. Conversely, let v be an Ł-valuation witnessing $\Gamma^{\text{Ł}}, \{p \vee \neg p \mid p \in \mathcal{X}\} \not\models_{\text{Ł}} \chi$. As we have just observed, this means that all variables are evaluated over $\{0,1\}$. As $\Gamma^{\text{Ł}}$ and $\chi^{\text{Ł}}$ are obtained from Γ and χ by replacing $\wedge$ and $\vee$ with $\odot$ and $\oplus$, it follows that $\Gamma \not\models_{\mathsf{CPL}} \chi$. □

We summarise complexity results from this section in Table 1.

4.1 Solution Recognition

We begin with solution recognition. First, we show that recognition of arbitrary, proper, and entailment-minimal solutions is DP-complete.

Theorem 4.2. *Given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ and an interval term τ, it is DP-complete to decide whether $\tau \in \mathcal{S}(\mathbb{P})$.*

Proof. Membership follows immediately from Proposition 2.3 and the fact that $\tau \in \mathcal{S}(\mathbb{P})$ iff $\Gamma, \tau \models^{\mathsf{cons}}_{\text{Ł}} \chi$. For hardness, we provide a reduction from the classical solution recognition which is DP-complete (Eiter and Gottlob 1995, §4). Let now $\mathbb{P} = \langle \Delta, \psi, \mathsf{H} \rangle$ be a *classical* abduction problem, $\mathcal{X} = \mathsf{Pr}[\Delta \cup \{\psi\}]$, and τ an $\mathcal{L}_{\mathsf{CPL}}$-term. Define $\mathbb{P}^{\text{Ł}} = \langle \Delta^{\sharp}_{\mathcal{X}}, \psi^{\text{Ł}}, \mathsf{H}^{\odot} \rangle$. We show that $\tau \in \mathcal{S}(\mathbb{P})$ iff $\tau^{\odot} \in \mathcal{S}(\mathbb{P}^{\text{Ł}})$. (Recall Convention 3 for the notions of $\Delta^{\sharp}_{\mathcal{X}}$, $\psi^{\text{Ł}}$, $\mathsf{H}^{\odot}$, and $\tau^{\odot}$.)

Assume that $\tau \in \mathcal{S}(\mathbb{P})$, i.e., $\Delta, \tau \models^{\mathsf{cons}}_{\mathsf{CPL}} \psi$ and let v be a *classical* valuation such that $v(\phi) = 1$ for every $\phi \in \Delta$ and $v(\tau) = 1$. Since $\odot$ and $\oplus$ behave on $\{0,1\}$ the same as $\wedge$ and $\vee$, it is clear that $v(\pi) = 1$ for every $\pi \in \Delta^{\sharp}_{\chi}$ and $v(\tau^{\odot}) = 1$ as well. Now assume further for the sake of contradiction that there is some Ł-valuation $v^{\text{Ł}}$ such that $v^{\text{Ł}}(\chi) = 1$ for every $\chi \in \Delta^{\sharp}_{\chi}$, $v^{\text{Ł}}(\tau^{\odot}) = 1$, but $v(\psi^{\text{Ł}}) \neq 1$. But $v^{\text{Ł}}$ must be *classical*, i.e., assign only values from $\{0,1\}$ to all $p \in \mathsf{Pr}[\Delta \cup \{\psi\}]$ because $p \vee \neg p \in \Delta^{\sharp}_{\chi}$ and $v^{\text{Ł}}(p \vee \neg p) = 1$ iff $v(p) \in \{0,1\}$. This would mean that $v^{\text{Ł}}$ witnesses $\Delta, \tau \not\models_{\mathsf{CPL}} \psi$, contrary to the assumption.

For the converse direction, let $\sigma = \bigodot_{i=1}^{m}(r_i \geq \overline{1}) \odot \bigodot_{j=1}^{n}(s_j \leq \overline{0})$ be an *interval* term composed from interval literals of $\mathsf{H}^{\odot}$ and let further $\sigma \in \mathcal{S}(\mathbb{P}^{\text{Ł}})$, i.e., $\Delta^{\sharp}_{\chi}, \sigma \models^{\mathsf{cons}}_{\text{Ł}} \psi$. We show that $\sigma_{\mathsf{cl}} \in \mathcal{S}(\mathbb{P})$, i.e., $\Delta, \sigma_{\mathsf{cl}} \models^{\mathsf{cons}}_{\mathsf{CPL}} \psi$. It is clear that $\Delta, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \psi$. To see that $\Delta, \sigma_{\mathsf{cl}} \not\models_{\mathsf{CPL}} \bot$, assume for contradiction that $\Delta, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \bot$. Then, by Lemma 4.1, we have that $\Delta^{\sharp}_{\chi}, \sigma^{\text{Ł}} \models_{\text{Ł}} \bot$. Now observe from Remark 2 that $\bigodot_{i=1}^{m}(r_i \geq \overline{1}) \odot \bigodot_{j=1}^{n}(s_j \leq \overline{0}) \simeq_{\text{Ł}} \bigodot\limits_{i=1}^{m} r_i \odot \bigodot_{j=1}^{n} \neg s_j$. Thus, we get $\Delta^{\sharp}_{\chi}, \sigma \models_{\text{Ł}} \bot$, contrary to the assumption. ◻

Theorem 4.3. *Given an* Ł*-abduction problem* $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ *and an interval term* τ*, it is* DP*-complete to decide whether* $\tau \in \mathcal{S}^{\mathsf{p}}(\mathbb{P})$.

Proof. First, we obtain hardness via a reduction from the arbitrary solution recognition in Łukasiewicz logic, which is DP-complete by Theorem 4.2. Namely, let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem. We show that $\tau \in \mathcal{S}(\mathbb{P})$ iff τ is a *proper solution* of $\mathbb{P}_{\mathsf{prp}} = \langle \Gamma \cup \{p\}, \chi \odot p, \mathsf{H} \rangle$ with $p \notin \mathsf{Pr}[\Gamma \cup \{\chi\}]$. Assume that $\tau \in \mathcal{S}(\mathbb{P})$. As $p \notin \mathsf{Pr}(\chi)$ and τ is Ł-satisfiable, it is clear that $\tau \not\models_{\text{Ł}} \chi \odot p$. It is also clear that $\Gamma, p, \tau \models_{\text{Ł}} \bot$ iff $\Gamma, \tau \models_{\text{Ł}} \bot$ and $\Gamma, p, \tau \models_{\text{Ł}} \chi \odot p$ iff $\Gamma, \tau \models_{\text{Ł}} \chi$. Conversely, let $\tau \in \mathcal{S}^{\mathsf{p}}(\mathbb{P}_{\mathsf{prp}})$. As $\Gamma, p, \tau \models^{\mathsf{cons}}_{\text{Ł}} \chi$, it is clear that $\Gamma, \tau \models^{\mathsf{cons}}_{\text{Ł}} \chi$.

For membership, given τ and $\mathbb{P}$, we (i) use an NP oracle to guess two Ł-valuations v_{sat} and v_{prp} and check that they witness $\Gamma, \tau \not\models_{\text{Ł}} \bot$ and $\tau \not\models_{\text{Ł}} \chi$. At the same time, we (ii) conduct a coNP check that $\Gamma, \tau \models_{\text{Ł}} \chi$. Note that this is possible because we do not need the result of (i) to do (ii). It follows that $\tau \in \mathcal{S}^{\mathsf{p}}(\mathbb{P})$ iff both checks succeed. ◻

To establish the complexity of $\models_{\text{Ł}}$-minimal solution recognition, we will use a reduction from the prime implicant recognition in classical logic, which is DP-complete (Marquis 1995, Proposition 115). We recall the notion of prime implicants in the next definition.

Definition 4.4. Let $\chi \in \mathcal{L}_{\mathsf{CPL}}$ and τ be an $\mathcal{L}_{\mathsf{CPL}}$-term. We say that τ is a *prime implicant of* χ iff $\tau \models_{\mathsf{CPL}} \chi$ and there is no other $\mathcal{L}_{\mathsf{CPL}}$-term τ' such that $\tau' \models_{\mathsf{CPL}} \chi$, $\tau \models_{\mathsf{CPL}} \tau'$, and $\tau' \not\models_{\mathsf{CPL}} \tau$.

Theorem 4.5. *Given an* Ł*-abduction problem* $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ *and an interval term* τ*, it is* DP*-complete to decide whether* $\tau \in \mathcal{S}^{\min}(\mathbb{P})$.

Proof. For hardness, we provide a reduction from the prime implicant recognition in classical logic, which is known to be DP-complete. Let w.l.o.g. χ be classically satisfiable and τ a prime implicant of χ. Thus, $\tau \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$ and there is no $\mathcal{L}_{\mathsf{CPL}}$-term σ such that $\sigma \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$, $\tau \models_{\mathsf{CPL}} \sigma$, and $\sigma \not\models_{\mathsf{CPL}} \tau$. We show that $\tau^{\odot}$ is an Ł-minimal solution to $\mathbb{P} = \left\langle \Gamma, \chi^{\text{Ł}} \odot q, \mathsf{H} \right\rangle$ with $q \notin \mathsf{Pr}(\chi)$ and

$$\Gamma = \{p \vee \neg p \mid p \in \mathsf{Pr}(\chi \wedge q)\} \cup \{q\} \qquad \mathsf{H} = \{p \geq \overline{1} \mid p \in \mathsf{Pr}(\chi)\} \cup \{p \leq \overline{0} \mid p \in \mathsf{Pr}(\chi)\}$$

First, all solutions of $\mathbb{P}$ are proper because $q \notin \mathsf{Pr}[\mathsf{H}]$. Now, consider $\tau^{\odot}$ (recall Convention 3). It is clear that $\Gamma, \tau^{\odot} \models^{\mathsf{cons}}_{\text{Ł}} \chi^{\text{Ł}} \odot q$. Indeed, τ was classically satisfiable, whence $\tau^{\odot}$ is Ł-satisfiable by the same valuation v that satisfied τ since $\odot$ and $\wedge$ behave identically on $\{0,1\}$. But it also means that $\Gamma \cup \{\tau\}$ is Ł-satisfiable. To see that, set $v(q) = 1$ and observe that $v(p \vee \neg p) = 1$ iff $v(p) \in \{0,1\}$. Furthermore, if for some Ł-valuation v', $v'(\tau^{\odot}) = 1$ and $v'(\phi) = 1$ for every $\phi \in \Gamma$, it means that all variables are evaluated over $\{0,1\}$, i.e., v' is a classical valuation.

Hence, $v'(\chi^{Ł} \odot q) = 1$, as required. It remains to see that there is no solution σ such that $\tau^{\odot} \models_{Ł} \sigma$ and $\sigma \not\models_{Ł} \tau^{\odot}$. Assume for the sake of contradiction that there is such a solution σ and consider σ_{cl} as specified in Convention 3.

We will show that $\sigma_{\mathsf{cl}} \not\models_{\mathsf{CPL}} \tau$, $\sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \chi$, and $\tau \models_{\mathsf{CPL}} \sigma_{\mathsf{cl}}$. Observe that $(\sigma_{\mathsf{cl}})^{\odot} = \sigma$. Now, let v be an Ł-valuation witnessing $\sigma \not\models_{Ł} \tau^{\odot}$. Since σ and $\tau^{\odot}$ are interval terms containing only literals of the form $r \leq \overline{0}$ and $r \geq \overline{1}$, it means that $v(\sigma) = 1$ and $v(\tau^{\odot}) = 0$. Now, we define a classical valuation v^{cl} as follows:

$$v^{\mathsf{cl}}(r) = \begin{cases} 1 & \text{if } v(s \geq \overline{1}) = 1 \text{ or } v(r \leq \overline{0}) = 0 \\ 0 & \text{if } v(r \leq \overline{0}) = 1 \text{ or } v(r \geq \overline{1}) = 0 \end{cases}$$

It is clear that v^{cl} witnesses $\sigma_{\mathsf{cl}} \not\models_{\mathsf{CPL}} \tau$. Finally, using the definition of Γ, that σ contains only literals of the form $r \leq \overline{0}$ and $r \geq \overline{1}$, and that all $\mathcal{L}_{Ł}^{\mathbb{Q}}$ connectives behave classically on $\{0, 1\}$, we obtain that $\Gamma, \sigma \models_{Ł}^{\mathsf{cons}} \chi \odot q$ entails $\sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \chi$ and $\tau^{\odot} \models_{Ł} \sigma$ entails $\tau \models_{\mathsf{CPL}} \sigma^{\mathsf{cl}}$. This, however, means that τ is not a prime implicant of χ, contrary to the assumption.

For the converse, let $\tau \in \mathcal{S}^{\min}(\mathbb{P})$. This means that $\Gamma, \tau \models_{Ł}^{\mathsf{cons}} \chi^{Ł} \odot q$ and there is no other proper solution τ' such that $\tau \models_{Ł} \tau'$, but $\tau' \not\models_{Ł} \tau$. We consider τ_{cl}. From Lemma 4.1 and the fact that $\tau \simeq_{Ł} \bigodot_{(r \geq \overline{1}) \in \tau} r \odot \bigodot_{(s \leq \overline{0}) \in \tau} \neg s$, we have $\tau_{\mathsf{cl}} \models_{\mathsf{CPL}}^{\mathsf{cons}} \chi$. Now assume for contradiction that there is some σ such that $\sigma \models_{\mathsf{CPL}} \chi$, $\tau \models_{\mathsf{CPL}} \sigma$, but $\sigma \not\models_{\mathsf{CPL}} \tau$. It is easy to check that in this case, $\sigma^{\odot} \in \mathcal{S}(\mathbb{P})$, $\tau \models_{Ł} \sigma^{\odot}$, and $\sigma^{\odot} \not\models_{Ł} \tau$, i.e., τ is not a $\models_{Ł}$-minimal solution, contrary to the assumption.

Now, consider membership. Given $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ and an interval term $\tau = \bigodot_{i=1}^{n} \lambda_i$, we use an NP-oracle to guess $O(n \cdot |\mathsf{H}|)$ valuations v_{sat}, v_{prp}, and v_{λ} (for each $\lambda \in \tau$) and verify whether they witness (i) $\Gamma, \tau \not\models_{Ł} \bot$, (ii) $\tau \not\models_{Ł} \chi$, and (iii) $\Gamma, \tau_{\lambda}^{\flat} \not\models_{Ł} \chi$ (recall Definition 3.13 for $\tau_{\lambda}^{\flat}$), respectively. By Proposition 3.14, it follows that it is sufficient to consider $\tau_{\lambda}^{\flat}$'s as candidates for solutions. Parallel to that (as we do not need the results of (i)–(iii)), we conduct a coNP check that (iv) $\Gamma, \tau \models_{Ł} \chi$. It follows from the definition of $\models_{Ł}$-minimal solutions that τ is a $\models_{Ł}$-minimal solution iff the NP and coNP checks succeed. ◻

In the case of theory-minimal solutions, we establish membership in Π_2^{P}. We expect that this case is indeed harder than entailment-minimality. The intuition behind this is that the presence of the theory means we cannot readily identify a polynomial number of candidates for better solutions to check. We leave the search for a matching lower bound for future work and remark that, to the best of our knowledge, the complexity of the analogous problem in CPL is also unknown.

Theorem 4.6. *It is in Π_2^{P} to decide, given an Ł-abduction problem $\mathbb{P}$ and an interval term τ, whether τ is a theory-minimal solution of $\mathbb{P}$.*

Proof. Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem, and σ is the interval term we are testing. We sketch a Σ_2^{P} procedure for the complementary problem of testing whether σ is *not* a theory-minimal solution. We first guess either (1) 'not a proper solution' or (2) another interval term σ'. In the case (1), due to the DP-completeness of proper solution recognition (Theorem 4.3), we make one or two calls to an NP oracle to verify that σ is indeed not a proper solution (in which case we return 'yes'). In the case (2), we can make four calls to an NP oracle: two to verify that σ' is a proper solution; one to verify that $\Gamma, \sigma \models_{Ł} \sigma'$, and another one to check that $\Gamma, \sigma' \not\models_{Ł} \sigma$ (returning 'yes' if the calls show this to be the case). By Proposition 3.5, these entailments can indeed be verified by an NP-oracle. The procedure we have just described will return 'yes' iff the input σ is not a theory-minimal solution, which yields the desired membership in Π_2^{P} for the original problem. ◻

4.2 Solution Existence

We now turn to establishing the complexity of the solution existence in Ł-abduction problems. Note that a problem may have solutions but no proper solutions. On the other hand, if a problem has proper solutions, it will have entailment- and theory-minimal solutions as well. Thus, we will consider the complexity of arbitrary and proper solution existence.

Theorem 4.7. *It is Σ^{P}_2-complete to decide whether an Ł-abduction problem has a (proper) solution.*

Proof. Membership follows immediately from Theorem 4.2 for arbitrary solutions and from Theorem 4.3 for proper solutions. Let us now show Σ^{P}_2-hardness.

We begin with the existence of arbitrary solutions. We construct a polynomial reduction from the solution existence to classical abduction problems, which is known to be Σ^{P}_2-complete. Given a classical abduction problem $\mathbb{P} = \langle \Delta, \psi, \mathsf{H} \rangle$ and $\mathcal{X} = \mathsf{Pr}[\Delta \cup \{\psi\}]$, we define an Ł-abduction problem $\mathbb{P}^{Ł} = \langle \Delta^{\sharp}_{\mathcal{X}}, \psi^{Ł}, \mathsf{H}^{\odot} \rangle$ (recall Convention 3). Let us now show that $\mathcal{S}(\mathbb{P}) \neq \varnothing$ iff $\mathcal{S}(\mathbb{P}^{Ł}) \neq \varnothing$.

Assume that $\tau \in \mathcal{S}(\mathbb{P})$, i.e., $\Delta, \tau, \models^{\mathsf{cons}}_{\mathsf{CPL}} \psi$. We show that $\tau^{\odot} \in \mathcal{S}(\mathbb{P}^{Ł})$. Let v be a classical valuation witnessing $\Delta, \tau \not\models_{\mathsf{CPL}} \bot$. It is clear that v also witnesses $\Delta^{\sharp}_{\mathcal{X}}, \tau^{\odot} \not\models_{Ł} \bot$. To see that Ł-entailment holds, observe that by Lemma 4.1, we have that $\Delta^{\sharp}_{\mathcal{X}}, \tau^{Ł} \models_{Ł} \psi^{Ł}$. Using Remark 2 and Convention 3, one can see that given an $\mathcal{L}_{\mathsf{CPL}}$-term τ, $\tau^{\odot}$ and $\tau^{Ł}$ are *weakly equivalent*. Thus, $\Delta^{\sharp}_{\mathcal{X}}, \tau^{\odot} \models_{Ł} \psi^{Ł}$, as required. Conversely, let $\sigma = \bigodot_{i=1}^{m}(s_i \geq \overline{1}) \odot \bigodot_{j=1}^{n}(r_j \leq \overline{0})$ be a solution to $\mathbb{P}^{Ł}$, i.e., $\Delta^{\sharp}_{\mathcal{X}}, \sigma \models^{\mathsf{cons}}_{Ł} \psi^{Ł}$. Again, using Lemma 4.1 and the fact that $\sigma \simeq_{Ł} \bigodot_{i=1}^{m} s_i \odot \bigodot_{j=1}^{n} \neg r_j$, one can check that $\sigma_{\mathsf{cl}} \in \mathcal{S}(\mathbb{P})$, as required.

To see that the existence of proper solutions is Σ^{P}_2-hard, we provide a polynomial reduction from the (arbitrary) solution existence to Ł-abduction problems. Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem. We show that $\mathcal{S}(\mathbb{P}) \neq \varnothing$ iff the following Ł-abduction problem has proper solutions: $\mathbb{P}_{\mathsf{prp}} = \langle \Gamma \cup \{p\}, \chi \odot p, \mathsf{H} \rangle$ with $p \notin \mathsf{Pr}[\Gamma \cup \{\chi\}]$. Assume that $\tau \in \mathcal{S}(\mathbb{P})$. As $p \notin \mathsf{Pr}(\chi)$ and τ is Ł-satisfiable, it is clear that $\tau \not\models_{Ł} \chi \odot p$. It is also clear that $\Gamma, p, \tau \models_{Ł} \bot$ iff $\Gamma, \tau \models_{Ł} \bot$ and $\Gamma, p, \tau \models_{Ł} \chi \odot p$ iff $\Gamma, \tau \models_{Ł} \chi$. Conversely, let $\tau \in \mathcal{S}^{\mathsf{p}}(\mathbb{P}_{\mathsf{prp}})$. As $\Gamma, p, \tau \models^{\mathsf{cons}}_{Ł} \chi$, it is clear that $\Gamma, \tau \models^{\mathsf{cons}}_{Ł} \chi$, as required. □

4.3 Relevance and Necessity of Hypotheses

Let us now proceed to the complexity of determining the relevance and necessity of hypotheses w.r.t. different sets of solutions to Ł-abduction problems. Namely, given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$, we will consider the complexity of determining whether an interval literal $\lambda \in \mathsf{H}$ is relevant (necessary) w.r.t. $\mathcal{S}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$, $\mathcal{S}^{\min}(\mathbb{P})$, and $\mathcal{S}^{\mathsf{Th}}(\mathbb{P})$.

We begin with the complexity of relevance and necessity w.r.t. arbitrary, proper, and $\models_{Ł}$-minimal solutions. The classical counterparts of these decision problems were considered by Eiter and Gottlob (1995). We show that the complexity of relevance and necessity w.r.t. (proper) solutions and $\models_{Ł}$-minimal solutions coincides with the complexity of the analogous problems in CPL.

Theorem 4.8. *It is Σ^{P}_2-complete (respectively, Π^{P}_2-complete) to decide, given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ and $\lambda \in \mathsf{H}$, whether λ is relevant (respectively, necessary) w.r.t. $\mathcal{S}(\mathbb{P})$. The same holds for relevance and necessity w.r.t. $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$ and $\mathcal{S}^{\min}(\mathbb{P})$.*

Proof. The membership is evident from Theorems 4.2 and 4.5 because it suffices to guess a (proper or $\models_{Ł}$-minimal) solution τ which can be verified in DP time and then check in linear time whether $\lambda \in \tau$. If $\lambda \in \tau$, then it is relevant. Otherwise, it is *non-necessary.*

For the hardness, we construct the following polynomial reductions:

(1) from the existence of (proper) solutions to Ł-abduction problems to the *relevance* of a hypothesis w.r.t. the set of all (proper) solutions;
(2) from the existence of (proper) solutions to Ł-abduction problems to the *non-necessity* of a hypothesis w.r.t. the set of all (proper) solutions.

Note briefly that in Item (2), we show that the problem *complementary* to the necessity of hypotheses is Σ^{P}_2-hard. This means that the necessity of hypotheses is Π^{P}_2-hard.

Now, given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H}\rangle$, we define the following Ł-abduction problem $\mathbb{P}^{\mathsf{r}} = \langle \Gamma^{\mathsf{r}}, \chi^{\mathsf{r}}, \mathsf{H}^{\mathsf{r}}\rangle$:

$$\Gamma^{\mathsf{r}} = \underbrace{\{t \rightarrow \psi \mid \psi \in \Gamma\}}_{\Psi} \cup \underbrace{\{t' \rightarrow \chi, t' \rightarrow \neg t, t \rightarrow t'', t' \rightarrow t''\}}_{\Theta} \qquad \chi^{\mathsf{r}} = t'' \odot \chi \qquad \mathsf{H}^{\mathsf{r}} = \mathsf{H} \cup \{t \geq \overline{1}, t' \geq \overline{1}\} \tag{8}$$

In (8), t, t', and t'' are propositional variables such that $t, t', t'' \notin \mathsf{Pr}[\mathbb{P}]$. Observe also that $\mathcal{S}^{\mathsf{p}}(\mathbb{P}^{\mathsf{r}}) = \mathcal{S}(\mathbb{P}^{\mathsf{r}})$ (recall Convention 2 for the notation of different types of solutions) because $t'' \notin \mathsf{H}^{\mathsf{r}}$.

We now prove that

(1) $\mathbb{P}$ has solutions iff $t \geq \overline{1}$ is relevant w.r.t. $\mathcal{S}(\mathbb{P}^{\mathsf{r}})$;
(2) $\mathbb{P}$ has solutions iff $t' \geq \overline{1}$ is not necessary w.r.t. $\mathcal{S}(\mathbb{P}^{\mathsf{r}})$.

We begin by showing that the following holds:

$$\mathcal{S}(\mathbb{P}^{\mathsf{r}}) = \underbrace{\left\{\varrho \odot (t \geq \overline{1}) \mid \varrho \in \mathcal{S}(\mathbb{P})\right\}}_{\mathsf{REL}} \cup \underbrace{\left\{\varrho' \odot (t' \geq \overline{1}) \mid \exists \mathsf{H}' \subseteq \mathsf{H} : \varrho' = \bigodot_{l \in \mathsf{H}'} l\right\}}_{\mathsf{NEC}} \tag{9}$$

First, let $\varrho \in \mathcal{S}(\mathbb{P})$. Then, $\Gamma, \varrho \models^{\mathsf{cons}}_{Ł} \chi$. It is clear from (8) that $\Gamma, \varrho \odot (t \geq \overline{1}) \models^{\mathsf{cons}}_{Ł} t'' \odot \chi$. Thus, $\mathsf{REL} \subseteq \mathcal{S}(\mathbb{P}^{\mathsf{r}})$. Similarly, observe from (8) that $(t' \geq \overline{1}) \in \mathcal{S}(\mathbb{P}^{\mathsf{r}})$ because $\Theta, t' \geq \overline{1} \models^{\mathsf{cons}}_{Ł} \neg t$, whence, $\Theta, t' \geq \overline{1} \models^{\mathsf{cons}}_{Ł} t \rightarrow \phi$ for every $\phi \in \mathcal{L}^{\mathbb{Q}}_{Ł}$. Furthermore, $\Theta, t' \geq \overline{1} \models^{\mathsf{cons}}_{Ł} t'' \odot \chi$, as required. Thus, for any interval term ϱ' over H, it holds that $\varrho' \odot (t' \geq \overline{1})$ is a solution to $\mathbb{P}^{\mathsf{r}}$. Hence, $\mathsf{NEC} \subseteq \mathcal{S}(\mathbb{P}^{\mathsf{r}})$.

Conversely, let $\sigma \in \mathcal{S}(\mathbb{P}^{\mathsf{r}})$. If $t' \geq \overline{1}$ occurs in σ, then $\sigma \in \mathsf{NEC}$. If $t' \geq \overline{1}$ does not occur in σ, we show that $\sigma = \varrho \odot (t \geq \overline{1})$ for some $\varrho \in \mathcal{S}(\mathbb{P})$. We reason for contradiction. Assume first that $t \geq \overline{1}$ does not occur in σ. Then set $v(t) = v(t') = v(t'') = 0$. It is clear that $v(\phi) = 1$ for every $\phi \in \Gamma^{\mathsf{r}}$ but $v(t'' \odot \chi) = 0$. Thus, $\sigma \notin \mathcal{S}(\mathbb{P}^{\mathsf{r}})$. Second, let $\sigma = \sigma' \odot (t \geq \overline{1})$ for some $\sigma' \notin \mathcal{S}(\mathbb{P})$. We have two cases: (i) $\Gamma, \sigma \models_{Ł} \bot$ and (ii) $\Gamma, \sigma \not\models_{Ł} \chi$. If (i) holds, it is immediate that $\Gamma^{\mathsf{r}}, \sigma' \odot (t \geq \overline{1}) \models_{Ł} \bot$. If (ii) holds, let v be an Ł-valuation witnessing $\Gamma, \sigma \not\models_{Ł} \chi$. We extend it to $\{t, t', t''\}$, setting $v(t) = 1$, $v(t') = v(t'') = 0$. It is clear that $v(\phi) = 1$ for every $\phi \in \Gamma^{\mathsf{r}}$ but $v(t'' \odot \chi) = 0$. Thus, v witnesses $\Gamma^{\mathsf{r}}, \sigma' \odot (t \geq \overline{1}) \not\models_{Ł} \chi^{\mathsf{r}}$.

It now follows that $\mathbb{P}$ has solutions iff $t \geq \overline{1}$ is relevant and $t' \geq \overline{1}$ is not necessary w.r.t. $\mathcal{S}(\mathbb{P}^{\mathsf{r}})$. Indeed, if $\mathbb{P}$ has a solution ϱ, then $\sigma = \varrho \odot (t \geq \overline{1})$ is a solution to $\mathbb{P}^{\mathsf{r}}$. As $t' \geq \overline{1}$ does not occur in σ, it is not necessary. Conversely, if $\mathcal{S}(\mathbb{P}) = \varnothing$, then $\mathsf{REL} = \varnothing$ (recall (9)). Thus, all solutions contain $t' \geq \overline{1}$ (i.e., it is necessary), and no solution contains $t \geq \overline{1}$ (i.e., it is not relevant).

For hardness w.r.t. $\mathcal{S}^{\min}(\mathbb{P})$, observe that $t \geq \overline{1}$ is relevant to $\mathbb{P}^{\mathsf{r}}$ iff it is relevant w.r.t. $\models_{Ł}$-minimal solutions. Similarly, $t' \geq \overline{1}$ is (not) necessary in $\mathbb{P}^{\mathsf{r}}$ iff it is (not) necessary w.r.t. $\models_{Ł}$-minimal solutions. ◻

We finish the section by presenting the membership results on the complexity of the recognition of relevant and necessary hypotheses w.r.t. theory-minimal solutions. The next statement is an easy consequence of Theorem 4.6. To the best of our knowledge, no tight complexity bounds have been established for this problem in the CPL-abduction.

Theorem 4.9. *It is in Σ^{P}_3 (respectively, Π^{P}_3) to decide, given an* Ł*-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H}\rangle$ and $\lambda \in \mathsf{H}$, whether λ is relevant (respectively, necessary) w.r.t. $\mathcal{S}^{\mathsf{Th}}(\mathbb{P})$.*

Proof. We consider the relevance. Recall from Theorem 4.6 that a theory-minimal solution can be recognised in Π^{P}_2. Thus, it suffices to guess a theory-minimal solution τ and then check that λ indeed occurs there. It follows

Table 2. Complexity of solution recognition and solution existence to Ł-abduction problems in clause fragments. $\mathsf{X} \in \{\mathcal{S}, \mathcal{S}^{\mathrm{p}}, \mathcal{S}^{\min}\}$.

	$\tau \in \mathsf{X}(\mathbb{P})$?		$\tau \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$?		$\mathcal{S}(\mathbb{P}) \neq \varnothing$? / $\mathcal{S}^{\mathrm{p}}(\mathbb{P}) \neq \varnothing$?	
simple-clause theories	P-c.	Th. 5.7	in coNP	Th. 5.8	NP-c.	Th. 5.9
interval-clause theories	DP-c.	Th. 5.15	in Π_2^{P}	Th. 4.6	Σ_2^{P}-c.	Th. 5.16
cover-free theories	P-c.	Th. 5.20	in coNP	Th. 5.23	NP-c.	Th. 5.21
monodirectional theories	P-c.	Th. 5.26	in coNP	Crl. 5.27	NP-c.	Th. 5.26
positive mon.-dir. theories	in P	Crl. 5.27	in coNP	Crl. 5.27	in P	Th. 5.26
weakly cover-free theories	in P	Crl. 5.27	in P	Crl. 5.27	in P	Th. 5.26
short mon.-dir. theories	in P	Crl. 5.27	in P	Crl. 5.27	in P	Th. 5.26

that the relevance of a hypothesis w.r.t. the set of theory-minimal solutions can be decided in Σ_3^{P}. In the same way, we can show that *non-necessity* of a hypothesis can also be decided in Σ_3^{P}. Hence, necessity is decidable in Π_3^{P}. $\square$

5 Abduction in Clause Fragments

Recall from Proposition 2.3 that the complexity of Łukasiewicz logic coincides with the complexity of CPL. On the other hand, while every classical formula can be equivalently represented as a set of $\mathcal{L}_{\mathsf{CPL}}$-clauses, arbitrary $\mathcal{L}_{Ł}$-formulas cannot be transformed into sets of simple clauses. In fact, the simple clause fragment of $\mathcal{L}_{Ł}$ *is decidable in linear time* (Bofill et al. 2019, Lemma 2). As Ł-satisfiability of *arbitrary* $\mathcal{L}_{Ł}$-formulas is NP-complete (Mundici 1987), it follows that *simple clauses are less expressive than general formulas.*

Proposition 5.1. *Let $\Gamma = \{\kappa_1, \ldots, \kappa_n\}$ be a finite set of simple clauses. It takes linear time to decide whether there is an* Ł*-valuation v such that $v(\kappa_i) = 1$ for every $i \in \{1, \ldots, n\}$.*

We note that due to (1), the following formulas are pairwise strongly equivalent for any $n \in \mathbb{N}$ and $k < n$:

$$\bigoplus_{i=1}^{n} l_i \qquad \neg l_1 \to \bigoplus_{i=2}^{n} l_i \qquad \bigodot_{i=1}^{n-1} \neg l_i \to l_n \qquad \bigodot_{i=1}^{k} \neg l_i \to \bigoplus_{j=n-k}^{n} l_j \tag{10}$$

Proposition 5.1 together with (10) means, in particular, that the satisfiability of logic programming under Łukasiewicz semantics is polynomial independent of whether it contains negation. In classical logic, the complexity of abduction in polynomial fragments is expectedly lower than in general (cf. (Creignou and Zanuttini 2006; Nordh and Zanuttini 2008) for details). Hence, it is instructive to establish whether the complexity of Ł-abduction will also be lower in a polynomial fragment.

In this section, we will consider the complexity of Ł-abduction when the theory is a set of clauses of the form $\bigoplus_{i=1}^{n} l_i$ or $\bigoplus_{i=1}^{n} \lambda_i$ with l's being *simple literals* and λ's *interval literals.* The results are shown in Tables 2 and 3.

5.1 Simple Clause Fragment

Let us now consider the complexity of abductive reasoning for the simple clause fragment of Ł. We begin with the definition of simple clause theories and abduction problems.

Definition 5.2. A *simple-clause theory* (SC theory) is a finite $\Gamma \subseteq \mathcal{L}_{Ł}^{\mathbb{Q}}$ such that every $\phi \in \Gamma$ is either a simple clause or an interval term.

Table 3. Complexity of deciding relevance and necessity of hypotheses in Ł-abduction problems in clause fragments. $\mathsf{Y} \in \{\mathcal{S}, \mathcal{S}^{\mathsf{P}}\}$.

	Y-r.		Y-n.		$\mathcal{S}^{\min}$-r.		$\mathcal{S}^{\min}$-n.	
simple-clause theories	NP-c.	Th. 5.10	coNP-c.	Th. 5.10	NP-c.	Th. 5.10	coNP-c.	Th. 5.10
interval-clause theories	Σ^{P}_2-c.	Th. 5.17	Π^{P}_2-c.	Th. 5.17	Σ^{P}_2-c.	Th. 5.17	Π^{P}_2-c.	Th. 5.17
cover-free theories	NP-c.	Th. 5.22	coNP-c.	Th. 5.22	NP-c.	Th. 5.22	coNP-c.	Th. 5.22
monodirectional theories	NP-c.	Crl. 5.27	coNP-c.	Crl. 5.27	NP-c.	Crl. 5.27	coNP-c.	Crl. 5.27
positive mon.-dir. theories	in P	Crl. 5.27	in P	Crl. 5.27	NP-c.	Crl. 5.27	in P	Crl. 5.27
weakly cover-free theories	in P	Crl. 5.27	in P	Crl. 5.27	in P	Crl. 5.27	in P	Crl. 5.27
short mon.-dir. theories	in P	Crl. 5.27	in P	Crl. 5.27	in P	Crl. 5.27	in P	Crl. 5.27

Definition 5.3 (Simple clause abduction). A *simple clause abduction problem* (SCA problem) is an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ such that Γ is a simple-clause theory and χ is an interval clause, simple clause, interval term, or a simple term.

In the definition above, note that Γ can contain interval terms. This enables us to express constraints on the values of variables. Similarly, different observations correspond to various types of constraints that we explain based on the theory.

The proofs in this section utilise the fact that systems of linear inequalities over $\mathbb{R}$ and $\mathbb{Q}$ can be solved in polynomial time. Thus, we will need to transform simple clauses and interval terms into (systems of) linear inequalities. Furthermore, hardness results are obtained by reductions from classical Horn abduction. We introduce the relevant notions below.

Definition 5.4. Let $\kappa = \bigoplus\limits_{i=1}^{m} p_i \oplus \bigoplus\limits_{j=1}^{n} \neg q_j$ be a simple clause and $\lambda = p \diamond \overline{c}$ an interval literal. We define linear inequalities $\kappa^{\mathbb{LI}}$ and $\lambda^{\mathbb{LI}}$ as follows:

$$\kappa^{\mathbb{LI}} = \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} (1 - q_j) \geq 1 \qquad\qquad \lambda^{\mathbb{LI}} = p \diamond c$$

Now let $\Gamma = \{\kappa_i \mid i \in \{1, \ldots, m\}\} \cup \{\tau_j \mid j \in \{1, \ldots, n\}\}$ be a simple-clause theory such that $\tau_j = \bigodot\limits_{i=1}^{k_j} \lambda^j_i$. We define the system of linear inequalities $\Gamma^{\mathbb{LI}}$ that corresponds to Γ as follows:

$$\Gamma^{\mathbb{LI}} = \{\kappa_i^{\mathbb{LI}} \mid i \in \{1, \ldots, m\}\} \cup \bigcup_{j=1}^{n} \{(\lambda^j_i)^{\mathbb{LI}} \mid i \in \{1, \ldots, k_j\}\}$$

Definition 5.5. Let $\Gamma = \{\{l^1_1, \ldots, l^1_{m_1}\}, \ldots, \{l^n_1, \ldots, l^n_{m_n}\}\}$ be a set of Horn clauses in the form of sets of literals (that is, $\{l^j_1, \ldots, l^j_{m_j}\}$ contains at most one positive literal for every $j \in \{1, \ldots, n\}$). We define its corresponding set of *simple clauses* $\Gamma^{\oplus}$ as follows:

$$\Gamma^{\oplus} = \left\{ \bigoplus_{i=1}^{m_j} l^j_i \;\middle|\; j \in \{1, \ldots, n\} \right\}$$

We begin with an important technical statement. Namely, we show that Ł-satisfiability of sets of simple clauses is P-complete under logspace reductions. This strengthens Proposition 5.1.

LEMMA 5.6. *Let $\Gamma = \{\kappa_1, \ldots, \kappa_n\}$ be a finite set of simple clauses. It is* P*-complete under logspace reductions to decide whether there is an* Ł*-valuation v such that $v(\kappa_i) = 1$ for every $i \in \{1, \ldots, n\}$.*

PROOF. Membership is established by Proposition 5.1. For hardness, we consider the following logspace reduction from classical Horn satisfiability, which is P-complete. Given a set of Horn clauses Γ represented as sets of literals, we construct $\Gamma^{\oplus}$ as shown in Definition 5.5. Note that we can construct $\Gamma^{\oplus}$ using logarithmic space because we transform every $\{l_1, \ldots, l_m\} \in \Gamma$ into $\bigoplus_{i=1}^{m} l_i$.

It now remains to prove that Γ is classically satisfiable iff there is an Ł-valuation v such that $v(\kappa) = 1$ for every $\kappa \in \Gamma^{\oplus}$. If Γ is satisfied by a classical valuation v, it is clear that v also satisfies $\Gamma^{\oplus}$. For the converse, let Γ be *classically unsatisfiable*. We show that then $\Gamma^{\oplus}$ is not Ł-satisfiable either. To see this, observe that if Γ is classically unsatisfiable, then, by (Chang 1970, Theorem 1), there is a unit resolution inference of the empty clause from Γ, i.e., an inference where in every instance of the resolution rule, at least one premise is a unit clause. Now note that unit resolution is sound in Łukasiewicz logic: both $\kappa \oplus p, \neg p \models_{Ł} \kappa$ and $\kappa \oplus \neg p, p \models_{Ł} \kappa$ are valid instances of Ł-entailment. This means that we can reuse classical resolution inference in Łukasiewicz logic and infer the empty clause. Hence, $\Gamma^{\oplus}$ is Ł-unsatisfiable. □

We are now ready to show that recognition of arbitrary, proper, and $\models_{Ł}$-minimal solutions is P-complete.

THEOREM 5.7. *Given an SCA problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ and an interval term σ, it is* P*-complete under logspace reductions to decide whether σ is an arbitrary, proper, or $\models_{Ł}$-minimal solution.*

PROOF. We begin with hardness. We provide a logspace reduction from the Ł-satisfiability of sets of simple clauses which we showed to be P-hard in Lemma 5.6. Let Γ be a set of simple clauses and consider $\mathbb{P} = \langle \Gamma \cup \{p\}, p, \mathsf{H} \rangle$ such that $\mathsf{H} = \{p \leq \overline{1}\}$ and $p \notin \mathsf{Pr}[\Gamma]$. It is clear that $p \leq \overline{1}$ is a (proper, entailment-minimal) solution iff Γ is Ł-satisfiable. This covers the hardness for the case of the observation being a simple term or a simple clause. For the case of the observation being an interval term or an interval clause, we can reduce the Ł-satisfiability of Γ to determining whether $p \leq \overline{1}$ is a (proper, entailment-minimal) solution to $\mathbb{P} = \langle \Gamma \cup \{p \geq \overline{1}\}, p \geq \overline{1}, \mathsf{H} \rangle$ such that $\mathsf{H} = \{p \leq \overline{1}\}$ and $p \notin \mathsf{Pr}[\Gamma]$.

Let us now provide a polynomial procedure that determines whether σ is an arbitrary solution to $\mathbb{P}$. We assume w.l.o.g. that $\sigma = \bigodot\limits_{i=1}^{m} (p_i \diamond \overline{\mathsf{c}}_i)$ and check that $\Gamma, \sigma \not\models_{Ł} \bot$ can be verified in polynomial time. To see this, observe that $\Gamma \cup \{\sigma\}$ is Ł-satisfiable iff the following system of inequalities *has a solution* over $[0, 1]$ (cf. Definition 5.4 for $\Gamma^{\mathbb{LI}}$):

$$\mathsf{S}_{\mathsf{sat}} = \Gamma^{\mathbb{LI}} \cup \{p_i \diamond c_i \mid i \in \{1, \ldots, m\}\} \tag{11}$$

It is clear that the size of $\mathsf{S}_{\mathsf{sat}}$ is polynomial in $\mathsf{lgth}[\mathbb{P}]$. Thus, it takes polynomial time to check whether $\Gamma, \sigma \not\models_{Ł} \bot$.

It now remains to see that $\Gamma, \sigma \models_{Ł} \chi$ can be verified in polynomial time. We reason by cases depending on whether χ is a simple term, a simple clause, an interval term, or an interval clause.

First, let $\chi = \bigodot\limits_{i=1}^{n} l_i$ be a simple term. It is clear that σ is a solution to $\mathbb{P} = \left\langle \Gamma, \bigodot\limits_{i=1}^{n} l_i, \mathsf{H} \right\rangle$ iff σ is a solution for every $\mathbb{P} = \langle \Gamma, l_i, \mathsf{H} \rangle$ with $i \in \{1, \ldots, n\}$. Thus, it suffices to consider two cases: $\chi = p$ and $\chi = \neg p$. Consider the following two systems of inequalities:

$$\mathsf{S}_p = \mathsf{S}_{\mathsf{sat}} \cup \{p < 1\} \qquad\qquad \mathsf{S}_{\neg p} = \mathsf{S}_{\mathsf{sat}} \cup \{p > 0\} \tag{12}$$

It is clear that $\Gamma, \sigma \models_{Ł} p$ iff S_p *does not* have a solution over $[0, 1]$ and that $\Gamma, \sigma \models_{Ł} \neg p$ iff $\mathsf{S}_{\neg p}$ *does not* have a solution over $[0, 1]$.

In the second case, $\chi = \bigoplus\limits_{i=1}^{m} s_i \oplus \bigoplus\limits_{j=1}^{n} \neg t_j$ is a simple clause. To verify that $\Gamma, \sigma \models_{Ł} \chi$, we need to check that the following system of inequalities does not have solutions over $[0, 1]$:

$$\mathsf{S}_{\oplus} = \mathsf{S}_{\mathsf{sat}} \cup \left\{ \sum_{i=1}^{m} s_i + \sum_{j=1}^{n} (1 - t_j) < 1 \right\} \tag{13}$$

In the third case, $\chi = \bigodot\limits_{i=1}^{n} (q_i \diamond \overline{\mathrm{d}}_i)$. Just as in the first case, it is clear that it suffices to check that $\Gamma, \sigma \models_{Ł} q \diamond \overline{\mathrm{d}}$ can be verified in polynomial time. One can check that the entailment $\Gamma, \sigma \models_{Ł} q \diamond \overline{\mathrm{d}}$ holds iff the following system of inequalities does not have solutions over $[0, 1]$:

$$\mathsf{S}^{\diamond}_{p} = \mathsf{S}_{\mathsf{sat}} \cup \{q \blacklozenge d\} \tag{14}$$

Finally, in the fourth case, $\chi = \bigoplus\limits_{i=1}^{n} (q_i \diamond \overline{\mathrm{d}}_i)$ is an interval clause. To verify that $\Gamma, \sigma \models_{Ł} \chi$, we check that the following system of inequalities does not have solutions over $[0, 1]$:

$$\mathsf{S}^{\diamond}_{\oplus} = \mathsf{S}_{\mathsf{sat}} \cup \{q_i \blacklozenge d_i \mid i \in \{1, \ldots, n\}\} \tag{15}$$

One can see from (12), (13), (14), and (15) that the sizes of S_p, $\mathsf{S}_{\neg p}$, $\mathsf{S}_{\oplus}$, $\mathsf{S}^{\diamond}_{p}$, and $\mathsf{S}^{\diamond}_{\oplus}$ are polynomial in $\mathsf{lgth}[\mathbb{P}]$. Thus, the verification of entailment in simple clause abduction problems takes polynomial time, as required.

To see that *proper* solutions to SCA problems can be recognised in polynomial time, observe that given σ and $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$, we need to check that (i) $\Gamma, \sigma \models^{\mathsf{cons}}_{Ł} \chi$ and then that (ii) $\sigma \not\models_{Ł} \chi$. We have just shown that (i) can be verified in polynomial time. It follows that (ii) can also be verified in polynomial time.

Finally, to check whether σ is an $\models Ł$-*minimal* solution to $\mathbb{P}$, we first need to check that (i) it is a proper solution (this takes us polynomial time) and then (ii) establish that there is no other proper solution σ' such that $\sigma \models_{Ł} \sigma'$ but $\sigma' \not\models_{Ł} \sigma$. Furthermore, (ii) can also be checked in polynomial time. By Proposition 3.14, it suffices to verify that $\Gamma, \sigma^{\flat}_{\lambda} \not\models^{\mathsf{cons}}_{Ł} \chi$ for every 'next weakest' term $\sigma^{\flat}_{\lambda}$. Furthermore, there are only polynomially many of those. Moreover, by Theorem 5.7, it takes polynomial time to check that $\Gamma, \sigma^{\flat}_{\lambda} \not\models^{\mathsf{cons}}_{Ł} \chi$. The result follows. □

For the recognition of *theory-minimal* solutions, we provide a coNP membership result.

Theorem 5.8. *It is in* coNP *to decide, given an SCA problem* $\mathbb{P}$ *and an interval term* τ*, whether* τ *is a theory-minimal solution of* $\mathbb{P}$*.*

Proof. We consider the complementary problem: given τ and $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$, verify that it is *not* a theory-minimal solution and show that it belongs to NP. Indeed, it suffices to guess a proper solution τ' to $\mathbb{P}$ such that $\Gamma, \tau \models_{Ł} \tau'$ and $\Gamma, \tau' \not\models_{Ł} \tau$. Theorem 5.7 shows that these checks and a verification that τ' is a proper solution can be done in polynomial time. Thus, it is in NP to check that τ *is not* a theory minimal solution to $\mathbb{P}$. □

As expected, *solution existence* for simple clause abduction problems is NP-complete.

Theorem 5.9. *Given a simple clause abduction problem* $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$*, it is* NP*-complete to decide whether it has (proper) solutions.*

Proof. The membership is immediate from Theorem 5.7. For hardness, we can use a reduction from classical Horn abduction. Let $\mathbb{P} = \langle \Gamma, p, \mathsf{H} \rangle$ be a classical Horn abduction problem such that $p \notin \mathsf{H}$ and clauses in Γ are finite sets of literals. Now, consider $\mathbb{P}^{\oplus} = \langle \Gamma^{\oplus}, p, \mathsf{H}^{\odot} \rangle$ (cf. Definition 5.5 for $\Gamma^{\oplus}$ and Convention 3 for $\mathsf{H}^{\odot}$). Let further, $\tau \in \mathcal{S}(\mathbb{P})$. We show that $\tau^{\odot} \in \mathcal{S}(\mathbb{P})$.

First, we have that $\Gamma, \tau \not\models_{\mathsf{CPL}} \bot$. It follows that $\Gamma^{\oplus}, \tau^{\odot} \not\models_{Ł} \bot$. Indeed, if some classical valuation v satisfies Γ and τ, then it also Ł-satisfies $\Gamma^{\oplus}$ and $\tau^{\odot}$ because $\oplus$ and $\odot$ behave classically on $\{0, 1\}$. It remains to see that

$\Gamma, \tau \models_{\mathsf{CPL}} p$ entails $\Gamma^{\oplus}, \tau^{\odot} \models_{Ł} p$. To see this, we reuse the argument from Lemma 5.6. Namely, as $\Gamma \cup \{\tau\}$ is a classical Horn theory, then there must be a unit resolution inference of p from $\Gamma \cup \{\tau\}$. Moreover, recall that $p{\geq}\overline{1} \simeq_{Ł} p$ and $q{\leq}\overline{0} \simeq_{Ł} \neg q$. Thus, we can reuse the classical inference in Łukasiewicz logic and also derive $\{p\}$ from $\Gamma^{\oplus} \cup \{\tau^{\odot}\}$. It follows that $\tau^{\odot} \in \mathcal{S}(\mathbb{P}^{\oplus})$.

Conversely, let $\sigma \in \mathcal{S}(\mathbb{P}^{\oplus})$ for some interval term σ. We show that $\sigma_{\mathsf{cl}} \in \mathcal{S}(\mathbb{P})$ (recall Convention 3 for σ_{cl}). It is clear that $\Gamma^{\oplus}, \sigma \models_{Ł} p$ entails $\Gamma, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} p$. Assume for contradiction that $\Gamma, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \bot$. Then there is a unit resolution inference of the empty clause from $\Gamma \cup \{\sigma_{\mathsf{cl}}\}$ that we can reuse in Łukasiewicz logic. But this would mean that $\Gamma^{\oplus}, \sigma \models_{Ł} \bot$, contrary to the assumption.

Finally, observe that $\mathbb{P}$ and $\mathbb{P}^{\oplus}$ do not have *non-proper* solutions because $p \notin \mathsf{H}$ and $(p{\geq}\overline{1}) \notin \mathsf{H}^{\odot}$. Thus, our reduction also works for the case of proper solution existence. ◻

The NP-completeness of relevance of hypotheses can be obtained using the reduction shown in (8), assuming that Γ is a set of *simple clauses*. Indeed, if ψ is a simple clause, then $t \to \psi$ is strongly equivalent to a simple clause $\neg t \oplus \psi$ (recall (10)).

Theorem 5.10. *It is* NP*-complete (respectively,* coNP*-complete) to decide, given an SCA problem* $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ *and* $\lambda \in \mathsf{H}$*, whether* λ *is relevant (respectively, necessary) w.r.t. all solutions, proper solutions, and entailment-minimal solutions.*

We finish the section with several remarks. First, the complexity of simple clause abduction coincides with that of *classical Horn abduction* (Creignou and Zanuttini 2006; Eiter and Gottlob 1995). Thus, Łukasiewicz abduction with clausal theories is *simpler than classical abduction* with clausal theories.

Second, we did not consider observations in the form of 'mixed terms' $\chi_{\mathsf{MT}} = \bigodot_{i=1}^{k} \lambda_i \odot \bigodot_{i=1}^{k} \mu_i$ or 'mixed clauses' $\chi_{\mathsf{MC}} = \bigoplus_{i=1}^{k} \lambda_i \oplus \bigoplus_{i=1}^{m} \mu_i$ (with λ's being *interval* literals and μ's *simple* literals) in Theorems 5.7–5.10. Note, however, that our results can be straightforwardly extended to such observations. Namely, to explain χ_{MT}, it suffices to find an interval term τ that explains all literals in χ_{MT}. For χ_{MC}, we proceed as follows. Let τ be an interval term and $\mathbb{P} = \langle \Gamma, \chi_{\mathsf{MC}}, \mathsf{H} \rangle$ be an Ł-abduction problem such that Γ is a set of simple clauses and interval terms. Now note that $\Gamma, \tau \models_{Ł}^{\mathsf{cons}} \chi_{\mathsf{MC}}$ iff $\Gamma, \tau, \bigodot_{i=1}^{k} \neg\lambda_i \models_{Ł}^{\mathsf{cons}} \bigoplus_{i=1}^{m} \mu_i$ (because $v(\bigodot_{i=1}^{k} \neg\lambda_i) \in \{0, 1\}$ for every Ł-valuation v). Since $\bigodot_{i=1}^{k} \neg\lambda_i$ is an interval term, we have that $\mathbb{P}' = \langle \Gamma \cup \{\bigodot_{i=1}^{k} \neg\lambda_i\}, \bigoplus_{i=1}^{m} \mu_i, \mathsf{H} \rangle$ is an SCA problem and that τ is a (proper, $\models_{Ł}$-minimal) solution to $\mathbb{P}'$ iff τ is a (proper, $\models_{Ł}$-minimal) solution to $\mathbb{P}$.

Third, one can easily see that simple clause abduction problems can be straightforwardly generalised to problems whose theories are *Łukasiewicz fuzzy logic programs* as presented by Vojtáš (1999, 2001) *while preserving the complexity.*

The next definition adapts the notion of a fuzzy logic program from (Vojtáš 1999, 2001).

Definition 5.11. A *Łukasiewicz fuzzy logic program* (Ł-FLP) is a finite set

$$\Gamma_{\mathsf{P}} = \{\kappa_i \diamond x_i \mid i \in \{1, \ldots, n\}, x_i \in [0, 1]_{\mathbb{Q}}, \diamond \in \{\leq, <, >, \geq\}\}$$

with κ_i's being simple clauses written as $\bigodot_{i=1}^{m} l_i \to l$. Constructions $\kappa_i \diamond x_i$ are called *fuzzy rules.* An Ł-valuation v *satisfies* Γ_{P} if $v(\kappa_i) \diamond x_i$ for every $(\kappa_i \diamond x_i) \in \Gamma_{\mathsf{P}}$.

An Ł*-FLP abduction problem* is a tuple $\mathbb{P} = \langle \Gamma_{\mathsf{P}}, \chi, \mathsf{H} \rangle$ with Γ_{P} being an Ł-FLP, χ a fuzzy rule or interval term, and H a set of interval literals. Solutions to $\mathbb{P}$ are defined as in Definition 3.7.

One can observe from Definition 5.3 that simple clause abduction problems are a particular case of Ł-FLP abduction problems (namely, when $x_i = 1$ for every i). Thus, the hardness results are preserved. For the membership, note that Ł-FLP abduction problems can be reduced to solving systems of linear inequalities in the same way as simple clause abduction problems. The following corollary is a straightforward consequence of Theorems 5.7, 5.9, and 5.10.

Corollary 5.12. *Given an Ł-FLP abduction problem $\mathbb{P} = \langle \Gamma_{\mathsf{P}}, \chi, \mathsf{H}\rangle$, an interval term τ, and $\lambda \in \mathsf{H}$, it holds that:*

(1) it is P*-complete under logspace reductions to determine whether τ is a (proper, $\models_{Ł}$-minimal) solution to $\mathbb{P}$;*
(2) it is NP*-complete to determine whether $\mathbb{P}$ has a (proper) solution;*
(3) it is NP*-complete (*coNP*-complete) to determine whether λ is relevant (necessary) w.r.t. all (proper, $\models_{Ł}$-minimal) solutions.*

5.2 Interval Clause Fragments

Results in Section 4 show that the complexity of Ł-abduction in the general case was not affected by whether we allow the use of interval literals *in theories or observations*. Recall from the proofs that we established hardness results without assuming that theories or observations contain interval literals. On the other hand, membership results hold even if theories and observations of Ł-abduction problems contain interval literals (recall from Definition 3.7 that theories and observations are defined over $\mathcal{L}_{Ł}^{\mathbb{Q}}$ and can contain interval literals).

Using interval literals in the formulations of abduction problems is quite convenient. In particular, they can reduce the size and facilitate the understanding (for a human) of an abduction problem while preserving its solutions. E.g., theories Γ_{lift} and Γ_{ship} of problems $\mathbb{P}_{\mathsf{lift}}$ and $\mathbb{P}_{\mathsf{ship}}$ from Examples 3.2 and 3.3 can be reformulated as follows:

$$\Gamma^{\mathbb{Q}}_{\mathsf{lift}} = \left\{(c{\geq}\overline{\tfrac{1}{4}}) \rightarrow g, (c{\leq}\overline{\tfrac{2}{3}}) \rightarrow b\right\} \qquad \Gamma^{\mathbb{Q}}_{\mathsf{ship}} = \left\{\neg(p \leftrightarrow s) \leftrightarrow d, (p \oplus s) \leftrightarrow l, (d{\leq}\overline{\tfrac{1}{5}}) \odot (l{\leq}\overline{\tfrac{3}{4}}) \odot (l{\geq}\overline{\tfrac{1}{2}}) \rightarrow c\right\}$$

One can straightforwardly check that for $\mathbb{P}^{\mathbb{Q}}_{\mathsf{lift}} = \langle \Gamma^{\mathbb{Q}}_{\mathsf{lift}}, g \odot b, \mathsf{H}_{\mathsf{lift}}\rangle$ and $\mathbb{P}^{\mathbb{Q}}_{\mathsf{ship}} = \langle \Gamma^{\mathbb{Q}}_{\mathsf{ship}}, c, \mathsf{H}_{\mathsf{ship}}\rangle$, we have $\mathcal{S}(\mathbb{P}_{\mathsf{lift}}) = \mathcal{S}(\mathbb{P}^{\mathbb{Q}}_{\mathsf{lift}})$ and $\mathcal{S}(\mathbb{P}_{\mathsf{ship}}) = \mathcal{S}(\mathbb{P}^{\mathbb{Q}}_{\mathsf{ship}})$ (and likewise for other types of solutions). Indeed, observe that $(c{\geq}\overline{\tfrac{1}{4}}) \simeq_{Ł} (c \oplus c \oplus c \oplus c)$. Hence, if τ is an interval term, then $(c{\geq}\overline{\tfrac{1}{4}}) \rightarrow g, \tau \models^{\mathsf{cons}}_{Ł} g$ iff $(c \oplus c \oplus c \oplus c) \rightarrow g, \tau \models^{\mathsf{cons}}_{Ł} g$. Similar reasoning establishes that $(c{\leq}\overline{\tfrac{2}{3}}) \rightarrow b, \tau \models^{\mathsf{cons}}_{Ł} b$ iff $(\neg c \oplus \neg c \oplus \neg c) \rightarrow b, \tau \models^{\mathsf{cons}}_{Ł} b$ and that solutions to $\mathbb{P}^{\mathbb{Q}}_{\mathsf{ship}}$ and $\mathbb{P}_{\mathsf{ship}}$ coincide too.

Moreover, using interval literals in $\oplus$ clauses simplifies the presentation of relations between values. E.g., 'if $v(p) \geq \frac{1}{2}$, then $v(q) \leq \frac{3}{4}$' can be put as $(p{\geq}\overline{\tfrac{1}{2}}) \rightarrow (q{\leq}\overline{\tfrac{3}{4}})$. Furthermore, interval literals increase the expressivity of Łukasiewicz logic. While there are $\mathcal{L}_{Ł}$-formulas that are *weakly* equivalent to interval literals $p{\leq}\overline{\mathsf{c}}$ or $p{\geq}\overline{\mathsf{d}}$ (recall Example 3.3 and (6)), *strict intervals* $p'{>}\overline{\mathsf{c}}'$ and $p'{<}\overline{\mathsf{d}}'$ can only be expressed using the Baaz Delta operator $\triangle$ (recall (5)).

5.2.1 Arbitrary Interval Clauses. We begin with an observation on the abductive reasoning in the interval-clause fragment. In Example 3.15, we sketched a procedure that generates solutions to *arbitrary* Ł-abduction problems. For that, we needed to use a reduction from Łukasiewicz logic to mixed-integer programming and MIP solvers. On the other hand, Ł-abduction in the interval-clause fragment can be straightforwardly translated to classical abduction.

Definition 5.13. Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H}\rangle$ be an Ł-abduction problem such that Γ is a set of interval clauses and χ is an interval clause or an interval term. For each interval literal λ occurring in $\mathbb{P}$, we let r_λ be a fresh propositional variable such that $r_\lambda \notin \mathsf{Pr}[\mathbb{P}]$. Furthermore,

- if $\kappa = \bigoplus_{i=1}^{n} \lambda$ is an interval clause, we set $\kappa_{\mathsf{CPL}} = \bigvee_{\lambda \in \kappa} r_\lambda$;
- if $\sigma = \bigodot_{i=1}^{n} \lambda_i$ is an interval term, we set $\sigma_{\mathsf{CPL}} = \bigwedge_{\lambda \in \sigma} r_\lambda$.

Now, we define the CPL counterpart $\mathbb{P}_{\mathsf{CPL}} = \langle \Gamma_{\mathsf{CPL}}, \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}, \mathsf{H}_{\mathsf{CPL}}\rangle$ of $\mathbb{P}$ as follows.

$$\Gamma_{\top} = \left\{ r_\lambda \vee r_{\lambda'} \,\middle|\, \begin{matrix}\lambda, \lambda' \text{ occur in } \mathbb{P} \\ Ł \models \lambda \oplus \lambda'\end{matrix} \right\} \qquad \Gamma_{\bot} = \left\{ \neg r_\lambda \vee \neg r_{\lambda'} \,\middle|\, \begin{matrix}\lambda, \lambda' \text{ occur in } \mathbb{P} \\ \lambda, \lambda' \models_{Ł} \bot\end{matrix} \right\} \quad \Gamma_{\subseteq} = \left\{ \neg r_\lambda \vee r_{\lambda'} \,\middle|\, \begin{matrix}\lambda, \lambda' \text{ occur in } \mathbb{P} \\ \lambda \models_{Ł} \lambda'\end{matrix} \right\}$$

$$\Gamma_{\mathsf{CPL}} = \{\kappa_{\mathsf{CPL}} \mid \kappa \in \Gamma\} \cup \Gamma_{\top} \cup \Gamma_{\bot} \cup \Gamma_{\subseteq}$$
$$\gamma_{\mathbb{P}} = \bigwedge_{\gamma_{\top} \in \Gamma_{\top}} \gamma_{\top} \wedge \bigwedge_{\gamma_{\bot} \in \Gamma_{\bot}} \gamma_{\bot} \wedge \bigwedge_{\gamma_{\subseteq} \in \Gamma_{\subseteq}} \gamma_{\subseteq}$$
$$\mathsf{H}_{\mathsf{CPL}} = \{r_{\lambda} \mid \lambda \in \mathsf{H}\}$$

Theorem 5.14. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem such that Γ is a set of interval clauses and χ is an interval clause or an interval term. Let further, $\mathbb{P}_{\mathsf{CPL}} = \langle \Gamma_{\mathsf{CPL}}, \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}, \mathsf{H}_{\mathsf{CPL}} \rangle$ be the CPL-counterpart of $\mathbb{P}$. Then*

(1) $\mathbb{P}_{\mathsf{CPL}}$ can be constructed in polynomial time w.r.t. $\mathsf{lgth}[\mathbb{P}]$;

(2) σ is a (proper, theory-minimal) solution to $\mathbb{P}$ iff σ_{CPL} is a (proper, theory-minimal) solution to $\mathbb{P}_{\mathsf{CPL}}$.

Proof. We begin with Item 1. To see that $\mathbb{P}_{\mathsf{CPL}}$ can be constructed in polynomial time, we proceed as follows. First, we observe that $\Gamma_{\top}$, $\Gamma_{\bot}$, and $\Gamma_{\subseteq}$ each have length $O(\mathsf{lgth}[\mathbb{P}]^2)$ because they contain, in the worst case, all pairs of interval literals occurring in $\mathbb{P}$. Similarly, notice that $\mathsf{lgth}(\gamma_{\mathbb{P}}) = O(|\Gamma_{\mathsf{CPL}}|)$ and $\mathsf{lgth}[\mathsf{H}_{\mathsf{CPL}}] = O(\mathsf{lgth}[\mathsf{H}])$. Thus, $\mathsf{lgth}[\mathbb{P}_{\mathsf{CPL}}] = O(\mathsf{lgth}[\mathbb{P}]^2)$. It remains to see that Γ_{CPL}, $\gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}$, and $\mathsf{H}_{\mathsf{CPL}}$ can all be constructed in polynomial time. For $\mathsf{H}_{\mathsf{CPL}}$ and χ_{CPL}, this follows immediately from Definition 5.13. For Γ_{CPL}, observe that we need to check whether (A) $\text{Ł} \models \lambda \oplus \lambda'$, (B) $\lambda, \lambda' \models_{\text{Ł}} \bot$, and (C) $\lambda \models_{\text{Ł}} \lambda'$ for every pair of interval literals λ and λ' occurring in $\mathbb{P}$. By Proposition 3.5, it takes polynomial time to verify (B) and (C). Furthermore, observe from Definition 2.2 that $\text{Ł} \models \lambda \oplus \lambda'$ iff $\neg\lambda \models_{\text{Ł}} \lambda'$, whence (A) can also be verified in polynomial time. It follows that Γ_{CPL} (and thus, $\gamma_{\mathbb{P}}$) can be constructed in polynomial (quadratic) time w.r.t. $\mathbb{P}$. This finishes the proof of Item 1.

Let us now proceed to Item 2. Assume that $\sigma \in \mathcal{S}(\mathbb{P})$. Observe, first, that $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models^{\mathsf{cons}}_{\mathsf{CPL}} \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}$ iff $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$. This is because $\gamma \in \Gamma_{\mathsf{CPL}}$ for every $\gamma \in \gamma_{\mathbb{P}}$. Thus, it suffices to show that $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$. Let v be an Ł-valuation witnessing that $\Gamma, \sigma \not\models_{\text{Ł}} \bot$. We construct the classical valuation v_{CPL} as follows:

$$v_{\mathsf{CPL}}(r_{\lambda}) = \begin{cases} 1 & \text{if } v(\lambda) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

It is clear from the construction of Γ_{CPL} and σ_{CPL} that v_{CPL} witnesses $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \not\models_{\mathsf{CPL}} \bot$. It remains to see that $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$.

Assume for the sake of contradiction that $\mathbf{v}$ is a classical valuation such that $\mathbf{v}(\phi) = 1$ for all $\phi \in \Gamma_{\mathsf{CPL}}$, $\mathbf{v}(\sigma_{\mathsf{CPL}}) = 1$, but $\mathbf{v}(\chi_{\mathsf{CPL}}) = 0$. We need to define an Ł-valuation $v_{\mathbf{v}}$ such that $v_{\mathbf{v}}(\lambda) = 1$ iff $\mathbf{v}(r_{\lambda}) = 1$ would hold for every λ occurring in $\mathbb{P}$. For every $p \in \mathsf{Pr}[\mathbb{P}]$, we define $\mathcal{V}_p$ — the interval of values that p can have so that $v_{\mathbf{v}}(p \diamond \overline{\mathsf{e}}) = 1$ iff $\mathbf{v}(r_{p \diamond \overline{\mathsf{e}}}) = 1$ for every interval literal of the form $p \diamond \overline{\mathsf{e}}$ occurring in $\mathbb{P}$. We do this depending on the comparison sign ($\leq$, $<$, $>$, or $\geq$) of $p \diamond \overline{\mathsf{e}}$ and on whether the corresponding variable $r_{p \diamond \overline{\mathsf{e}}}$ is true or false under $\mathbf{v}$. Taking the intersection of all such intervals produces the set of values that satisfy all constraints imposed by $\mathbf{v}$. (Below, $r_{p \diamond \overline{\mathsf{c}}}, r_{p \diamond \overline{\mathsf{d}}} \in \mathsf{Pr}[\mathbb{P}_{\mathsf{CPL}}]$.)

$$\begin{aligned}\mathcal{V}_p = & \bigcap_{\mathbf{v}(r_{p \geq \overline{\mathsf{c}}})=1} [c, 1] \cap \bigcap_{\mathbf{v}(r_{p > \overline{\mathsf{c}}})=1} (c, 1] \cap \bigcap_{\mathbf{v}(r_{p \leq \overline{\mathsf{c}}})=0} (c, 1] \cap \bigcap_{\mathbf{v}(r_{p < \overline{\mathsf{c}}})=0} [c, 1] \cap \\ & \bigcap_{\mathbf{v}(r_{p \geq \overline{\mathsf{d}}})=0} [0, d) \cap \bigcap_{\mathbf{v}(r_{p > \overline{\mathsf{d}}})=0} [0, d] \cap \bigcap_{\mathbf{v}(r_{p \leq \overline{\mathsf{d}}})=1} [0, d] \cap \bigcap_{\mathbf{v}(r_{p < \overline{\mathsf{d}}})=1} [0, d)\end{aligned} \tag{17}$$

We show that $\mathcal{V}_p \neq \varnothing$ for each $p \in \mathsf{Pr}[\mathbb{P}]$. Assume for the sake of contradiction that there is some $p \in \mathsf{Pr}[\mathbb{P}]$ such that $\mathcal{V}_p = \varnothing$. This means that one of the following holds.

(1) There are $\triangleright, \triangleright' \in \{\geq, >\}$, $\triangleleft, \triangleleft' \in \{\leq, <\}$, and $c, d \in [0, 1]_{\mathbb{Q}}$ such that $c > d$ and
   (a) $v(r_{p \triangleright \overline{\mathsf{c}}}) = 1$ and $v(r_{p \triangleleft \overline{\mathsf{d}}}) = 1$, or
   (b) $v(r_{p \triangleright \overline{\mathsf{c}}}) = 1$ and $v(r_{p \triangleright' \overline{\mathsf{d}}}) = 0$, or
   (c) $v(r_{p \triangleleft \overline{\mathsf{c}}}) = 0$ and $v(r_{p \triangleright \overline{\mathsf{d}}}) = 0$, or

(d) $v(r_{p\lhd\overline{c}}) = 0$ and $v(r_{p\lhd'\overline{d}}) = 1$.

(2) There are $\rhd, \rhd' \in \{\geq, >\}$, $\lhd, \lhd' \in \{\leq, <\}$, and $c, d \in [0,1]_{\mathbb{Q}}$ such that $c \geq d$, $\rhd \neq \rhd'$, $\lhd \neq \lhd'$, and
  (a) $v(r_{p\rhd\overline{c}}) = 1$ and $v(r_{p\lhd'\overline{d}}) = 1$, or
  (b) $v(r_{p\rhd\overline{c}}) = 1$ and $v(r_{p\rhd'\overline{d}}) = 0$, or
  (c) $v(r_{p\lhd\overline{c}}) = 0$ and $v(r_{p\rhd'\overline{d}}) = 0$ (with $\langle\lhd, \rhd'\rangle \in \{\langle<, \geq\rangle, \langle\leq, >\rangle\}$), or
  (d) $v(r_{p\lhd\overline{c}}) = 0$ and $v(r_{p\lhd'\overline{d}}) = 1$.

We consider the first case since the second one can be dealt with in a similar way. In the case (1.a), note that $p \rhd \overline{c}, p \lhd \overline{d} \models_{Ł} \bot$. Thus, $\neg r_{p\rhd\overline{c}} \vee \neg r_{p\lhd\overline{d}} \in \Gamma_{\mathsf{CPL}}$. Hence, $\mathbf{v}(\neg r_{p\rhd\overline{c}} \vee \neg r_{p\lhd\overline{d}}) = 0$, and $\mathbf{v}$ does not satisfy Γ_{CPL} contrary to the assumption. In the case (1.b), note that $(p \rhd \overline{c}) \models_{Ł} (p \rhd' \overline{d})$, whence $\neg r_{p\rhd\overline{c}} \vee r_{p\rhd'\overline{d}} \in \Gamma_{\mathsf{CPL}}$. Again, $\mathbf{v}$ does not satisfy Γ_{CPL}. In the case (1.c), we have that $Ł \models (p \lhd \overline{c}) \oplus (p \rhd \overline{d})$, whence $r_{p\lhd\overline{c}} \vee r_{p\rhd\overline{d}} \in \Gamma_{\mathsf{CPL}}$, and again, $\mathbf{v}$ does not satisfy Γ_{CPL}, contrary to the assumption. Finally, in the case (1.d), we have $(p \lhd' \overline{d}) \models_{Ł} (p \lhd \overline{c})$, whence $\neg r_{p\lhd'\overline{d}} \vee r_{p\lhd\overline{c}} \in \Gamma_{\mathsf{CPL}}$. Just as in the case (1.b), we have that $\mathbf{v}$ does not satisfy Γ_{CPL}.

Now, for every p, we set $v_{\mathbf{v}}(p) = x$ for some $x \in \mathcal{V}_p$. It is easy to see that $v_{\mathbf{v}}(\theta) = 1$ for all $\theta \in \Gamma$ and $v_{\mathbf{v}}(\sigma) = 1$ but $v_{\mathbf{v}}(\chi) = 0$. Indeed, observe from (17) that $\mathbf{v}(r_\lambda) = 1$ iff $v_{\mathbf{v}}(\lambda) = 1$ and recall from Definition 2.2 that $\oplus$ behaves like $\vee$ on $\{0, 1\}$. Thus, $\Gamma, \sigma \not\models_{Ł} \chi$ which contradicts the assumption that $\sigma \in \mathcal{S}(\mathbb{P})$. Thus, $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$. It follows now that σ_{CPL} is a solution to $\mathbb{P}_{\mathsf{CPL}}$.

Conversely, let σ_{CPL} be a solution to $\mathbb{P}_{\mathsf{CPL}}$, that is, $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$. Assume that $\mathbf{v}$ is a classical valuation witnessing $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \not\models_{\mathsf{CPL}} \bot$. We define $v_{\mathbf{v}}$ using (17) as in the previous part of the proof, which gives us that $v_{\mathbf{v}}$ witnesses $\Gamma, \sigma \not\models_{Ł} \bot$. It remains to see that $\Gamma, \sigma \models_{Ł} \chi$. To do that, assume for contradiction that there is some Ł-valuation witnessing $\Gamma, \sigma \not\models_{Ł} \chi$. We define v_{CPL} as shown in (16). It is clear that v_{CPL} would witness $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \not\models_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$, contrary to the assumption. Hence, $\sigma \in \mathcal{S}(\mathbb{P})$, as required.

Let us now briefly consider the remaining cases of σ being a proper and a theory-minimal solution to $\mathbb{P}$. If σ is a proper solution to $\mathbb{P}$, then $\sigma \not\models_{Ł} \chi$. Using the same reasoning as above, one can show that $\sigma_{\mathsf{CPL}} \not\models_{\mathsf{CPL}} \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}$. For this, we note that the deduction theorem for classical logic implies that $\sigma_{\mathsf{CPL}} \not\models_{\mathsf{CPL}} \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}$ iff $\Gamma_\top, \Gamma_\bot, \Gamma_\subseteq \not\models_{\mathsf{CPL}} \chi_{\mathsf{CPL}}$. For the converse direction, assume that σ is a solution, but not a proper one, then $\sigma \models_{Ł} \chi$, and we can show that $\sigma_{\mathsf{CPL}} \models_{\mathsf{CPL}} \gamma_{\mathbb{P}} \rightarrow \chi_{\mathsf{CPL}}$, which means that σ_{CPL} is not a proper solution to $\mathbb{P}_{\mathsf{CPL}}$.

If σ is a theory-minimal solution to $\mathbb{P}$, then there is no other proper solution ϱ such that $\Gamma, \sigma \models_{Ł} \varrho$ and $\Gamma, \varrho \not\models_{Ł} \sigma$. Now, assume for the sake of contradiction that there is some proper solution ϱ to $\mathbb{P}_{\mathsf{CPL}}$ such that $\Gamma, \sigma_{\mathsf{CPL}} \models \varrho$ and $\Gamma, \varrho \not\models_{\mathsf{CPL}} \sigma_{\mathsf{CPL}}$. Now define $\varrho^{\diamond Ł} = \bigodot_{r_\lambda \in \varrho} \lambda$. Again, we can show that $\varrho^{\diamond Ł}$ is a proper solution to $\mathbb{P}$ such that $\Gamma, \sigma \models_{Ł} \varrho^{\diamond Ł}$ and $\Gamma, \varrho^{\diamond Ł} \not\models_{Ł} \sigma$. Hence, σ is not a theory-minimal solution to $\mathbb{P}$. For the converse, we assume that σ is a proper solution but not a theory-minimal one. Then there is some solution ϱ such that $\Gamma, \sigma \models_{Ł} \varrho$ and $\Gamma, \varrho \not\models_{Ł} \sigma$. One can show that ϱ_{CPL} is a proper solution to $\mathbb{P}_{\mathsf{CPL}}$ such that $\Gamma_{\mathsf{CPL}}, \sigma_{\mathsf{CPL}} \models_{\mathsf{CPL}} \varrho_{\mathsf{CPL}}$ and $\Gamma, \varrho_{\mathsf{CPL}} \not\models_{Ł} \sigma_{\mathsf{CPL}}$. Hence, σ_{CPL} is not a theory-minimal solution either. ◻

Theorem 5.14 shows that we can use classical techniques of abductive reasoning to solve Ł-abduction problems with interval-clause theories and observations. Note, however, that the translation in Definition 5.13 does not work for $\models_{Ł}$-minimal solutions. This happens because any interval literal will be translated as a different propositional variable. Thus, while we can compare $\lambda = p{\geq}\overline{\tfrac{3}{4}}$ and $\mu = p{\geq}\overline{\tfrac{5}{8}}$ w.r.t. $\models_{Ł}$-entailment, their CPL-counterparts are propositional variables r_λ and r_μ. To compare those, we need to 'know' that $r_{p\geq\overline{3/4}}$ 'should imply' $r_{p\geq\overline{5/8}}$. But this cannot be expressed via a conjunction of literals. On the other hand, proper and *theory-minimal* solutions are preserved precisely because this information is not lost. One can also see that to preserve *arbitrary* solutions, it suffices to consider the problem $\langle\Gamma_{\mathsf{CPL}}, \chi_{\mathsf{CPL}}, \mathsf{H}_{\mathsf{CPL}}\rangle$.

As expected from Theorem 5.14, the complexity of abductive reasoning with arbitrary interval-clause theories coincides with that of classical abduction. The next statements provide the hardness results for solution recognition, solution existence, and relevance / necessity of hypotheses.

Theorem 5.15. *Given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$, such that Γ is a set of interval clauses and λ an interval literal, and an interval term τ, it is* DP*-complete to decide whether τ is a (proper) solution to $\mathbb{P}$.*

Proof. Membership follows from Theorem 4.2. We show the hardness via a reduction from the classical solution recognition. Assume that $\mathbb{P} = \langle \Gamma, p, \mathsf{H} \rangle$ is a *classical abduction problem* with $\Gamma = \{\kappa_i \mid i \in \{1, \ldots, n\}\}$ being a set of disjunctive clauses, $p \notin \mathsf{H}$, and that τ is a solution to $\mathbb{P}$. We define $\mathbb{P}_{\mathbf{int}} = \langle \Gamma_{\mathbf{int}}, p \geq \overline{1}, \mathsf{H}_{\mathbf{int}} \rangle$ and $\tau_{\mathbf{int}}$ as follows:

$$\Gamma_{\mathbf{int}} = \left\{ \bigoplus_{q_i \in \kappa_i} (q_i \geq \overline{1}) \oplus \bigoplus_{\neg r_i \in \kappa_i} (r_i < \overline{1}) \mid i \in \{1, \ldots, n\} \right\}$$
$$\mathsf{H}_{\mathbf{int}} = \{s \geq \overline{1} \mid s \in \mathsf{H}\} \cup \{t < \overline{1} \mid \neg t \in \mathsf{H}\}$$
$$\tau_{\mathbf{int}} = \bigodot_{s' \in \tau} (s' \geq \overline{1}) \odot \bigodot_{\neg t' \in \tau} (t' < \overline{1})$$

Observe briefly that since $p \notin \mathsf{H}$ and $p \geq \overline{1} \notin \mathsf{H}_{\mathbf{int}}$, all solutions to $\mathbb{P}$ and $\mathbb{P}_{\mathbf{int}}$ must be *proper*. Now let $\tau \in \mathcal{S}(\mathbb{P})$, i.e., $\Gamma, \tau \models^{\mathsf{cons}}_{\mathsf{CPL}} p$. Clearly, $\Gamma_{\mathbf{int}}, \tau_{\mathbf{int}} \not\models_{\text{Ł}} \bot$. To see that $\Gamma_{\mathbf{int}}, \tau_{\mathbf{int}} \models_{\text{Ł}} p$, assume for contradiction that that there is an Ł-valuation v that witnesses $\Gamma_{\mathbf{int}}, \tau_{\mathbf{int}} \not\models_{\text{Ł}} p \geq \overline{1}$. Consider a classical valuation v' such that $v'(q) = 1$ iff $v(q) = 1$ and $v'(q) = 0$ iff $v(q) \neq 1$. One can easily see that v' witnesses $\Gamma, \tau \not\models_{\mathsf{CPL}} p$, contrary to the assumption that $\tau \in \mathcal{S}(\mathbb{P})$. For the converse direction, let $\sigma \in \mathcal{S}(\mathbb{P}_{\mathbf{int}})$. A solution for $\mathbb{P}$ can be obtained by replacing $r \geq \overline{1}$ with r and $r < 1$ with $\neg r$. The reasoning is similar. □

The Σ^{P}_2-hardness of solution existence for Ł-abduction in the interval-clause fragment can be obtained by the reduction used in Theorem 5.15.

Theorem 5.16. *Given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$ such that Γ is a set of interval clauses and λ is an interval literal, it is Σ^{P}_2-complete to decide whether there is a (proper) solution to $\mathbb{P}$.*

As expected, determining the relevance of $\lambda \in \mathsf{H}$ is also Σ^{P}_2-hard. The next statement can be obtained by a reduction shown in (8). The only difference is that instead of *variables* t, t', t'', we take *interval literals* $t \geq \overline{1}$, $t' \geq \overline{1}$, and $t'' \geq \overline{1}$.

Theorem 5.17. *It is Σ^{P}_2-complete (respectively, Π^{P}_2-complete) to decide, given an Ł-abduction problem $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ such that Γ is a set of interval clauses and $\lambda \in \mathsf{H}$, whether λ is relevant (respectively, necessary) w.r.t. $\mathcal{S}(\mathbb{P})$. The same holds for relevance and necessity w.r.t. $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$ and $\mathcal{S}^{\min}(\mathbb{P})$.*

Note that the hardness results obtained in this section entail the hardness results in Section 4. On the other hand, one can observe from the proofs in Section 4 that the hardness results in Table 1 hold even if the theory and the observation of an abduction problem *do not* contain interval literals.

5.2.2 Interval Rules Fragments. As we see from the results in the previous section, interval-clause theories behave as classical disjunctive-clause theories. Thus, it makes sense to investigate restrictions on them that make them behave as more computationally simple classes of clause theories. In this section, we consider several such cases. Moreover, as we discussed above, interval clauses can be rewritten in the implicative form to represent statements of the form 'if the value of p is at least $\frac{2}{3}$, then the value of q is less than $\frac{1}{4}$'. Thus, we will represent interval clauses in the rule form as shown below.

Definition 5.18 (Interval rules). Let κ be an interval clause. We say that κ *has a rule form* iff $\kappa = \bigodot_{i=1}^{n}(p_i \diamond \overline{\mathsf{c}}_i) \rightarrow \bot$ or $\kappa = \bigodot_{i=1}^{n}(p_i \diamond \overline{\mathsf{c}}_i) \rightarrow (q \diamond \overline{\mathsf{d}})$. In the former case, we say that κ is a *negative* interval rule and in the latter case that it is a *positive* interval rule.

First, we can restrict interval clause (or interval rules) theories so that solution existence and relevance of hypotheses become NP-complete (and necessity of hypotheses becomes coNP-complete). Namely, we prohibit interval literals whose disjunction is 'covering' the entire $[0, 1]$ interval in the implicative representation of interval clauses. E.g., $p \leq \overline{\frac{2}{3}}$ and $p \geq \overline{\frac{1}{4}}$ cannot both occur in a cover-free theory because $[0, \frac{2}{3}] \cup [\frac{1}{4}, 1] = 1$. This restriction replicates the behaviour of *classical Horn theories*.

Definition 5.19 (Cover-free theories and abduction problems).

- Let Γ be a set of interval rules. Γ is *cover-free* (CF) if no pair of interval literals λ and λ' over the same variable such that $Ł \models \lambda \oplus \lambda'$ occurs in Γ.
- An Ł-abduction problem $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$, with $\tau = \bigodot_{i=1}^{k} \lambda_i$ being an interval term is cover-free iff $\Gamma \cup \{\lambda_i \mid i \in \{1, \ldots, k\}\} \cup \mathsf{H}$ is cover-free.

Note that in the definition above, we demand that *the union* of the theory, observation, and the set of hypotheses be cover-free. We need this to ensure that $\Gamma \cup \{\lambda'_i \mid i \in \{1, \ldots, m\}\}$ remains cover-free for every possible solution candidate $\sigma = \bigodot_{i=1}^{m} \lambda'_i$.

The following theorems can be shown by reductions between Horn abduction problems and Ł-abduction problems with CF theories. We will also rely on the proof of Theorem 5.14.

THEOREM 5.20. *Given a CF Ł-abduction problem $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$, it is* P*-complete under logspace reductions to check whether an interval term σ is a (proper, entailment-minimal) solution to $\mathbb{P}$.*

PROOF. For membership, we construct a reduction to the classical *Horn* abduction for which solution recognition is polynomial. We adapt the translation from Definition 5.13. Let $\Gamma = \{\kappa_1, \ldots, \kappa_m, \kappa'_1, \ldots \kappa'_{m'}\}$ be a set of interval rules of the form $\kappa = \bigodot_{j=1}^{n}(p_j \diamond \overline{\mathsf{c}}_j) \rightarrow (q_i \diamond \overline{\mathsf{d}}_i)$ and $\kappa' = \bigodot_{j=1}^{n'}(p'_j \diamond \overline{\mathsf{c}}_j) \rightarrow \bot$. Now, for each interval literal λ, let r_λ be a fresh propositional variable. We define

$$\kappa_{\mathcal{H}} \coloneqq \bigwedge_{j=1}^{n_i} r_{p_j \diamond \overline{\mathsf{c}}_j} \rightarrow r_{q_i \diamond \overline{\mathsf{d}}_i} \qquad\qquad \kappa'_{\mathcal{H}} \coloneqq \bigwedge_{j=1}^{n'_i} r_{p'_j \diamond \overline{\mathsf{c}}_j} \rightarrow \bot \tag{18}$$

We consider the following *classical* abduction problem: $\mathbb{P}_{\mathcal{H}} = \langle \Gamma_{\mathcal{H}}, \tau_{\mathsf{CPL}}, \mathsf{H}_{\mathsf{CPL}} \rangle$ defined as described below (recall Definition 5.13 for $\Gamma_\bot$, $\Gamma_\subseteq$, τ_{CPL}, and $\mathsf{H}_{\mathsf{CPL}}$).

$$\Gamma_{\mathcal{H}} = \{\kappa_{\mathcal{H}} \mid \kappa \in \Gamma\} \cup \{\kappa'_{\mathcal{H}} \mid \kappa' \in \Gamma\} \cup \Gamma_\bot \cup \Gamma_\subseteq \tag{19}$$

As one can see (recall the proof of Theorem 5.14), $\mathbb{P}_{\mathcal{H}}$ is polynomial (quadratic) in $\mathsf{lgth}[\mathbb{P}]$: to construct it, we check whether $\lambda, \lambda' \models_Ł \bot$ and $\lambda \models_Ł \lambda'$ for every pair of interval literals λ and λ' occurring in $\mathbb{P}$. Thus, $\mathsf{lgth}[\mathbb{P}_{\mathcal{H}}] = O(\mathsf{lgth}[\mathbb{P}]^2)$. Note, moreover, that by Proposition 3.5, these checks can be conducted in polynomial time, whence $\mathbb{P}_{\mathcal{H}}$ can be constructed in polynomial time. Furthermore, $\Gamma_{\mathcal{H}}$ is a set of Horn formulas and only contains variables of the form r_λ with λ being an interval term occurring in $\mathbb{P}$. Using the same reasoning as in the proof of Theorem 5.14, we can show that σ is a solution to $\mathbb{P}$ iff σ_{CPL} is a solution to $\mathbb{P}_{\mathcal{H}}$.

The only difference is that now $\Gamma_\top = \varnothing$ (recall Definition 5.13) because $\mathbb{P}$ is a CF problem. Hence, when showing that $\mathcal{V}_p \neq \varnothing$ for all $p \in \mathsf{Pr}[\mathbb{P}]$ (recall (17)), the cases (1.c) and (2.c) will be tackled differently. For the case (1.c), observe that we will have $Ł \models (p \triangleleft \overline{\mathsf{c}}) \oplus (p \triangleright \overline{\mathsf{d}})$. For the case (2.c), we will have $Ł \models (p \triangleleft \overline{\mathsf{c}}) \oplus (p \triangleright' \overline{\mathsf{d}})$. In both cases, it follows that Γ is not a CF theory, contrary to the assumption.

Thus, we have that recognition of arbitrary solutions to problems whose theories are sets of cover-free clauses is polynomial. As our observation is also an interval term, it follows from Proposition 3.5 that verifying $\sigma \not\models_{Ł} \tau$ takes polynomial time. This means that determining whether σ is a *proper* solution also takes polynomial time. Finally, to see that recognition of *entailment-minimal* solutions is polynomial as well, recall from the proof of Theorem 4.5 that there are only polynomially many candidate interval terms we need to check for a given σ.

For hardness, we construct a logspace reduction of the solution recognition for classical Horn abduction problems to the solution recognition for Łukasiewicz abduction problems with cover-free theories. Namely, let $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$ be a classical Horn abduction problem with Γ represented as Horn implications $\eta = \bigwedge_{i=1}^{m} p_i \rightarrow q$ and $\theta = \bigwedge_{i=1}^{m'} p'_i \rightarrow \bot$ and $\mathsf{H} \cap \mathsf{Pr}(\tau) = \varnothing$. We define

$$\eta^{\mathsf{C}} \coloneqq \bigodot_{i=1}^{m} (p_i \geq \overline{1}) \rightarrow (q \geq \overline{1}) \qquad\qquad \theta^{\mathsf{C}} \coloneqq \bigodot_{i=1}^{m'} (p'_i \geq \overline{1}) \rightarrow \bot \tag{20}$$

and set $\mathbb{P}^{\mathsf{C}} = \langle \Gamma^{\mathsf{C}}, \tau^{\odot}, \mathsf{H}^{\odot} \rangle$ as follows (recall Convention 3 for $\tau^{\odot}$ and $\mathsf{H}^{\odot}$):

$$\Gamma^{\mathsf{C}} = \left\{ \eta^{\mathsf{C}} \mid \eta \in \Gamma \right\} \cup \left\{ \theta^{\mathsf{C}} \mid \theta \in \Gamma \right\} \tag{21}$$

It is clear that Γ^{C} is cover-free and that $\mathbb{P}^{\mathsf{C}}$ can be obtained from $\mathbb{P}$ using logarithmic space only. Now assume that $\sigma \in \mathcal{S}(\mathbb{P})$. We show that $\sigma^{\odot} \in \mathcal{S}(\mathbb{P}^{\mathsf{C}})$. First, it is clear that $\Gamma^{\mathsf{C}}, \sigma^{\odot} \not\models_{Ł} \bot$ because $\Gamma, \sigma \not\models_{\mathsf{CPL}} \bot$. To check the entailment, observe that there is a unit resolution inference of literals comprising τ from $\Gamma \cup \{\sigma\}$. It is clear that classical unit resolution is sound for interval clauses: if $v(p \geq \overline{1}) = 1$ and $v(((p \geq \overline{1}) \odot \tau) \rightarrow \lambda) = 1$, then $v(\tau \rightarrow \lambda) = 1$; if $v(p \leq \overline{0}) = 1$ and $v(\tau \rightarrow (p \geq \overline{1})) = 1$, then $v(\tau \rightarrow \bot) = 1$. Thus, reusing the classical inference in Ł, we obtain $\Gamma^{\mathsf{C}}, \sigma^{\odot} \models_{Ł} \tau^{\odot}$, as required. Conversely, let $\sigma \in \mathcal{S}(\mathbb{P}^{\mathcal{H}})$ and define σ_{cl} as shown in Convention 3. It is clear that $\Gamma, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \tau$. To see that $\Gamma, \sigma_{\mathsf{cl}} \not\models_{\mathsf{CPL}} \bot$, assume for contradiction that $\Gamma, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \bot$. Then there is a unit resolution inference of the empty clause from $\Gamma \cup \{\sigma_{\mathsf{cl}}\}$. However, as we can reuse it in Łukasiewicz logic, this would mean that $\Gamma^{\mathsf{C}}, \sigma \models_{Ł} \bot$, which contradicts the assumption.

As $\mathsf{H} \cap \mathsf{Pr}(\tau) = \varnothing$, it follows that all solutions to $\mathbb{P}$ and $\mathbb{P}^{\mathsf{C}}$ are proper. Hence, the reduction is applicable to both arbitrary and proper solutions. Finally, to see that recognition of entailment-minimal solutions is P-hard, one can easily check that if $\sigma \in \mathcal{S}^{\min}(\mathbb{P})$, then $\sigma^{\odot} \in \mathcal{S}^{\min}(\mathbb{P}^{\mathsf{C}})$. Conversely, if $\sigma' \in \mathcal{S}^{\min}(\mathbb{P}^{\mathsf{C}})$, then $\sigma'_{\mathsf{cl}} \in \mathcal{S}^{\min}(\mathbb{P})$. □

The NP-completeness of solution existence for abduction problems with CF theories follows from Theorem 5.20. For hardness, we reuse the reduction shown in (21).

Theorem 5.21. *Given a CF Ł-abduction problem $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$, it is* NP-*complete to decide whether $\mathbb{P}$ has (proper) solutions.*

Determining the relevance of hypotheses in CF theories is NP-complete. The proof is the same as for Theorem 5.17 (but we reduce from hypotheses relevance for Horn theories).

Theorem 5.22. *It is* NP-*complete (respectively,* coNP-*complete) to decide, given a CF Ł-abduction problem $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$ and $\lambda \in \mathsf{H}$, whether λ is relevant (respectively, necessary) w.r.t. $\mathcal{S}(\mathbb{P})$. The same holds for relevance and necessity w.r.t. $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$ and $\mathcal{S}^{\min}(\mathbb{P})$.*

The next statement follows from Theorem 5.20 because it takes polynomial time to check whether $\Gamma, \sigma \models_{Ł} \tau$ given interval terms σ and τ and a CF theory Γ.

Theorem 5.23. *Given a CF Ł-abduction problem $\mathbb{P} = \langle \Gamma, \tau, \mathsf{H} \rangle$ and an interval term σ, it is in* coNP *to decide whether $\sigma \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$.*

As we have seen, abduction with cover-free theories has the same complexity as the classical Horn abduction. CF theories, however, have a drawback. Namely, consider the following example:

$$\Gamma_1 = \left\{ \underbrace{(p{\geq}\overline{\tfrac{1}{3}}) \odot (q{\geq}\overline{\tfrac{2}{3}}) \rightarrow (r{\leq}\overline{\tfrac{1}{4}})}_{\phi_1}, \underbrace{(p{\leq}\overline{\tfrac{1}{4}}) \odot (q{\geq}\overline{\tfrac{3}{4}}) \rightarrow (r{\geq}\overline{\tfrac{1}{2}})}_{\chi_1} \right\}$$

$$\Gamma_2 = \left\{ \underbrace{(p{\geq}\overline{\tfrac{1}{3}}) \odot (q{\geq}\overline{\tfrac{2}{3}}) \rightarrow (r{\leq}\overline{\tfrac{1}{4}})}_{\phi_1}, \underbrace{(p{\leq}\overline{\tfrac{1}{4}}) \odot (r < \overline{\tfrac{1}{2}}) \rightarrow (q < \overline{\tfrac{3}{4}})}_{\chi_2} \right\} \tag{22}$$

It is clear that $\chi_1 \equiv_{\text{Ł}} \chi_2$. On the other hand, Γ_1 is cover-free while Γ_2 is not because $\text{Ł} \models (q{\geq}\overline{\tfrac{2}{3}}) \oplus (q{<}\overline{\tfrac{3}{4}})$. Thus, a theory that is not cover-free might be equivalent to a cover-free one, and this may be non-trivial to establish.

Thus, it makes sense to consider easily recognisable subclasses of cover-free theories. These subclasses of cover-free theories resemble subclasses of (classical) Horn theories. As expected, we will see that abduction in some of them *becomes polynomially tractable*.

Definition 5.24. Let $\Gamma = \{\kappa_1, \ldots, \kappa_m\}$ be a set of interval rules.

- Γ is *weakly cover-free* (WCF) iff it is cover-free (recall Definition 5.19) and does not contain positive rules.
- Γ is *monodirectional* (MIC) iff for every pair of interval literals $p \diamond \overline{\mathsf{c}}$ and $q \diamond' \overline{\mathsf{d}}$ occurring in Γ, it holds that either $\diamond, \diamond' \in \{<, \leq\}$ or $\diamond, \diamond' \in \{>, \geq\}$, and there is no $p \in \mathsf{Pr}[\Gamma]$ such that $p{\geq}\overline{0}$ or $p{\leq}\overline{1}$ occurs in Γ.
  - Γ is *positive monodirectional* (PMC) iff it is monodirectional and does not contain negative rules.
  - Γ is *short monodirectional* (SMC) iff it is monodirectional and all positive rules have the form $q \diamond \overline{\mathsf{d}}$ or $p \diamond \overline{\mathsf{c}} \rightarrow q \diamond \overline{\mathsf{d}}$.

An Ł-abduction problem $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$ is WCF (MIC, PMC, SMC) iff $\Gamma \cup \{\lambda\} \cup \mathsf{H}$ is WCF (MIC, PMC, SMC).

Before establishing the formal results, we observe that weakly cover-free and monodirectional theories are indeed cover-free. First, weakly cover-free theories are cover-free. Second, one can see that if $\diamond, \diamond' \in \{\leq, <\}$ or $\diamond, \diamond' \in \{\geq, >\}$, then $\text{Ł} \not\models (p \diamond \overline{\mathsf{c}}) \oplus (p \diamond' \overline{\mathsf{d}})$, unless one of these literals is $p{\geq}\overline{0}$ or $p{\leq}\overline{1}$. But occurrences of such literals are prohibited in MIC theories. Hence, all MIC theories are CF theories.

One can also see that weakly cover-free and monodirectional theories are straightforwardly recognisable. Furthermore, notice from the reduction shown in (19) and Definition 5.24 that MIC theories can be translated to Horn theories and PMC theories to *definite* Horn theories. Moreover, WCF theories and SMC theories can be translated to a specific fragment of the classical Horn language — $\mathsf{IHS\text{-}B}^-$ (implicative hitting set-bounded) — introduced by Khanna et al. (2002).

Definition 5.25.

- An *$\mathsf{IHS\text{-}B}^-$ clause* can have only the following types: p, $\neg p \vee q$, or $\neg p_1 \vee \ldots \vee \neg p_m$.
- An *$\mathsf{IHS\text{-}B}^-$ theory* is a set of $\mathsf{IHS\text{-}B}^-$ clauses.

Represented implicatively, an $\mathsf{IHS\text{-}B}^-$ clause has one of the following forms: p, $p \rightarrow q$, or $p_1 \wedge \ldots \wedge p_n \rightarrow \bot$. It is shown by Creignou and Zanuttini (2006) and Nordh and Zanuttini (2008) that solution existence in $\mathsf{IHS\text{-}B}^-$ abduction problems is polynomial if the observation is a single literal. Moreover, CPL-satisfiability of $\mathsf{IHS\text{-}B}^-$-formulas (and hence, solution recognition), while non-trivial, is very simple and can be computed in uniform AC^0, by setting $v(p) = 1$ for every variable appearing in a positive unit clause and $v(q) = 0$, otherwise.[8] Furthermore,

[8]We are grateful to Emil Jeřábek who brought this to our attention.

solution existence in definite Horn abduction problems is also polynomially tractable if the observation is a *positive* literal (cf. (Bylander et al. 1991; Friedrich et al. 1990) and (Eiter and Gottlob 1995, Corollary 5.5) for details). Let us now consider the complexity of abductive reasoning in subclasses of cover-free theories.

THEOREM 5.26.

(1) Given a WCF (PMC, SMC) Ł*-abduction problem* $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$*, it can be decided in polynomial time whether* $\mathbb{P}$ *has (proper) solutions.*

(2) Given an MIC Ł*-abduction problem* $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$ *and an interval term* τ*, it is* P*-complete under logspace reductions to decide whether* τ *is a (proper, entailment-minimal) solution to* $\mathbb{P}$*.*

(3) Given an MIC Ł*-abduction problem* $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$*, it is* NP*-complete to decide whether* $\mathbb{P}$ *has (proper) solutions.*

PROOF. We begin with Item 1. Let $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$ be an Ł-abduction problem with $\Gamma \cup \{\lambda\}$ a WCF (PMC, or SMC) theory. We define the CPL-abduction problem $\mathbb{P}_{\mathcal{H}} = \langle \Gamma_{\mathcal{H}}, \lambda_{\mathsf{CPL}}, \mathsf{H}_{\mathsf{CPL}} \rangle$ as shown in (19) and Definition 5.13. Now, observe that if $\Gamma \cup \{\lambda\}$ is a WCF or a SMC theory, then $\lambda_{\mathsf{CPL}} = p_\lambda$ and $\Gamma_{\mathcal{H}} \cup \{p_\lambda\}$ is an IHS-B$^-$ theory. This is because Γ either does not contain any positive rules at all (if it is WCF) or only rules of the form $p \diamond \overline{\mathsf{c}} \rightarrow q \diamond \overline{\mathsf{d}}$ with only one interval literal in the body. From the proof of Theorem 5.20, we know that σ is a solution to $\mathbb{P}$ iff σ_{CPL} is a solution to $\mathbb{P}_{\mathcal{H}}$. Conversely (and again, relying on the reasoning from Theorem 5.20), ϱ is a solution to $\mathbb{P}_{\mathcal{H}}$ iff $\varrho^{\diamond Ł} = \bigodot_{r_\lambda \in \varrho} \lambda$ is a solution to $\mathbb{P}$. This gives us a desired polynomial reduction. Since the existence of (proper) solutions to CPL-abduction problems whose theories are sets of IHS-B$^-$ clauses is decidable in polynomial time, it follows that (proper) solution existence is polynomially decidable for WCF and SMC Ł-abduction problems.

Consider now the case when $\Gamma \cup \{\lambda\}$ is a PMC theory. Then $\Gamma_{\mathcal{H}}$ is a definite Horn theory. Using the reasoning from the previous paragraph, we reduce the solution existence for PMC Ł-abduction problems to the solution existence for definite Horn CPL-abduction problems. Since the solution existence to definite Horn CPL-abduction problems is polynomially decidable when the observation is a propositional variable, it follows that solution existence for PMC Ł-abduction problems is also polynomially decidable.

To finish Item 1, let us prove that the existence of *proper* solutions to WCF, PMC, and SMC Ł-abduction problems is also polynomially decidable. Since λ is an interval literal, a solution τ is *not proper* iff there is some interval literal $\mu \in \tau$ such that $\mu \models_{Ł} \lambda$. Now, given a set of interval literals H and an interval literal λ, let $\mathsf{H}^{-\lambda} = \mathsf{H} \setminus \{\mu \mid \mu \in \mathsf{H}, \mu \models_{Ł} \lambda\}$. It follows that $\mathbb{P} = \langle \Gamma, \lambda, \mathsf{H} \rangle$ and $\mathbb{P}^{-\lambda} = \langle \Gamma, \lambda, \mathsf{H}^{-\lambda} \rangle$ have the same set of *proper solutions* and that all solutions to $\mathbb{P}^{-\lambda}$ are proper. Furthermore, by Proposition 3.5, we have that given any λ and H, $\mathsf{H}^{-\lambda}$ can be constructed in polynomial time. Thus, the existence of proper solutions to WCF, PMC, and SMC Ł-abduction problems is polynomially decidable, as required.

Consider now Item 2. Since monodirectional Ł-abduction problems are cover-free, we have from Theorem 5.20 that solution recognition is polynomially decidable. To show the P-hardness of solution recognition, we reuse the reduction from Theorem 5.20. Namely, given a classical Horn abduction problem $\mathbb{P} = \langle \Gamma, p, \mathsf{H} \rangle$, we construct an Ł-abduction problem $\mathbb{P}^{\mathsf{C}} = \langle \Gamma^{\mathsf{C}}, p^{\odot}, \mathsf{H}^{\odot} \rangle$ as shown in (21) (recall Convention 3 for $\tau^{\odot}$ and $\mathsf{H}^{\odot}$). One can see that $\mathbb{P}^{\mathsf{C}}$ is indeed an MIC Ł-abduction problem. Now, using the reasoning from Theorem 5.20, we can show that if τ is a (proper, $\models_{\mathsf{CPL}}$-minimal) solution to $\mathbb{P}$, then $\tau^{\odot}$ is a (proper, $\models_{Ł}$-minimal) solution to $\mathbb{P}^{\mathsf{C}}$. And conversely, if σ is a (proper, $\models_{Ł}$-minimal) solution to $\mathbb{P}^{\mathsf{C}}$, then σ_{cl} is a (proper, $\models_{\mathsf{CPL}}$-minimal) solution to $\mathbb{P}$.

Item 3 can be shown in a similar way, but we use Theorem 5.21 to obtain that solution existence is decidable in NP time. For NP-hardness, we reuse the reduction from Theorem 5.20, as shown in the previous paragraph. The result now follows. □

An important property of IHS-B$^-$-abduction in classical logic is that $\models_{\mathsf{CPL}}$-minimal solutions have *at most one literal* if the observation is a single propositional variable (Creignou and Zanuttini 2006, Proposition 12).[9]

[9]On the other hand, if the observation is an $\mathcal{L}_{\mathsf{CPL}}$-term, solution existence is NP-complete — cf. the theory of the abduction problem in (Eiter and Gottlob 1995, Theorem 5.3). This discrepancy is expected, as it is not guaranteed that an $\mathcal{L}_{\mathsf{CPL}}$-term can be explained by a single literal.

This means that the relevance of a hypothesis can be decided in polynomial time (because it suffices to check all hypotheses one-by-one) and that *theory-minimal* solutions can also be recognised in polynomial time (because it suffices to check the entailment $\Gamma, h \models_{\mathsf{CPL}} h'$ for *all pairs* of hypotheses h and h'). From this fact and from Theorem 5.26, it follows that the recognition of theory-minimal solutions and relevance of hypotheses in WCF and SMC Ł-abduction problems are also polynomial.

On the other hand, this property fails for *definite Horn* theories. Still, one can show (cf. (Creignou and Zanuttini 2006; Friedrich et al. 1990) for details) that if a classical definite Horn abduction problem $\mathbb{P} = \langle\Gamma, p, \mathsf{H}\rangle$ has a solution, then $\bigwedge_{q\in\mathsf{H}} q$ is (also) a solution to $\mathbb{P}$. From this, it follows that the relevance of hypotheses w.r.t. all or proper solutions as well as the necessity w.r.t. all, proper or $\models_{\mathsf{CPL}}$-minimal solutions are decidable in polynomial time (Eiter and Gottlob 1995, Proposition 5.1.2). Note, however, that *the relevance of hypotheses w.r.t.* $\models_{\mathsf{CPL}}$*-minimal solutions* is NP-complete (Eiter and Gottlob 1995, Theorem 5.1.3).

As expected from the results in this section, these computational properties of abduction in subclasses of Horn theories transfer to the Ł-abduction in subclasses of cover-free theories. The following corollary summarises the discussion from the two preceding paragraphs. Note that the coNP-membership of *theory-minimal* solution recognition follows from Theorem 5.23. Furthermore, the hardness results in items (2) and (3) follow from the fact that Horn theories can be translated to *monodirectional* interval-clause theories (recall (20)).

Corollary 5.27.

(1) Given a WCF or SMC Ł*-abduction problem* $\mathbb{P} = \langle\Gamma, \mu, \mathsf{H}\rangle$*, an interval term* τ*, and* $\lambda \in \mathsf{H}$*, it takes polynomial time to decide whether*
  - *(a)* λ *is relevant (necessary) w.r.t.* $\mathcal{S}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$*, and* $\mathcal{S}^{\min}(\mathbb{P})$*;*
  - *(b) it* $\tau \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$*.*

(2) Given a PMC Ł*-abduction problem* $\mathbb{P} = \langle\Gamma, \mu, \mathsf{H}\rangle$*, an interval term* τ*, and* $\lambda \in \mathsf{H}$*,*
  - *(a) it takes polynomial time to decide whether* λ *is relevant w.r.t.* $\mathcal{S}(\mathbb{P})$ *and* $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$*, or is necessary w.r.t.* $\mathcal{S}(\mathbb{P})$*,* $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$*,* $\mathcal{S}^{\min}(\mathbb{P})$*;*
  - *(b) it is* NP*-complete to decide whether* λ *is relevant w.r.t.* $\mathcal{S}^{\min}(\mathbb{P})$*;*
  - *(c) it is in* coNP *to decide whether* $\tau \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$*.*

(3) Given an MIC Ł*-abduction problem* $\mathbb{P} = \langle\Gamma, \mu, \mathsf{H}\rangle$*, an interval term* τ*, and* $\lambda \in \mathsf{H}$*,*
  - *(a) it is* NP*-complete (*coNP*-complete) to decide whether* λ *is relevant (necessary) w.r.t.* $\mathcal{S}(\mathbb{P})$*,* $\mathcal{S}^{\mathsf{p}}(\mathbb{P})$*, and* $\mathcal{S}^{\min}(\mathbb{P})$*;*
  - *(b) it is in* coNP *to decide whether* $\tau \in \mathcal{S}^{\mathsf{Th}}(\mathbb{P})$*.*

We finish the section with a remark on our translation of interval-clause theories into $\mathcal{L}_{\mathsf{CPL}}$-clause theories.

Remark 3. Note that applying the translation from Definition 5.13 to *non-cover-free* sets of rules results in non-Horn theories. For example, let

$$\Gamma = \{p{\geq}\overline{\tfrac{1}{4}} \rightarrow q{\leq}\overline{\tfrac{2}{3}}, p{\leq}\overline{\tfrac{1}{3}} \odot q{\geq}\overline{\tfrac{3}{4}} \rightarrow s{\leq}\overline{\tfrac{1}{2}}\}$$

The translation will give us the following theory (we rewrite formulas as $\mathcal{L}_{\mathsf{CPL}}$-clauses):

$$\Gamma_{\mathsf{CPL}} = \{\neg r_{p\geq\overline{1/4}} \vee r_{q\leq\overline{2/3}}, \neg r_{p\leq\overline{1/5}} \vee \neg r_{q\geq\overline{3/4}} \vee r_{s\leq\overline{1/2}}\} \cup \{r_{p\geq\overline{1/4}} \vee r_{p\leq\overline{1/3}}\} \cup \{\neg r_{q\leq\overline{2/3}} \vee \neg r_{q\geq\overline{3/4}}\}$$

As one can see, Γ_{CPL} is not a Horn theory. On the other hand, abduction in non-cover-free interval-clause theories is not necessarily intractable. A simple example would be theories containing *bijunctive* interval clauses (i.e., clauses with at most two interval literals). As one can see from Definition 5.13, the translation does not change the shape of the clause in the classical theory, and thus, the number of literals in κ_{CPL} and κ coincides. Thus, if Γ is a bijunctive interval-clause theory, then Γ_{CPL} is a bijunctive $\mathcal{L}_{\mathsf{CPL}}$ theory. As solution existence for CPL-abduction problems with bijunctive theories is polynomially tractable (Creignou and Zanuttini 2006, Proposition 10), so is solution existence for Ł-abduction problems whose theories are sets of bijunctive interval clauses.

6 Abduction in Łukasiewicz Logic With Degree-Preserving Entailment

We have so far assumed the definition of entailment corresponding to the *absolute truth preservation*. That is, if the premises are absolutely true (have value 1), then so must be the conclusion (cf. Definition 2.2). In many contexts involving reasoning with truth degrees, however, we cannot ensure the absolute truth of premises. Moreover, we might need to utilise a premise even if we know that it is not absolutely true. For example, a constraint on the value of a parameter can be *soft*, a rule that we use is not always applicable, information might be contradictory, bear a degree of uncertainty, or be imperfect in some other way. Similarly, the observation itself might bear a degree of uncertainty. In such contexts, it is reasonable to alter the notion of entailment as follows: ϕ entails χ iff $v(\phi) \leq v(\chi)$ in all valuations. That is, the value of the conclusion should not be lower than the value of the premise. Such an entailment is called *degree-preserving* (Ertola et al. 2015). Degree-preserving entailment relations were first proposed by Wojcicki (1988). Fuzzy logics with degree-preserving entailment relations have been extensively investigated since then (Bou et al. 2009; Coniglio et al. 2021). Degree-preserving *Łukasiewicz logic* (denoted $\mathsf{Ł}_{\leq}$) was first considered by Font et al. (2006) and then further studied by Ertola et al. (2015).

In this section, we will consider abductive reasoning in Łukasiewicz logic with degree-preserving entailment. We will consider the same reasoning tasks as we did in Section 4 in both full logic and its clausal fragments. Let us first introduce the needed formal notions.

Definition 6.1 (Degree-preserving entailment). Let $\Gamma = \{\phi_1, \ldots, \phi_n\}$ and $\Gamma \cup \{\chi\} \subseteq \mathscr{L}_{\mathsf{Ł}}^{\mathbb{Q}}$. We define degree-preserving entailment ($\Gamma \models_{\mathsf{Ł}_{\leq}} \chi$) and consistent degree-preserving entailment ($\Gamma \models^{\mathsf{cons}}_{\mathsf{Ł}_{\leq}} \chi$) as follows:

$$\Gamma \models_{\mathsf{Ł}_{\leq}} \chi \text{ iff } \forall v : \max\left(0, \sum_{i=1}^{n} v(\phi_i) - (n-1)\right) \leq v(\chi) \qquad \Gamma \models^{\mathsf{cons}}_{\mathsf{Ł}_{\leq}} \chi \text{ iff } \Gamma \models_{\mathsf{Ł}_{\leq}} \chi \text{ and } \Gamma \not\models_{\mathsf{Ł}_{\leq}} \bot$$

Γ is *positively* Ł*-satisfiable* iff there is an Ł-valuation v such that $\max\left(0, \sum\limits_{i=1}^{n} v(\phi_i) - (n-1)\right) > 0$. The notion of validity is preserved from Definition 2.2.

Note that the deduction theorem holds w.r.t. $\models_{\mathsf{Ł}_{\leq}}$: $\phi \models_{\mathsf{Ł}_{\leq}} \chi$ iff $\phi \rightarrow \chi$ is Ł-valid. Thus, the contraposition is also valid: given a finite set of formulas Γ, $\Gamma, \phi \models_{\mathsf{Ł}_{\leq}} \chi$ iff $\Gamma, \neg\chi \models_{\mathsf{Ł}_{\leq}} \neg\phi$. This, however, comes at a cost. Recall from (4) that in the truth-preserving entailment, sets of premises could be equivalently represented as *weak* and as *strong conjunctions* of formulas. This happens because $\phi \odot \chi \simeq_{\mathsf{Ł}} \phi \wedge \chi$. On the other hand, in the degree-preserving entailment, preservation of the absolute truth is not enough. So, if we want to preserve the general form of the deduction theorem

$$\phi, \chi \models_{\mathsf{Ł}_{\leq}} \psi \text{ iff } \phi \models_{\mathsf{Ł}_{\leq}} \chi \rightarrow \psi, \tag{23}$$

we have to interpret sets of premises as their *strong conjunctions*. Indeed, observe that $p, \neg p \models_{\mathsf{Ł}_{\leq}} q$, whence $p \rightarrow (\neg p \rightarrow q)$ and $(p \odot \neg p) \rightarrow q$ are Ł-valid. On the contrary, $p \wedge \neg p \not\models_{\mathsf{Ł}_{\leq}} q$ (consider $v(p) = \frac{1}{2}$ and $v(q) = 0$).

Let us remark on the relation between $\models_{\mathsf{Ł}}$ and $\models_{\mathsf{Ł}_{\leq}}$.

Remark 4. One can check that $\Gamma \models_{\mathsf{Ł}_{\leq}} \chi$ entails $\Gamma \models_{\mathsf{Ł}} \chi$, and hence, $\Gamma \models_{\mathsf{Ł}_{\leq}} \chi$ entails $\Gamma_{\mathsf{cl}} \models_{\mathsf{CPL}} \chi_{\mathsf{cl}}$ (cf. Convention 3). The converse does not hold in the general case: $p \wedge \neg p \models_{\mathsf{Ł}} q$ (because $p \wedge \neg p$ is not Ł-satisfiable) but $p \wedge \neg p \not\models_{\mathsf{Ł}_{\leq}} q$ (because $p \wedge \neg p$ is *positively satisfiable*).

On the other hand, if all formulas in $\Gamma \cup \{\chi\}$ are obtained as $\{\neg, \odot, \oplus, \rightarrow\}$-combinations of *interval literals*, we also have that $\Gamma \models_{\mathsf{Ł}} \chi$ entails $\Gamma \models_{\mathsf{Ł}_{\leq}} \chi$. This happens because such formulas can only have values in $\{0, 1\}$.

Moreover, it is clear that in contrast to the truth-preserving entailment, the degree-preserving entailment is not trivialised by premises whose value is less than 1. Indeed, $p, p < \overline{1} \models_{\mathsf{Ł}} q$ but $p, p < \overline{1} \not\models_{\mathsf{Ł}_{\leq}} q$. On the other hand, as p and $p \odot p$ are not strongly equivalent (recall Remark 1), sometimes, one has to *repeat* premises to obtain a desired entailment. In particular, one can check that

$$p, p \to q, p \to r \models_{Ł} q \odot r \qquad p, p \to q, p \to r \not\models_{Ł_\leq} q \odot r \qquad p \odot p, p \to q, p \to r \models_{Ł_\leq} q \odot r$$

Indeed, set $v(p) = v(q) = v(r) = \frac{1}{2}$ and denote $\Gamma = \{p, p \to q, p \to r\}$. By Definition 6.1, v witnesses $\Gamma \not\models_{Ł_\leq} q \odot r$ because $v(p \odot (p \to q) \odot (p \to r)) = \frac{1}{2}$ but $v(q \odot r) = 0$

Before proceeding to abduction, let us observe that the complexity of checking degree-preserving entailment and positive satisfiability is the same as that of Ł-entailment and Ł-satisfiability.

Proposition 6.2. *Given a finite* $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{Ł}^{\mathbb{Q}}$*:*

(1) it is coNP*-complete to verify whether* $\Gamma \models_{Ł_\leq} \chi$*;*

(2) it is NP*-complete to verify whether* Γ *is positively satisfiable.*

Proof. The first item follows from Proposition 2.3 by mutual polynomial-time reductions between Ł-validity and degree-preserving entailment. Namely, let $\Gamma = \{\phi_1, \ldots, \phi_m\}$. Using m applications of (23), we have that $\Gamma \models_{Ł_\leq} \chi$ iff $\phi_1 \to (\ldots \to (\phi_m \to \chi))$ is Ł-valid. Thus, we have coNP-membership by the reduction of the degree-preserving entailment to Ł-validity, which is coNP-complete (Proposition 2.3). To see that the coNP-hardness holds, note that χ is Ł-valid iff $p \to p \models_{Ł_\leq} \chi$.

For the second item, observe that Γ is *positively* Ł*-satisfiable* iff $\Gamma \not\models_{Ł_\leq} \bot$. The result now follows. □

6.1 Abduction Problems with $Ł_\leq$-solutions

Let us now consider abductive reasoning in Łukasiewicz logic with degree-preserving entailment. We begin with a suitable reformulation of the notion of solution to an abduction problem.

Definition 6.3 ($Ł_\leq$-solutions). Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem. An interval term τ is an $Ł_\leq$*-solution to* $\mathbb{P}$ iff $\Gamma, \tau \models^{\mathsf{cons}}_{Ł_\leq} \chi$.

- An $Ł_\leq$-solution τ is *proper* iff $\tau \not\models_{Ł_\leq} \chi$.
- A proper $Ł_\leq$-solution τ is $\models_{Ł_\leq}$*-minimal* iff there is no proper $Ł_\leq$-solution σ such that $\tau \models_{Ł_\leq} \sigma$ but $\sigma \not\models_{Ł_\leq} \tau$.
- A proper $Ł_\leq$-solution τ is *theory-minimal* if there is no proper $Ł_\leq$-solution σ such that $\Gamma, \tau \models_{Ł_\leq} \sigma$ and $\Gamma, \sigma \not\models_{Ł_\leq} \tau$.

Note, furthermore, that even in the degree-preserving entailment, we cannot use *simple terms* as solutions. The following remark illustrates the need to use interval terms.

Remark 5. Recall from Examples 3.2 and 3.3 that solutions to Ł-abduction problems cannot usually be represented as simple terms because for a simple term τ to have value 1, all variables that occur positively must have value 1 and all variables that occur negatively must have value 0. In the degree-preserving entailment, a premise does not have to have value 1. Still, simple terms are not sufficient for $Ł_\leq$-solutions. Indeed, no simple term τ such that $\mathsf{Pr}(\tau) = \{c\}$ can be a solution to $\mathbb{P}_{\mathsf{lift}} = \langle \Gamma_{\mathsf{lift}}, \chi_{\mathsf{lift}} \rangle$ (recall Example 3.2): $\Gamma_{\mathsf{lift}} = \{(c \oplus c \oplus c \oplus c) \to g, (\neg c \oplus \neg c \oplus \neg c) \to b\}$, $\chi_{\mathsf{lift}} = g \odot b$.

To see that, note that either $\tau_1 = c \odot \ldots \odot c$ or $\tau_2 = \neg c \odot \ldots \odot \neg c$ (with c, respectively $\neg c$, being repeated several times). In the first case, we set $v_1(c) = 1$, $v_1(g) = 1$, and $v_1(b) = 0$; in the second case, we set $v_2(c) = 0$, $v_2(g) = 0$, and $v_2(b) = 1$. One can see that v_1 witnesses $\Gamma_{\mathsf{lift}}, \tau_1 \not\models_{Ł_\leq} \chi_{\mathsf{lift}}$ and v_2 witnesses $\Gamma_{\mathsf{lift}}, \tau_2 \not\models_{Ł_\leq} \chi_{\mathsf{lift}}$. On the other hand, $(\overline{c \geq \frac{1}{4}}) \odot (\overline{c \leq \frac{2}{3}})$ is indeed a (theory-minimal) $Ł_\leq$-solution to $\mathbb{P}_{\mathsf{lift}}$, just as it was in the case of truth-preserving entailment.

In addition to that, some abduction problems that do not have solutions w.r.t. the truth-preserving entailment still have $Ł_\leq$-solutions. The converse statement also holds: there are Ł-abduction problems that have solutions w.r.t. the truth-preserving entailment, but no $Ł_\leq$-solutions.

Example 6.4. Let $\Gamma = \{p \rightarrow q, q{<}\overline{1}\}$. Intuitively, Γ states that p implies q, but there is some uncertainty about q (because it cannot be absolutely true). In other words, q is manifested *to some degree*. Now, it is clear from Definitions 2.2 and 3.7 that there is no set of hypotheses H such that $\mathbb{P} = \langle\Gamma, q, \mathsf{H}\rangle$ has solutions. On the other hand, $p{\geq}\overline{1}$ is an $\text{Ł}_{\leq}$-solution to $\mathbb{P}$. Note, moreover, that $\Gamma, p{\geq}\overline{1} \models_{\text{Ł}} \bot$ but $\Gamma, p{\geq}\overline{1} \not\models_{\text{Ł}_{\leq}} \bot$. In particular, if $v(p) = 1$ and $v(q) = x < 1$, then $v(p \rightarrow q) = x$. Thus, to explain q, we might (implicitly) assume the implication $p \rightarrow q$ also bears some uncertainty (i.e., not absolutely true).

Now, consider the following abduction problem $\mathbb{P}' = \langle\Gamma', t, \mathsf{H}'\rangle$:

$$\Gamma' = \{p \rightarrow q, q \rightarrow r, q \rightarrow s, r \rightarrow (s \rightarrow t)\} \qquad\qquad \mathsf{H} = \{p{\geq}\overline{1}\}$$

One can easily see that $p{\geq}\overline{1}$ is indeed a solution w.r.t. $\models_{\text{Ł}}$. On the other hand, $\Gamma', p{\geq}\overline{1} \not\models_{\text{Ł}_{\leq}} t$ is witnessed by the following valuation: $v(p) = 1$, $v(q) = v(r) = v(s) = \frac{1}{2}$, $v(t) = 0$. Moreover, there is no set of hypotheses H'' with $\mathsf{Pr}[\mathsf{H}''] = \{p\}$ such that $\mathbb{P}'' = \langle\Gamma'', t, \mathsf{H}''\rangle$ would have a solution. Indeed, let $v(p) = x < 1$, setting $v(q) = v(r) = v(s) = x$ and $v(t) = 0$, we have that $v(\bigodot_{\phi\in\Gamma} \phi) = 1 - \max(0, x + x - 1) + 0 > 0$, but $v(t) = 0$, thus, falsifying the entailment.

The following convention introduces notation for the different types of solutions in Definition 6.3.

Convention 4. Given an abduction problem $\mathbb{P}$, we will use $\mathcal{S}_{\leq}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}_{\leq}(\mathbb{P})$, $\mathcal{S}^{\min}_{\leq}(\mathbb{P})$, and $\mathcal{S}^{\mathsf{Th}}_{\leq}(\mathbb{P})$ to denote the sets of all $\text{Ł}_{\leq}$-solutions, all proper $\text{Ł}_{\leq}$-solutions, all $\models_{\text{Ł}_{\leq}}$-minimal $\text{Ł}_{\leq}$-solutions, and all theory-minimal $\text{Ł}_{\leq}$-solutions of $\mathbb{P}$, respectively.

As in the case with abductive reasoning w.r.t. the truth-preserving entailment, we will use reductions from classical abduction to obtain hardness results. Recall from Section 4 that we relied heavily on the fact that the law of excluded middle $p \vee \neg p$ has value 1 iff $v(p) \in \{0, 1\}$ and that Łukasiewicz connectives behave classically on $\{0, 1\}$. This, unfortunately, is not enough in the case of the degree-preserving entailment because even if $v(\phi) < 1$, v can still witness $\phi \models_{\text{Ł}_{\leq}} \chi$. To circumvent this issue, we recall the result of Mundici (1987).

Definition 6.5. Let $i, n \geq 1$, $t \geq 2$, and $\{p_1, \ldots, p_n\} \subseteq \mathsf{Pr}$. We define the following formulas:

$$\mathsf{LEM}(p_i) \coloneqq p_i \vee \neg p_i \qquad \mathsf{LEM}^t(p_i) \coloneqq \underbrace{\mathsf{LEM}(p_i) \odot \ldots \odot \mathsf{LEM}(p_i)}_{t \text{ times}} \qquad \mathsf{BiVal}^n_t \coloneqq \bigodot_{i=1}^{n} \mathsf{LEM}^t(p_i)$$

The idea of the proof of the next statement is that a sufficiently long *strong conjunction* of instances of $p \vee \neg p$ makes values of variables *almost* classical. Namely, $v(\mathsf{LEM}^t(p_i)) > 0$ iff $\max(v(p_i), 1 - v(p_i)) > \frac{t-1}{t}$.

Proposition 6.6 (Mundici (1987, Lemma 3.3)). *Let $\phi \in \mathcal{L}_{\mathsf{CPL}}$. Then $\mathsf{CPL} \models \phi$ iff $\text{Ł} \models \mathsf{BiVal}^{|\mathsf{Pr}(\phi)|}_{\mathsf{lgth}(\phi)} \rightarrow \phi^{\text{Ł}}$.*

Remark 6. We observe first that the original result of Mundici (1987) differs slightly from Proposition 6.6 because it uses $\mathsf{BiVal}^{|\mathsf{Pr}(\phi)|}_{\mathsf{lgth}(\phi)} \rightarrow \phi$ instead of $\mathsf{BiVal}^{|\mathsf{Pr}(\phi)|}_{\mathsf{lgth}(\phi)} \rightarrow \phi^{\text{Ł}}$ (i.e., does not replace $\wedge$ and $\vee$ with $\odot$ and $\oplus$). This is not a problem because we can w.l.o.g. assume that ϕ contains only $\neg$ and $\rightarrow$ (in which case, $\phi = \phi^{\text{Ł}}$) and treat $\wedge$ and $\vee$ as defined connectives (recall that $\wedge$ and $\vee$ *in classical logic* are defined using $\neg$ and $\rightarrow$ in the same way as $\odot$ and $\oplus$ in Łukasiewicz logic). In particular, an $\mathcal{L}_{\mathsf{CPL}}$-term $\sigma = l_1 \wedge \ldots \wedge l_n$ is CPL-equivalent to $\sigma^{\rightarrow} = \neg(l_1 \rightarrow (\ldots \rightarrow (l_{n-1} \rightarrow \neg l_n)))$. One can observe from Convention 3 that $(\sigma^{\rightarrow})^{\text{Ł}} = \sigma^{\rightarrow}$.

Moreover, it is important to note that if $n \geq n'$ or $t \geq t'$, then $\mathsf{BiVal}^n_t \models_{\text{Ł}_{\leq}} \mathsf{BiVal}^{n'}_{t'}$. Thus, for $\mathsf{Pr}(\phi) \subseteq \Xi$ and $\mathsf{lgth}(\phi) \leq t$, it follows that ϕ is a classical tautology iff $\text{Ł} \models \mathsf{BiVal}^{|\Xi|}_t \rightarrow \phi^{\text{Ł}}$.

Theorem 6.7. *Given an Ł-abduction problem $\mathbb{P} = \langle\Gamma, \chi, \mathsf{H}\rangle$ and an interval term τ, it is* DP*-complete to determine whether τ is an $\text{Ł}_{\leq}$-solution to $\mathbb{P}$.*

Proof. We begin with the complexity of determining whether τ is an arbitrary solution. Membership follows from Proposition 6.2. For hardness, we construct a polynomial reduction from the Sat-UnSat problem. Let w.l.o.g. ϕ and χ be two formulas over $\{\neg, \rightarrow\}$ such that and $\mathsf{Pr}(\phi) \cap \mathsf{Pr}(\chi) = \varnothing$. Let further, $p \notin \mathsf{Pr}(\phi) \cup \mathsf{Pr}(\chi)$. We show that ϕ is CPL-satisfiable and χ is CPL-unsatisfiable iff $p{\geq}\overline{1}$ is an $Ł_{\leq}$-solution to $\mathbb{P} = \langle \Gamma, \neg\chi, \{p{\geq}\overline{1}\}\rangle$ with

$$\Gamma = \left\{\phi, \mathsf{BiVal}^{|\mathsf{Pr}(\phi)|}_{\mathsf{lgth}(\phi)}, \mathsf{BiVal}^{|\mathsf{Pr}(\chi)|}_{\mathsf{lgth}(\neg\chi)}\right\}$$

First, assume that ϕ is CPL-satisfiable and χ is CPL-unsatisfiable (i.e., $\neg\chi$ is CPL-valid). To see that Γ is positively satisfiable, take a classical valuation v such that $v(\phi) = 1$ and set $v(q) = 1$ for $q \in \mathsf{Pr}(\chi) \cup \{p\}$. Furthermore, by Proposition 6.6, we have that $\mathsf{BiVal}^{|\mathsf{Pr}(\chi)|}_{\mathsf{lgth}(\neg\chi)} \rightarrow \neg\chi$ is Ł-valid. Hence, $\mathsf{BiVal}^{|\mathsf{Pr}(\chi)|}_{\mathsf{lgth}(\neg\chi)} \models_{Ł_{\leq}} \neg\chi$, and, as Γ is $Ł_{\leq}$-satisfiable, it follows that $\Gamma, p{\geq}\overline{1} \models^{\mathsf{cons}}_{Ł_{\leq}} \neg\chi$, as required.

For the converse direction, we consider two cases. If ϕ is CPL-unsatisfiable (i.e., $\neg\phi$ is classically valid), then $\mathsf{BiVal}^{|\mathsf{Pr}(\phi)|}_{\mathsf{lgth}(\phi)} \rightarrow \neg\phi$ is Ł-valid by Proposition 6.6. But in this case, Γ is not positively satisfiable either because $v(\eta \odot (\eta \rightarrow \neg\theta) \odot \theta) = 0$ for every Ł-valuation v and every $\eta, \theta \in \mathcal{L}^{\mathbb{Q}}_{Ł}$. Thus, $\mathbb{P}$ has *no solutions*. Now let both ϕ and χ be CPL-satisfiable. As $\mathsf{Pr}(\phi) \cap \mathsf{Pr}(\chi) = \varnothing$, there is a CPL valuation v such that $v(\phi) = v(\chi) = 1$. Now, defining $v(p) = 1$, we obtain that v witnesses $\Gamma, p{\geq}\overline{1} \not\models_{Ł_{\leq}} \neg\chi$ because $v(\neg\chi) = 0$ but $v(p{\geq}\overline{1}) = 1$ and $v(\psi) = 1$ for every $\psi \in \Gamma$. ◻

The next statement can be shown in the same way as Theorem 4.3.

Theorem 6.8. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H}\rangle$ be an Ł-abduction problem and τ an interval term. Then it is* DP*-complete to determine whether τ is a proper $Ł_{\leq}$-solution to $\mathbb{P}$.*

As expected from Proposition 6.2, recognition of $\models_{Ł_{\leq}}$-minimal solutions is also DP-complete.

Theorem 6.9. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H}\rangle$ be an Ł-abduction problem and τ an interval term. Then it is* DP*-complete to check whether $\tau \in \mathcal{S}^{\min}_{\leq}(\mathbb{P})$.*

Proof. We begin with a sketch of the membership proof. Observe from Remark 4 that given two *interval terms* τ and σ, it holds that $\tau \models_{Ł} \sigma$ iff $\tau \models_{Ł_{\leq}} \sigma$ (and thus, is decidable in polynomial time by Proposition 3.5). Thus, we can rely on the technique from Theorem 4.5. Indeed, given τ and $\mathbb{P}$, we can check in DP whether it is a proper $Ł_{\leq}$-solution and then we need to consider polynomially many 'next weakest' terms τ' (recall Definition 3.13) and check that none of them is a proper solution to $\mathbb{P}$.

Let us now consider hardness. We construct a polynomial reduction from the recognition of prime implicants in classical logic. Let w.l.o.g. χ be CPL-satisfiable and contain only $\neg$ and $\rightarrow$. Let further, $\mathsf{Pr}(\chi) = \{p_1, \ldots, p_n\}$. We show that an $\mathcal{L}_{\mathsf{CPL}}$-term τ is a prime implicant of χ iff $\tau^{\odot}$ (recall Convention 3) is an entailment-minimal solution of $\mathbb{P} = \langle\{\mathsf{BiVal}^{n}_{t}, q\}, \chi \odot q, \mathsf{H}\rangle$ with $\mathsf{H} = \{p{\geq}\overline{1} \mid p \in \mathsf{Pr}(\chi)\} \cup \{p{\leq}\overline{0} \mid p \in \mathsf{Pr}(\chi)\}$, $t = \mathsf{lgth}(\tau \rightarrow \chi)$, and $q \notin \mathsf{Pr}(\chi)$.

Let τ be a prime implicant of χ. Since χ is CPL-satisfiable, it follows that $\tau \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$ and there is no other $\mathcal{L}_{\mathsf{CPL}}$-term τ' such that $\tau' \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$, $\tau \models_{\mathsf{CPL}} \tau'$, but $\tau' \not\models_{\mathsf{CPL}} \tau$. It follows that $\tau^{\odot}, \mathsf{BiVal}^{n}_{t}, q \not\models_{Ł_{\leq}} \bot$. Let us now show that $\tau^{\odot}, \mathsf{BiVal}^{n}_{t}, q \models_{Ł_{\leq}} \chi$. We reason as follows:

$$\begin{aligned}
\tau \models_{\mathsf{CPL}} \chi \text{ iff } & \mathsf{CPL} \models \tau \rightarrow \chi \\
& \text{iff } Ł \models \mathsf{BiVal}^{n}_{t} \rightarrow (\tau^{Ł} \rightarrow \chi) && \text{(Proposition 6.6)} \\
& \text{iff } \mathsf{BiVal}^{n}_{t}, \tau^{Ł} \models_{Ł_{\leq}} \chi \\
& \text{iff } \mathsf{BiVal}^{n}_{t}, \tau^{Ł}, q \models_{Ł_{\leq}} \chi \odot q \\
& \text{then } \mathsf{BiVal}^{n}_{t}, \tau^{\odot}, q \models_{Ł_{\leq}} \chi \odot q && (\text{because } \tau^{\odot} \models_{Ł_{\leq}} \tau^{Ł})
\end{aligned}$$

As $q \notin \mathsf{Pr}(\tau^{\odot})$, it is clear that $\tau^{\odot} \not\models_{Ł_{\leq}} \chi \odot q$, i.e., that $\tau^{\odot}$ is a proper $Ł_{\leq}$-solution to $\mathbb{P}$. It remains to see that $\tau^{\odot}$ is entailment-minimal. Assume for contradiction that there is a proper solution τ' such that $\tau^{\odot} \models_{Ł_{\leq}} \tau'$ but $\tau' \not\models_{Ł_{\leq}} \tau^{\odot}$. Then we have $\mathsf{BiVal}^{n}_{t}, \tau', q \models_{Ł_{\leq}} \chi \odot q$, whence, $\tau'_{\mathsf{cl}} \models_{Ł_{\leq}} \chi$, and, moreover, $(\tau^{\odot})_{\mathsf{cl}} \models_{\mathsf{CPL}} \tau'_{\mathsf{cl}}$ but $\tau'_{\mathsf{cl}} \not\models_{\mathsf{CPL}} (\tau^{\odot})_{\mathsf{cl}}$. As $(\tau^{\odot})_{\mathsf{cl}} = \tau$, this means that τ is not a prime implicant of χ, contrary to the assumption.

Conversely, let σ be an $\models_{Ł_{\leq}}$-minimal solution to $\mathbb{P}$. We check that σ_{cl} (recall Convention 3) is a prime implicant of χ. First, it is clear that $\sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \chi$. We need to check that there is no other σ' such that $\sigma' \models_{\mathsf{CPL}} \chi$, $\sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \sigma'$, but $\sigma' \not\models_{\mathsf{CPL}} \sigma_{\mathsf{cl}}$. We assume that there is such σ' and reason for the contradiction. Observe that $\sigma' \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$ because otherwise, $\sigma' \models_{\mathsf{CPL}} \sigma$. Thus, applying Proposition 6.6 and the fact that $\mathsf{lgth}(\sigma'^{\odot}) \leq \mathsf{lgth}(\sigma)$, we have that $\mathsf{BiVal}^{n}_{t}, \sigma'^{\odot}, q \models^{\mathsf{cons}}_{Ł_{\leq}} \chi \odot q$. But then it is clear that $\sigma \models_{Ł_{\leq}} \sigma'^{\odot}$ but $\sigma'^{\odot} \not\models_{Ł_{\leq}} \sigma$, contrary to the assumption that σ is $\models_{Ł_{\leq}}$-minimal. □

We finish the section by showing that the complexity of determining the existence of $Ł_{\leq}$-solutions and the relevance of hypotheses is the same as in the truth-preserving entailment. We provide brief sketches of the proofs.

Theorem 6.10. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem. Then, it is Σ^{P}_{2}-complete to determine whether $\mathbb{P}$ has a (proper) $Ł_{\leq}$-solution.*

Proof sketch. Σ^{P}_{2}-membership of arbitrary solution existence follows from Theorem 6.7. For proper solutions, we use Theorem 6.8. For Σ^{P}_{2}-hardness, we construct a polynomial reduction from solution existence in classical logic. Let $\mathbb{P} = \langle \{\phi\}, \chi, \mathsf{H} \rangle$ be a classical abduction problem such that $\mathsf{H} \subseteq \mathsf{Pr}[\{\phi, \chi\}]$. Let further, $\Xi = \mathsf{Pr}[\{\phi, \chi\}]$, $\mathsf{lgth}(\phi) = l_{\phi}$, and $\mathsf{lgth}(\chi) = l_{\chi}$. We consider $\mathbb{P}^{Ł_{\leq}} = \langle \{\phi^{Ł}, \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|}\}, \chi^{Ł}, \mathsf{H}^{\odot} \rangle$ (cf. Convention 3) and show that $\mathcal{S}(\mathbb{P}) \neq \varnothing$ iff $\mathcal{S}_{\leq}(\mathbb{P}^{Ł_{\leq}}) \neq \varnothing$.

Let $\tau \in \mathcal{S}(\mathbb{P})$, i.e., $\phi, \tau \models^{\mathsf{cons}}_{\mathsf{CPL}} \chi$. We show that $\tau^{\odot} \in \mathcal{S}_{\leq}(\mathbb{P}^{Ł_{\leq}})$. It is clear that $\phi^{Ł}, \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|}, \tau \not\models_{Ł_{\leq}} \bot$. Using Proposition 6.6 and Remark 6 and reasoning as in Theorem 6.9, one can check that the following entailment holds — $\phi^{Ł}, \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|}, \tau^{\odot} \models_{Ł_{\leq}} \chi^{Ł}$. Conversely, let $\sigma \in \mathcal{S}_{\leq}(\mathbb{P}^{Ł_{\leq}})$. It follows that $\phi, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \chi$. To see that $\phi, \sigma_{\mathsf{cl}} \not\models_{\mathsf{CPL}} \bot$, assume for contradiction that $\phi, \sigma_{\mathsf{cl}} \models_{\mathsf{CPL}} \bot$ (i.e., $\mathsf{CPL} \models \phi \rightarrow \neg\tau$). Using Proposition 6.6, we obtain that $Ł \models \mathsf{BiVal}^{|\mathsf{Pr}(\phi \rightarrow \neg\tau)|}_{\mathsf{lgth}(\phi \rightarrow \neg\tau)} \rightarrow (\phi^{Ł} \rightarrow \neg\tau^{Ł})$. Now observe that

$$\mathsf{Pr}(\phi \rightarrow \neg\tau) \subseteq \mathsf{Pr}[\{\phi, \chi\}] \qquad \text{and} \qquad \mathsf{lgth}(\phi \rightarrow \neg\tau) \leq l_{\phi} + l_{\chi} + |\mathsf{H}|$$

Thus, we have $Ł \models \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|} \rightarrow (\phi^{Ł} \rightarrow \neg\tau^{Ł})$, whence, $\phi^{Ł}, \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|}, \tau^{Ł} \models_{Ł_{\leq}} \bot$. As $\tau^{\odot} \models_{Ł_{\leq}} \tau^{Ł}$, it follows that $\phi^{Ł}, \mathsf{BiVal}^{|\Xi|}_{l_{\phi}+l_{\chi}+|\mathsf{H}|}, \tau^{\odot} \models_{Ł_{\leq}} \bot$, contrary to the assumption.

To see that the existence of *proper* $Ł_{\leq}$-solutions is Σ^{P}_{2}-hard, let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem and consider $\mathbb{P}_{\mathsf{prp}} = \langle \Gamma \cup \{p\}, \chi \odot p, \mathsf{H} \rangle$ with $p \notin \mathsf{Pr}[\Gamma \cup \{\chi\}] \cup \mathsf{H}$. It is clear that $\mathbb{P}$ has $Ł_{\leq}$-solutions iff $\mathbb{P}_{\mathsf{prp}}$ has *proper* $Ł_{\leq}$-solutions (note that since $p \notin \mathsf{Pr}[\mathsf{H}]$, $\mathbb{P}_{\mathsf{prp}}$ does not have non-proper solutions). This establishes the required polynomial reduction from the existence of $Ł_{\leq}$-solutions to the existence of *proper* $Ł_{\leq}$-solutions. □

Theorem 6.11. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem and $\lambda \in \mathsf{H}$. Then it is Σ^{P}_{2}-complete (Π^{P}_{2}-complete) to determine whether λ is relevant (necessary) w.r.t. $\mathcal{S}_{\leq}(\mathbb{P})$, $\mathcal{S}^{\mathsf{p}}_{\leq}(\mathbb{P})$, and $\mathcal{S}^{\min}_{\leq}(\mathbb{P})$.*

Proof sketch. Σ^{P}_{2}-membership for relevance and Π^{P}_{2}-membership for necessity follow from Theorem 6.8 as it suffices to guess a solution that contains (or does not contain, if we check *non-necessity*) λ. For hardness, we can use the reduction from Theorem 4.8. Namely, given an Ł-abduction problem $\mathbb{P}$, we define $\mathbb{P}^{\mathsf{r}}$ as shown in (8). □

6.2 Complexity of Abduction in Clause Fragments

As we have seen in Section 5, there are several clause fragments of Łukasiewicz logic where abduction under the truth-preserving entailment is simpler than in the general case and sometimes even tractable. In this section, we will consider the complexity of abduction in clause fragments under the degree-preserving entailment. From Remark 4, it is clear that the complexity of abduction in *interval clause* fragment under both truth- and degree-preserving entailment is the same. Thus, we will only consider the simple clause fragment.

Corollary 6.12. *Results in Tables 2 and 3 for interval-clause fragments (arbitrary interval-clause, cover-free and weakly cover-free, monodirectional, positive, and short monodirectional theories in the tables) hold for Łukasiewicz logic with the degree-preserving entailment.*

Proof sketch. Recall from Remark 4 and Definition 2.2 that if ϕ and χ are composed from interval literals using $\{\neg, \odot, \oplus, \rightarrow\}$, then $v(\phi), v(\chi) \in \{0, 1\}$. From here, it follows immediately that $\phi \models_{Ł} \chi$ iff $\phi \models_{Ł_{\leq}} \chi$. Now let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem with Γ being a set of interval clauses. As solutions to Ł-abduction problems are interval terms, it follows that τ is a (proper, entailment-, or theory-minimal) solution to $\mathbb{P}$ iff τ is a (proper, entailment-, or theory-minimal) $Ł_{\leq}$-solution to $\mathbb{P}$.

This means that to establish whether τ is an $Ł_{\leq}$-solution to $\mathbb{P}$, it suffices to check whether it is a standard solution to $\mathbb{P}$ (i.e., a solution defined w.r.t. the truth-preserving entailment). And conversely, to check that τ is a standard solution, it suffices to verify that τ is an $Ł_{\leq}$-solution. Similarly, $\mathbb{P}$ has standard (proper) solutions iff $\mathbb{P}$ has (proper) $Ł_{\leq}$-solutions, and $\lambda \in \mathsf{H}$ is relevant (necessary) w.r.t. (proper, entailment-, or theory-minimal) solutions of $\mathbb{P}$ iff it is relevant (necessary) w.r.t. (proper, entailment-, or theory-minimal) $Ł_{\leq}$-solutions of $\mathbb{P}$. The result now follows. □

We note again that just as was the case with the results in Section 4, the hardness results in Section 6.1 follow from Corollary 6.12. Still, our proofs in the previous section show that hardness results can be obtained even when neither the theory nor the observation in the abduction problem contains interval literals.

In Section 5.1, we have utilised the fact that sets of simple clauses correspond to systems of linear inequalities. Recall from Remark 1 that in the truth-preserving entailment, sets of premises could be represented either as strong or as weak conjunctions of premises. Thus, it makes sense to investigate the complexity of both representations of simple clause theories w.r.t. degree-preserving entailment.

Definition 6.13. Let $\kappa_1, \ldots, \kappa_m$ and $\kappa'_1, \ldots, \kappa'_{m'}$ be simple clauses and $\lambda_1, \ldots, \lambda_n$ be interval literals. We define:

- a *weak* simple clause (WSC) theory to be a finite set $\Gamma = \left\{\bigwedge_{i=1}^{m} \kappa_i, \lambda_1, \ldots, \lambda_n\right\}$;
- a *strong* simple clause (SSC) theory to be a finite set $\Gamma = \left\{\bigodot_{i=1}^{m} \kappa_i, \lambda_1, \ldots, \lambda_n\right\}$;
- a *mixed* simple clause (MSC) theory to be a finite set $\Gamma = \left\{\bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\right\}$.

We note briefly that *every* formula of the form $\bigwedge_{i=1}^{m} \kappa_i$ is positively satisfiable. Indeed, observe that if $0 < v(p) < 1$ for every $p \in \mathsf{Pr}(\bigwedge_{i=1}^{m} \kappa_i)$, then each clause has a positive value because, in this case, $v(\neg p) > 0$. On the other hand, since interval literals are evaluated over $\{0, 1\}$, it does not matter whether we join them with $\wedge$ or $\odot$ (recall Definition 2.2 and Remark 4). Thus, we can w.l.o.g. assume that weak, strong, and mixed simple clause theories contain *sets* of interval literals.

In the remainder of the section, we will show that solution recognition in all types of simple clause theories is polynomial, and solution existence as well as relevance of hypotheses are in NP. As we are providing upper bounds, it suffices to consider only the case of abduction problems with MSC theories. We begin with a technical lemma that establishes that the positive satisfiability of MSC theories is polynomial.

Lemma 6.14. *Given a mixed simple clause theory $\Gamma = \left\{\bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\right\}$, it takes polynomial time to establish whether $\Gamma \not\models_{Ł_{\leq}} \bot$.*

Proof. We show how to reduce the positive satisfiability of MSC theories to the existence of solutions over $[0, 1]$ to a system of linear inequalities. The idea is similar to the one in Theorem 5.7.

Recall from Definition 6.1 that $\{\phi_1, \ldots \phi_k\}$ is positively satisfiable iff there is an Ł-valuation v such that $\max\left(0, \sum_{i=1}^{k} v(\phi_i) - (k-1)\right) > 0$. This means that there are some $y_1, \ldots, y_k \in [0,1]$ such that $v(\phi_i) \geq y_i$ and $\sum_{i=1}^{k} y_i - (k-1) > 0$. Let us now use this fact. For every simple clause $\kappa = \bigoplus_{i=1}^{k} p_i \oplus \bigoplus_{i=1}^{l} \neg q_i$, we set

$$\kappa^{\mathbb{LI}\leq} = \sum_{i=1}^{k} p_i + \sum_{i=1}^{l} (1 - q_i) \tag{24}$$

Now, given $\Gamma = \left\{\bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\right\}$, let $y_1, \ldots, y_m$ and y' be new variables ranging over $[0,1]$. We set (recall Definition 5.4 for $\lambda^{\mathbb{LI}}$):

$$\Gamma^{\mathbb{LI}\leq} = \left\{\kappa_i^{\mathbb{LI}\leq} \geq y_i \mid i \in \{1, \ldots, m\}\right\} \cup \left\{\kappa_i'^{\mathbb{LI}\leq} \geq y' \mid i \in \{1, \ldots, m'\}\right\} \cup \left\{y' + \sum_{i=1}^{m} y_i - m > 0\right\} \cup \left\{\lambda_i^{\mathbb{LI}} \mid i \in \{1, \ldots, n\}\right\}$$

We can now show that $\Gamma^{\mathbb{LI}\leq}$ has a solution over $[0,1]$ iff Γ is positively satisfiable.

Let v be an Ł-valuation such that $v\left(\bigodot_{i=1}^{m} \kappa_i \odot \bigwedge_{i=1}^{m'} \kappa'_i\right) > 0$ and $v(\lambda_i) = 1$ for all $i \in \{1, \ldots, n\}$. Observe also that $v(\bigwedge_{i=1}^{m'} \kappa'_i) \geq y'$ iff $v(\kappa'_i) \geq y'$ for every $i \in \{1, \ldots, m'\}$. We can now use the values of propositional variables under v as the values of variables in $\Gamma^{\mathbb{LI}\leq}$ and set $y_i = v(\kappa_i)$ and $y' = \min\{v(\kappa'_i) \mid i \in \{1, \ldots, m'\}\}$. This means that v induces a solution to $\Gamma^{\mathbb{LI}\leq}$.

For the converse direction, let $\mathbf{v}$ be a solution to $\Gamma^{\mathbb{LI}\leq}$. One can see that $\mathbf{v}(\lambda_i) = 1$ for all $i \in \{1, \ldots, n\}$. Furthermore, by Definition 2.2 and (24), we have (1) $y_i \in [0,1]$ and $\kappa_i^{\mathbb{LI}\leq} \geq y_i$ entail $\mathbf{v}(\kappa_i) \geq y_i$ for all $i \in \{1, \ldots, m\}$, and (2) $y' \in [0,1]$ and $\kappa_j'^{\mathbb{LI}\leq} \geq y'$ for all $j \in \{1, \ldots, m'\}$ entail that $\mathbf{v}(\bigwedge_{i=1}^{m'} \kappa'_j) \geq y'$. Thus, as $y' + \sum\limits_{i=1}^{m} y_i - m > 0$, it follows that $\mathbf{v}\left(\bigodot_{i=1}^{m} \kappa_i \odot \bigwedge_{i=1}^{m'} \kappa'_i\right) > 0$, as required. As $\mathsf{lgth}[\Gamma^{\mathbb{LI}\leq}] = \mathcal{O}[\Gamma]$, its positive satisfiability is decidable in polynomial time. □

Theorem 6.15. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ with Γ being an MSC theory and χ a simple clause, simple term, interval clause, or an interval term. Then, given an interval term τ, it takes polynomial time to determine whether it is a (proper, $\models_{\text{Ł}_\leq}$-minimal) $\text{Ł}_\leq$-solution to $\mathbb{P}$.*

Proof. We begin with the case of τ being an arbitrary solution. For this, we need to check that $\Gamma, \tau \not\models_{\text{Ł}_\leq} \bot$ and $\Gamma, \tau \not\models_{\text{Ł}_\leq} \chi$. Let $\tau = \bigodot_{i=1}^{k} p_i \diamond \overline{\mathsf{c}}_i$ be an interval term. It is clear that $\Gamma, \tau \not\models_{\text{Ł}_\leq} \bot$ iff $\Gamma \cup \{p_i \diamond \overline{\mathsf{c}}_i \mid i \in \{1, \ldots, k\}\}$ is positively satisfiable. As Γ is an MSC theory, by Lemma 6.14, we have that $\Gamma, \tau \not\models_{\text{Ł}_\leq} \bot$ can be checked in polynomial time. It remains to see that $\Gamma, \tau \not\models_{\text{Ł}_\leq} \chi$ can also be verified in polynomial time. We recall briefly that $\Gamma, \tau \models_{\text{Ł}_\leq} \chi$ iff $\Gamma, \tau, \neg\chi \models_{\text{Ł}_\leq} \bot$ and reason by cases, depending on the form of χ.

If $\chi = \bigodot_{i=1}^{l} s_i \odot \bigodot_{i=1}^{l'} \neg t_i$ is a simple term, $\neg\chi$ is strongly equivalent to a simple clause $\chi' = \bigoplus_{i=1}^{l} \neg s_i \oplus \bigoplus_{i=1}^{l'} t_i$ (recall (1)). Thus, we can check the positive satisfiability of the following MSC theory: $\Gamma \cup \{\tau, \chi'\}$. If $\chi = \bigoplus_{i=1}^{l} \neg s_i \oplus \bigoplus_{i=1}^{l'} t_i$ is a simple clause, then $\neg\chi$ is strongly equivalent a simple term $\chi'' = \bigodot_{i=1}^{l} s_i \odot \bigodot_{i=1}^{l'} \neg t_i$. Thus, if $\Gamma = \{\bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\}$, then $\Gamma, \tau, \neg\chi \models_{\text{Ł}_\leq} \bot$ iff $\{\chi'' \odot \bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\} \cup \{p_1 \diamond \overline{\mathsf{c}}_1, \ldots, p_k \diamond \overline{\mathsf{c}}_k\}$ is positively unsatisfiable. Again, by Lemma 6.14, this can be checked in polynomial time because a simple term is a strong conjunction of (unitary) simple clauses.

If $\chi = \bigoplus_{i=1}^{l} q_i \diamond \overline{\mathsf{d}}_i$ is an interval clause, then to check $\Gamma, \tau, \neg\chi \models_{\text{Ł}_\leq} \bot$, it suffices to verify that

$$\left\{\bigodot_{i=1}^{m} \kappa_i, \bigwedge_{i=1}^{m'} \kappa'_i, \lambda_1, \ldots, \lambda_n\right\} \cup \{p_1 \diamond \overline{\mathsf{c}}_1, \ldots, p_k \diamond \overline{\mathsf{c}}_k\} \cup \left\{q_1 \blacklozenge \overline{\mathsf{d}}_1, \ldots, q_l \blacklozenge \overline{\mathsf{d}}_l\right\}$$

is positively unsatisfiable. Again, as this is an MSC theory, we can do this in polynomial time. Finally, if $\chi = \bigodot_{i=1}^{l} q_i \diamond \overline{\mathrm{d}}_i$, we observe that $\Gamma, \tau \models_{Ł_\leq} \chi$ iff $\Gamma, \tau \models_{Ł_\leq} q_i \blacklozenge \overline{\mathrm{d}}_i$ for all $i \in \{1, \ldots, k\}$. By the previous case, it is clear that $\Gamma, \tau \models_{Ł_\leq} q_i \blacklozenge \overline{\mathrm{d}}_i$ can be verified in polynomial time.

To determine whether τ is a proper solution, we also need to check that $\tau \not\models_{Ł_\leq} \chi$. From the previous part of the proof, it is clear that this can be done in polynomial time. To check that τ is a $\models_{Ł_\leq}$-minimal solution, we need to verify that there is no other solution τ' such that $\tau \models_{Ł_\leq} \tau'$ but $\tau' \not\models_{Ł_\leq} \tau$. By Proposition 3.5 and Theorem 6.9, this can be verified in polynomial time. The result now follows. □

Using Theorem 6.15, we can immediately obtain that solution existence and hypothesis relevance for abduction problems with MSC theories can be decided in nondeterministic polynomial time. Consequently, the necessity of hypotheses is decidable in coNP.

Theorem 6.16. *Let $\mathbb{P} = \langle \Gamma, \chi, \mathsf{H} \rangle$ be an Ł-abduction problem such that Γ is an MSC theory and χ is a simple term, simple clause, interval term, or an interval clause. Let further $\lambda \in \mathsf{H}$.*

(1) It is in NP *to decide whether there are (proper) $Ł_\leq$-solutions to $\mathbb{P}$.*
(2) It is in NP *to decide whether λ is relevant w.r.t. all, proper, or $\models_{Ł_\leq}$-minimal solutions.*
(3) It is in coNP *to decide whether λ is necessary w.r.t. all, proper, or $\models_{Ł_\leq}$-minimal solutions.*

We finish the section with a short discussion of the results. As one can see, we have not provided matching lower bounds for simple clause abduction in the degree-preserving entailment. The issue is that the reduction from classical Horn abduction that we utilised in Section 5.1 will not work. In particular, observe that

$$p, p \rightarrow q, q \rightarrow r, q \rightarrow s, r \rightarrow (s \rightarrow t) \models_{\mathsf{CPL}} t \qquad \text{but} \qquad p, p \rightarrow q, q \rightarrow r, q \rightarrow s, r \rightarrow (s \rightarrow t) \not\models_{Ł_\leq} t$$

This means that to replicate the classical resolution inference of t, we would need to repeat p and $p \rightarrow q$ twice.

One might hope that for a classical Horn theory $\Gamma = \{\phi_1, \ldots, \phi_m\}$, it would suffice to repeat each formula m times to be able to replicate its resolution refutation in $Ł_\leq$, i.e., to consider $\Gamma^m = \{\underbrace{\phi^{Ł} \odot \ldots \odot \phi^{Ł}}_{m \text{ times}} \mid \phi \in \Gamma\}$ (recall Convention 3 for $\phi^{Ł}$). Unfortunately, as the next statement shows, this is not generally the case.[10]

Proposition 6.17. *Consider the following family of Horn theories Γ_n:*

$$\Gamma_n = \{p_{0,0}, p_{0,1}, \neg p_{n,0}\} \cup \{(p_{i,0} \wedge p_{i,1}) \rightarrow p_{i+1,0} \mid i \in \{0, \ldots, n-1\}\} \cup \{(p_{i,0} \wedge p_{i,1}) \rightarrow p_{i+1,1} \mid i \in \{0, \ldots, n-1\}\}$$

It follows that for every $n \geq 5$ and every $m < 2^{n-1}$, Γ_n is CPL*-unsatisfiable but Γ_n^m is positively satisfiable.*

Proof. One can easily check that $\Gamma_n \models_{\mathsf{CPL}} \bot$. We show that Γ_n^m's are positively satisfiable. Observe that Γ_n contains $2n + 3$ formulas, each containing at most three variables.

We construct the satisfying valuation for Γ as follows: $v(p_{i,0}) = v(p_{i,1}) = \frac{2^{n-i}-1}{2^{n-i}}$. Now observe that this gives:

$$\begin{aligned}
v(\neg p_{n,0}) &= 1 && \text{(because } v(p_{n,0}) = 0\text{)}\\
\forall i \in 0, \ldots, n-1 : v((p_{i,0} \odot p_{i,1}) \rightarrow p_{i+1,0}) &= 1 && \text{(because } v(p_{i,0} \odot p_{i,1}) = v(p_{i+1,0}) = v(p_{i+1,1})\text{)}\\
v((\underbrace{p_{0,0} \odot \ldots \odot p_{0,0}}_{2^{n-1} \text{ times}}) \odot (\underbrace{p_{0,1} \odot \ldots \odot p_{0,1}}_{2^{n-1} \text{ times}})) &= \tfrac{1}{2^{n-1}} && \text{(because } v(p_{0,0}) = v(p_{0,1}) = \tfrac{2^n-1}{2^n}\text{)}
\end{aligned}$$

From here, it follows that $v\left(\bigodot_{\phi \in \Gamma_n^m} \phi\right) = \frac{1}{2^{n-1}} > 0$, i.e., Γ_n^m is positively satisfiable. Finally, it is clear that for every $1 \leq m' < m$, v also positively satisfies $\Gamma_n^{m'}$. The result now follows. □

[10]We are grateful to Emil Jeřábek for providing this example.

7 Conclusion and Future Work

We have studied abductive reasoning in Łukasiewicz logic and its fragments. We proposed and motivated interval literals that we used as solutions to Ł-abduction problems and conducted a complexity analysis of the main reasoning tasks: solution recognition, solution existence, and relevance and necessity of hypotheses w.r.t. different types of solutions. Our results (Table 1) show that the complexity of abduction in arbitrary theories is not higher than in the classical case (Creignou and Zanuttini 2006; Eiter and Gottlob 1995; Nordh and Zanuttini 2008; Pfandler et al. 2015; Pichler and Woltran 2010). In clausal fragments (Tables 2 and 3), the complexity of abduction either coincides with that of the general case (for *arbitrary interval-clause theories*), with that of classical Horn abduction (for cover-free and simple clause theories), or is polynomial (for weakly cover-free theories). In addition to that, we considered abduction in Łukasiewicz logic with the degree-preserving entailment and proved that its complexity coincides with the complexity of abduction in Łukasiewicz logic with the truth-preserving entailment in the general case (Theorems 6.7–6.11) and in the case of interval-clause fragments (Corollary 6.12).

Several questions remain open. First, we do not know the exact complexity of the recognition of theory-minimal solutions and the relevance of hypotheses w.r.t. theory-minimal solutions. One way to approach this would be to establish the complexity of the closely related notion of theory prime implicants in CPL (Marquis 1995). Second, we did not obtain matching lower bounds for abduction in simple clause fragments of Łukasiewicz logic with the degree-preserving entailment. Taking into account that there seems to be no simple translation of the Horn fragment of CPL to simple clause fragments of $Ł_{\leq}$ (Proposition 6.17), it is reasonable to conjecture that solution existence in (weak, strong, or mixed) simple clause fragments of $Ł_{\leq}$ can be tractable. Third, observe from Example 6.4 that 'standard' and $Ł_{\leq}$-solutions to abduction problems are incomparable. Thus, it makes sense to establish whether there are some fragments of $\mathcal{L}_{Ł}^{\mathbb{Q}}$ (other than the interval-clause fragment) where 'standard' and $Ł_{\leq}$-solutions to abduction problems coincide.

Another important task is to generalise the notion of abduction problems. In our current framework, we restrict the set of hypotheses to be a finite set of *interval literals* to guarantee the finiteness of the set of solutions. Note, however, that it is known that satisfiability and entailment in the (infinite-valued) Łukasiewicz logic can be reduced to satisfiability and entailment in the *finite-valued* Łukasiewicz logics (Aguzzoli and Ciabattoni 2000; Aguzzoli and Gerla 2002a,b).[11] As is well-known, propositional abduction corresponds to the Σ_2-fragment of the *quantified* propositional logic. Quantified propositional Łukasiewicz logic is a fragment of the first-order theory of MV-algebras which admits quantifier elimination (Baaz and Veith 1998). Thus, it is reasonable to assume that the infinite-valued quantified propositional Łukasiewicz logic can also be reduced to its finite-valued variant. In this case, we would be able to construct the 'minimally sufficient' set of boundary values for hypotheses using the shape of the abduction problem.

To the best of our knowledge, existing algorithms for solution generation to abduction problems in fuzzy logics construct solutions in the form of 'standard' fuzzy sets (i.e., singular values of variables). As interval terms are a more general form of solutions, it would also be interesting to present an algorithm that would generate solutions to Ł-abduction problems in the form of interval terms. An obvious starting point would be to formalise the procedure outlined in Example 3.15. Another promising approach would be to use the constraint tableaux calculus (recall Proposition 2.3) augmented with additional rules for $\odot$ and $\oplus$. Here, we hope to utilise tableaux proofs to infer constraints on the values of variables occurring in the set of hypotheses and then use these constraints to construct solutions for abduction problems. We leave a detailed study and software implementation of solution generation algorithms for Ł-abduction problems for future research.

Furthermore, recall from Section 3 that interval literals can be interpreted as *interval-valued* fuzzy sets. There are multiple algorithms that construct solutions in the form of standard fuzzy sets (i.e., those where statements $x \in X$ are evaluated as *numbers* in $[0, 1]$) for abduction problems formulated in fuzzy logics. Thus, a natural

[11] An n-valued Łukasiewicz logic is a logic with the set of values $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and the connectives defined as in Definition 2.2.

continuation of the point in the previous paragraph would be to use the 'finitisation' of abduction in Łukasiewicz logic for the development of algorithms for generating solutions in the form of interval-valued fuzzy sets. Another step would be to generalise our results to *interval-valued fuzzy logics* (i.e., logics where formulas are evaluated as intervals in $[0, 1]$) such as those by Ornelas and Kreinovich (2007) and C. L. Walker and E. A. Walker (2009).

Furthermore, fuzzy logics have been applied to (deductive) reasoning about uncertainty, in particular, about probability measures (Baldi et al. 2020; Hájek and Tulipani 2001) and belief functions (Godo et al. 2003). Probabilistic and possibilistic abduction received much attention (Dubois, Gilio, et al. 2008; Dubois and Prade 1992; Poole 1993; Sato et al. 2011), but was studied from the standpoint of classical logic. Thus, it makes sense to apply fuzzy logics to *abductive* reasoning about uncertainty.

Finally, Ł-abduction problems can often be interpreted in terms of reasoning about constraints on the values of variables. Similarly, their solutions are also sets of constraints (cf. Examples 3.2 and 3.3 as well as the discussion in Section 5.2). This means that (at least a fragment of) Ł-abduction can be interpreted in terms of abductive constraint logic programming (ACLP) (Kakas et al. 2000). To the best of our knowledge, the connection between abduction in fuzzy logic and ACLP has not been explored. Thus, it would be instructive to establish the correspondence between fuzzy abduction and abductive constraint logic programming.

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