Efficient Constructions of EFX Allocations for Two Types of Indivisible Goods

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We study the problem of fairly and efficiently allocating indivisible goods among agents with additive valuations. We focus on envy-freeness up to any good (EFX)—an important fairness notion in fair division of indivisible goods. A central open question in this field is whether EFX allocations always exist for any number of agents. While prior work has established EFX existence for settings with at most three distinct valuations (HV, Ghosal, et al. 2025) and for two types of goods (Gorantla et al. 2023), the general case remains unresolved.

In this paper, we extend the existent knowledge by proving that EFX allocations satisfying Pareto optimality (PO) always exist and can be computed in quasilinear time when there are two types of goods, given that the valuations are positive. This result strengthens the existing work of (Gorantla et al. 2023), which only guarantees the existence of EFX allocations without ensuring Pareto optimality. Our findings demonstrate a fairly simple and efficient algorithm constructing an EFX+PO allocation.

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1 Introduction

Fair division of indivisible goods is a core research area in algorithmic game theory and computational social choice, focusing on the equitable allocation of discrete items—such as property, licenses, or humanitarian aid—among agents with heterogeneous preferences. The online platform Spliddit (spliddit.org) offers practical implementations of fair division algorithms, addressing real-world allocation challenges including rent distribution among roommates, taxi fare splitting, and fair assignment of goods between individuals. A central goal is to achieve fairness and effectiveness guarantees that balance efficiency and equity, even when exact solutions are theoretically or computationally out of reach.

The gold standard of fairness, envy-freeness (EF) (Foley 1967), ensures no agent prefers another's allocation over their own. Yet, EF allocations often fail to exist for indivisible goods, and even when they do, identifying

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them is NP-complete as shown by (Lipton et al. 2004) even with two agents. This limitation has spurred the study of meaningful relaxations. The initial relaxation of envy-freeness is the concept of envy-freeness up to one good (EF1), which was informally introduced by (Lipton et al. 2004) and later formally defined by (Budish 2011). Under EF1, an agent i may envy agent j, provided there is at least one good in j's bundle such that, if removed, i's envy toward j would disappear. EF1 allocations always exist for indivisible goods under general monotone valuations and can be computed in polynomial time using the envy cycle elimination method (Lipton et al. 2004). For additive valuations, (Caragiannis, Kurokawa, et al. 2019) showed that the Round-Robin algorithm yields an EF1 allocation. For the same valuation class, (Caragiannis, Kurokawa, et al. 2019) proved that the maximum Nash welfare (MNW) solution is Pareto-efficient and envy-free up to one good. Among these, envy-freeness up to any good (EFX), introduced by (Caragiannis, Kurokawa, et al. 2019), has emerged as a compelling alternative. EFX softens the strictness of EF: while an agent may envy another's bundle, this envy vanishes upon the removal of any single good from the envied bundle. The existence of EFX is still open for n agents with arbitrary additive valuations.

Recent years have seen significant progress in answering this question, though a general solution remains elusive. For identical valuations, (Plaut and Roughgarden 2020) showed that a leximin++ solution guarantees EFX. For two agents, (Plaut and Roughgarden 2020) showed that the cut and choose algorithm guarantees EFX. They also extended their analysis beyond identical valuations, demonstrating that the Envy-Cycle Elimination algorithm also guarantees EFX allocations for ordered instances—cases where agents share identical preference rankings over goods but may differ in their cardinal valuations. (Chaudhury, Garg, and Mehlhorn 2024) extended this result to three agents, while (Mahara 2023) and (HV, Ghosal, et al. 2025) proved existence for settings with two and three types of agents, respectively. On the other side, (Gorantla et al. 2023) demonstrated that EFX allocations exist when goods belong to just two distinct types, even providing a partial characterization of when exact envy-freeness (EF) is achievable in such cases. A previous paper established the existence of EFX and PO allocations under lexicographic preferences (Hosseini et al. 2021). Moreover, (Garg and Murhekar 2023) proved that an EFX and fPO allocation exists and can be computed in polynomial time for bivalued instances. Independently, (Jin and Tao 2026) characterize Pareto-optimal allocations under personalized bi-valued utilities—a generalization of our two-type setting—and pose as an open question whether an EFX and PO allocation always exists there. Our result resolves this question affirmatively for the subclass with a common partition of goods into two types.

Pareto optimality is a fundamental efficiency concept in fair division, ensuring no agent can improve their allocation without harming others. (Amanatidis, Birmpas, et al. 2021) showed that when goods take on at most two possible values, the Maximum Nash Welfare (MNW) solution guarantees EFX and Pareto optimality (PO), also they show a polynomial-time algorithm for EFX based on matching. However, (Plaut and Roughgarden 2020) also revealed a sobering limitation: if agents can assign zero value to goods, EFX and PO may be incompatible. This leaves an open problem: Does an EFX and PO allocation always exist when all valuations are positive?

Our Contribution. We significantly advance the understanding of fair and efficient allocations for two types of goods. First, we prove that EFX+PO allocation always exist when all agents have positive utilities, while demonstrating that the stronger EFX+fPO combination remains impossible. This improves upon (Gorantla et al. 2023), who established EFX existence without Pareto optimality.

We provide an efficient algorithmic solution: our approach computes such an allocation in $O(n \log n + \log m)$ time for n agents and m goods. It works as fast as $O(\log n + \log m)$ time, if the agents are ordered by their relative valuations of good types. Additionally, we introduce a class of *proper* allocations and prove that every *proper* allocation is Pareto optimal, offering new structural insights into the interplay between fairness and efficiency.

Example: EFX and fPO are not always compatible. Consider a simple example with two types of goods (croissant and coffee) and two agents (Alice (1) and Bob (2)). There are two croissants (a) and two coffees (b), with valuations

$v_1(a) = v_2(a) = 1$ and $v_1(b) = 10$ and $v_2(b) = 9$. In the unique EFX allocation, each agent receives one croissant and one coffee. While this allocation is envy-free up to any good, it fails to be fractionally Pareto optimal (fPO): a fractional allocation where Alice gets 1.1 coffee and Bob gets 0.9 coffee plus two croissant would strictly improve the allocation, Alice has the same utility, but Bob increases his utility (11 for Alice and 10.1 for Bob). However, in the indivisible setting, this EFX allocation is Pareto optimal (PO), as no discrete reallocation can improve one agent's utility without harming the other. This demonstrates that there may not be allocations that satisfy EFX and fPO simultaneously. Nevertheless, we show that an EFX+PO allocation always exists.

Table 1. An example that does not admit EFX+fPO allocations.

				
	1	1	10	10
	1	1	9	9

Additional Related Work. This is an extended and improved version of the work appeared in proceedings of the AAAI 2026 conference (Davidiuk et al. 2026). The fair division of indivisible goods has witnessed remarkable theoretical and algorithmic progress in recent years, as documented in several comprehensive surveys (Amanatidis, Aziz, et al. 2023; Aziz et al. 2022; Nguyen and Rothe 2023).

One promising direction involves EFX relaxation by allowing a small subset of goods to remain unassigned—often interpreted as charitable donations—while maintaining fairness guarantees. While trivially satisfying envy-freeness by leaving all goods unallocated is meaningless, meaningful progress has been made in bounding both the number of discarded items and their welfare impact. (Caragiannis, Gravin, et al. 2019) demonstrated that an EFX allocation exists for a subset of goods while preserving at least half of the Maximum Nash Welfare. (Chaudhury, Kavitha, et al. 2021) developed an algorithm for computing partial EFX allocations in which at most $n - 1$ goods remain unassigned and no agent strictly prefers the set of unallocated goods to their own bundle. Subsequently, (Mahara 2024) improved the general bound to $n - 2$ unassigned goods, while (Berger et al. 2022) established that only one good needs to remain unassigned for instances with four agents. (Ghosal et al. 2025) generalized these results, proving that for agents with at most k distinct valuations, EFX allocations exist with $k - 2$ unassigned goods.

Approximate EFX notions have also gained attention. Formally, an allocation is α-EFX if, for every pair of agents i and j and every good $g \in X_j$,

$$v_i(X_i) \geq \alpha \cdot v_i\big(X_j \setminus \{g\}\big).$$

(Plaut and Roughgarden 2020) established the existence of 0.5-EFX allocations, and this bound was later improved to 0.618 by (Amanatidis, Markakis, et al. 2020). Most recently, (Amanatidis, Filos-Ratsikas, et al. 2024) showed that $\frac{2}{3}$-EFX allocations exist for up to seven agents or when agents have no more than three distinct valuations. (HV, Mehta, et al. 2026) extended these results, showing that a $\frac{2}{3}$-EFX allocation exists for any number of agents with at most four distinct valuations, and generalizing the sublinear-charity guarantee to instances with an arbitrary number k of distinct valuations. Related $\frac{2}{3}$-EFX and ordinal-assumption guarantees have also been obtained by (Markakis and Santorinaios 2023) and, more recently, by (Filos-Ratsikas et al. 2026). Building on this line of work, (Chaudhury, Garg, Mehlhorn, et al. 2024) elegantly bridged the concepts of approximation and charity, demonstrating that a $(1 - \varepsilon)$-EFX allocation can be efficiently computed while donating only a sublinear number

of goods, all while preserving high Nash welfare for $\varepsilon \in (0, 0.5]$; (Akrami, Alon, et al. 2024) obtain a related near-optimal charity guarantee via the rainbow cycle number.

Setting. The setting consists of a set of n agents $N = [n] = \{1, \dots, n\}$, and a multiset M of indivisible goods that contains copies of two different goods: g_1 with multiplicity m_1 and g_2 with multiplicity m_2. For integers $a \le b$, we write

$$[a, b] = \{a, a + 1, \dots, b\};$$

if $a > b$, then $[a, b]$ is empty. Each agent $i \in N$ has a valuation function $v_i \colon 2^M \to \mathbb{R}_{\ge 0}$, which quantifies the utility that i derives from any subset of goods. We focus on additive valuations, meaning the value of a subset of goods is simply the sum of the values of its individual items. Formally, for any subset $S \subseteq M$, the valuation function satisfies $v_i(S) = \sum_{g \in S} v_i(\{g\})$. An allocation $X = (X_1, X_2, \dots, X_n)$ is a partition of M into n disjoint subsets, called bundles, where each agent i receives the bundle X_i. We match allocation X with the multiplicity of goods in each agent's bundle $\{(x_{i,1}, x_{i,2}) \mid i \in [n]\}$, so we can assume that agents utility equals $v_i(X_i) = x_{i,1} \cdot v_{i,1} + x_{i,2} \cdot v_{i,2}$, where $v_{i,j} = v_i(\{g_j\})$ for each $i \in [n]$, $j \in [2]$.

Definition 1 (EFX)**.** An allocation X is envy-free up to any good (EFX) if, for every pair of agents $i, j \in N$ and every good $g \in X_j$, it holds that

$$v_i(X_i) + v_i(\{g\}) \ge v_i(X_j).$$

We say that agent i *strongly envies* agent j if the EFX condition is violated for this pair, that is, if there exists a good $g \in X_j$ such that

$$v_i(X_i) < v_i\big(X_j \setminus \{g\}\big).$$

Definition 2 (PO)**.** An allocation X is Pareto optimal (PO) if there is no allocation Y such that $v_i(Y_i) \ge v_i(X_i)$ for all $i \in N$ and $v_j(Y_j) > v_j(X_j)$ for some $j \in N$. Equivalently, we will say that such an allocation is not Pareto dominated by any other allocation.

Organization of the paper. The remainder of the paper is organized as follows. Section 2 introduces the setting and key preliminaries, including proper allocations, prioritized equitable allocations (PEA), split allocations, and reallocations. Section 3 presents our algorithm and its running-time guarantees. While Section 2 and Section 3 formulate the main constructions and their properties that are enough to understand the algorithm and why it is correct, subsequent sections are devoted solely to proofs of these properties. Section 4 proves that every proper allocation is Pareto optimal. Section 5 studies envy in split allocations and establishes the key properties of envy directions used by the algorithm. Section 6 analyzes the specific reallocation construction and completes the proof of our main EFX+PO existence result. Finally, Section 7 concludes and discusses open questions.

2 The Setup

To show our main result, we have to define several crucial concepts. In this section, we give necessary definitions and formulate fundamental properties of our allocation structures. They serve as a backbone of our algorithm.

Most of the properties in this section are given without proofs to ease the understanding of the construction and the algorithm itself. Complete proofs of such properties and claims can be found in subsequent sections.

2.1 Input Preprocessing

Our first goal is to make the input data much more convenient and reflecting the actual allocation properties.

By n we denote the number of agents, by m_1 and m_2 we denote the total quantities of items of first and second types respectively. We naturally assume that $n \ge 1$ and $m_1, m_2 \ge 1$. Agents are identified with integer numbers in $[n]$. For $i \in [n]$, $j \in [2]$, let $v_{i,j}$ denote the utility of an item of type j as estimated by agent i.

We then divide the agents in two groups, the first group consists of agents i that have $v_{i,2} \leq v_{i,1}$ (agents that prefer a good of type 1 to a good of type 2 or value them equally), the second consists of agents i that have $v_{i,2} \geq v_{i,1}$[1]. Note that there may be several ways to divide the agents, choose any of them arbitrarily. Let n_1, n_2 denote the obtained group sizes. If $m_1 n_2 < m_2 n_1$, we interchange the good types and the groups (that is why we defined the groups symmetrically). Then $m_1 n_2 \geq m_2 n_1$ holds. Since $m_1, m_2 > 0$ and $n_1 + n_2 = n \geq 1$, this implies $n_2 > 0$.

We assume that $v_{i,j} > 0$ for each $i \in [n]$, $j \in [2]$, as motivated in the introduction. Then, we normalize the agent utilities: we replace $(v_{i,1}, v_{i,2})$ with $(1, v_{i,2}/v_{i,1})$ for each agent $i \in [n]$. Finally, we re-enumerate the agents in the way that $v_{i,2} \leq v_{i+1,2}$ holds for each $i \in [n-1]$. By definition of n_1 and n_2, we have that $\forall i \in [n_1] : v_{i,2} \leq 1$ and $v_{n_1+1,2} \geq 1$.

The other thing crucial to our further constructions is that the total number of goods is at least the number of agents.

Observation 1. *If $m_1 + m_2 \leq n$, give one good of the first type to each agent i with $i \leq m_1$. Give one good of the second type to each agent i with $i \geq n - m_2 + 1$. This allocation is EFX+PO.*

Proof. The observed allocation X is simply EFX, since each agent receives at most one good.

To see that the allocation is Pareto-optimal, assume the contrary, and there is an allocation Y that Pareto-dominates X. First, if an agent received one item in X (so the agent's utility in X is positive), the agent should receive at least one item in Y. Hence, each agent gets at most one item in Y. The total utility over all agents in Y equals $|S_1| + \sum_{i \in S_2} v_{i,2} = m_1 + \sum_{i \in S_2} v_{i,2}$, where S_1 and S_2 are the sets of agent numbers that get a good of type 1 and type 2 in Y respectively.

As soon as Y Pareto-dominates X, the total utility of all agents in X should be less than $m_1 + \sum_{i \in S_2} v_{i,2}$. This is impossible because $\sum_{i \in S_2} v_{i,2}$ is upper-bounded by $\sum_{i=n-m_2+1}^{n} v_{i,2}$ — the total utility provided by goods of the second type in X. □

We are ready to summarize our restrictions on the input, that all hold true after the input is preprocessed. We provide them below.

Preprocessed Input Restrictions.

- For each $i \in [n]$, $v_{i,1} = 1$ and $v_{i,2} > 0$;
- For each $i \in [n-1]$, $v_{i,2} \leq v_{i+1,2}$;
- $n_1 + n_2 = n$, with $v_{i,2} \leq 1$ for every $i \leq n_1$ and $v_{i,2} \geq 1$ for every $i > n_1$;
- $m_1 n_2 \geq m_2 n_1$ and $m_2 > 0$ and $n_2 > 0$;
- $m_1 + m_2 \geq n$.

Throughout the rest of the paper, we always assume that the input satisfies the above constraints. Our results might not hold true if these constraints are not satisfied.

2.2 Proper Allocations

In our work, the Pareto-optimality is achieved in all of the constructed allocations, both intermediate ones (that might not be EFX) and the resulting one (that is necessarily EFX). All of our allocations fall under the notion given below.

Definition 3 (Proper allocation)**.** An allocation of goods $\{(x_{i,1}, x_{i,2})\}$ is *proper*, if there exists $t \in [n]$ such that

- for each $i \in [t-1]$, $x_{i,2} = 0$, and
- for each $i \in [t+1, n]$, $x_{i,1} < v_{t,2}$.

[1]Agents with $v_{i,2} = v_{i,1}$ may be assigned to either group.

Intuitively, proper allocations ask that the leftmost agent with at least one good of the second type (agent t) cannot trade this good for enough ($\lceil v_{t,2} \rceil$) goods of the first type with an agent to the right of it. For example, the allocation in Observation 1 is proper (choose $t = m_1$). As announced above, in Section 4 we will prove that all proper allocations are PO, as formulated below.

Theorem 1. *If an allocation is proper, then it is Pareto-optimal.*

While it may seem easy from the definition that a proper allocation cannot be improved (in Pareto sense) via a single exchange between two agents, in our proof of Theorem 1 we carefully show that longer sequences of exchanges cannot lead to a Pareto-dominating solution as well.

It is interesting to ask whether Theorem 1 holds in the other direction, that is, *is every PO allocation proper?* We observe that this is true under additional reasonable assumptions on valuations, following the "single exchange" intuition mentioned above.

Observation 2. *If $\lceil v_{i,2} \rceil < v_{i+1,2}$ holds for each $i \in [n-1]$, then any Pareto-optimal allocation is proper.*

Proof. The proof is by contradiction. Assume that there is an allocation $X = \{(x_{i,1}, x_{i,2})\}$ that is not proper but is Pareto-optimal. Note that $x_{i,2} > 0$ holds for at lest one choice of $i \in [n]$, since $m_2 > 0$.

Now define t as the minimum possible integer such that $x_{t,2} > 0$. Since X is not proper, there exists $i > t$ such that $x_{i,1} \geq v_{t,2}$. Obtain a new allocation Y by modifying X with one exchange: take 1 good of type 2 from agent t and give it to agent i, and take $\lceil v_{t,2} \rceil$ goods of type 1 from agent i and give it to agent t. The utility of t changes by $-1 \cdot v_{t,2} + \lceil v_{t,2} \rceil \cdot 1 \geq 0$. The utility of i changes by $-\lceil v_{t,2} \rceil \cdot 1 + 1 \cdot v_{i,2} > 0$. The last inequality holds because $i > t$, so $\lceil v_{t,2} \rceil < v_{t+1,2} < \ldots < v_{i,2}$ by the observation statement.

That is, the exchange in the above paragraph does not decrease utility of agent t and increases the utility of agent i, while utilities of other agents remain the same. This contradicts the Pareto-optimality of X. □

Unfortunately, the condition on valuations in the observation above cannot be lifted, as in general there can exist Pareto-optimal allocations that are not proper. To see that, consider two agents with $v_{1,2} = 1.1$ and $v_{2,2} = 1.2$, let $m_1 = 2$ and $m_2 = 1$, that is, there are two goods of type 1 and only one good of type 2. Consider an allocation $X = ((0,1),(2,0))$, where agent 1 receives the second-type good and agent 2 receives all goods of the first type. Note that X is not proper, since $t = 1$ and $t = 2$ both fail. On the other hand, it is easy to verify that there is no allocation that Pareto-dominates X.

2.3 Prioritized Equitable Allocations

The following notion is basic to our allocation constructions. It formally defines the process of dividing the identical goods between specific agents in (almost) equal parts, where some specific agents should receive the greater part.

The notion below defines this process formally. Note that we use it for *partial* allocation of goods, while *complete* allocations are formed via performing several partial allocations consecutively.

Definition 4 (Prioritized equitable allocation, PEA)**.** For integers $c \in [2], q \in [m_c], s \in [n]$, and a sequence of distinct agents $a_1, a_2, \ldots, a_s \in [n]$, we say that we give q goods of type c *equitably prioritized* to the agents $a_1, a_2, \ldots, a_s$, if

- for each $i \in [s]$, agent a_i receives either $\lfloor q/s \rfloor$ or $\lceil q/s \rceil$ goods of type c;
- for each $1 \leq i < j \leq s$, agent a_j receives at least as many goods of type c as a_i.

Equivalently, if r is the remainder of division of q by s, then agents $a_1, a_2, \ldots, a_{s-r}$ receive $\lfloor q/s \rfloor$ goods each, and agents $a_{s-r+1}, \ldots, a_s$ receive $\lceil q/s \rceil$ goods each.

Table 2. Expressing (t,k)-split-allocation as a series of prioritized equitable allocations.

(t,k)-split-allocation as PEA sum		
type (c)	Quantity (q)	Agent sequence $(a_1,\ldots,a_s)$
2	k	t
2	m_2-k	$t+1,\ldots,n$
1	$\min\{m_1,\ \lceil kv_{t,2}\rceil\cdot(t-1)\}$	$1,\ldots,t-1$
1	$\max\{0,\ m_1-\lceil kv_{t,2}\rceil\cdot(t-1)\}$	$1,\ldots,t$

2.4 Split Allocations

We move on to the central allocation construction of our algorithm, the *split allocations*. Simply speaking, in a split allocation, only a specified agent t (the "split point") can receive goods of both types simultaneously. Agents $i<t$ or agents $i>t$ receive only goods of the first type or the second type respectively.

A split allocation is uniquely identified by two integers: $t\in[n]$ and $k\in[m_2]$ — the number of goods of the second type given to agent t. These integers should fit under specific constraints. We define all valid pairs of integers $\mathcal{T}\subset[n]\times[m_2]$:

$$\mathcal{T}=\left\{(t,k):v_{t,2}\geq 1, t<n, k\leq\frac{m_2}{n-t+1}\right\}\cup\{(n,m_2)\}.$$

Note that $\mathcal{T}$ is not empty since $n_2>0$ and $m_2>0$. We now give formal definition to split allocations.

Definition 5 (Split allocations)**.** For $(t,k)\in\mathcal{T}$, the (t,k)-*split-allocation* is defined by

(1) Give k goods of the second type to agent t.
(2) Give (m_2-k) remaining goods of the second type equitably prioritized to agents $t+1,t+2,\ldots,n$.
(3) Let $p=\lceil k\cdot v_{t,2}\rceil$. Give $\min\{p(t-1),m_1\}$ goods of the first type equitably prioritized to agents $1,2,\ldots,t-1$.
(4) Give remaining $\max\{0,m_1-p(t-1)\}$ goods of the first type equitably prioritized to agents $1,2,\ldots,t$.

The restriction to pairs $(t,k)\in\mathcal{T}$ is intentional: split allocations are defined only for these admissible pairs. Steps 2–4 distribute same-type goods equitably within the groups on either side of the split point. Consequently, no strong envy can arise within either $[t-1]$ or $[t+1,n]$. Agent t is the only agent who may receive both types of goods, and possible strong envy involving agent t is analyzed separately below.

We summarize that the (t,k)-split-allocation is expressed as a sequence of four prioritized equitable allocations in Table 2. Note that the agent t can receive 0 goods of the first type, as demonstrated in Table 3.

Table 3. Structure of (t,k)-split-allocations with $m_1\leq p(t-1)$. The number in i^{th} column and j^{th} row equals $x_{i,j}$, the number of goods of type j given to agent i.

	1	…	$t-1$	t	$t+1$	…	n
1	$\left\lfloor\frac{m_1}{t-1}\right\rfloor$	…	$\left\lceil\frac{m_1}{t-1}\right\rceil$	0	0	…	0
2	0	…	0	k	$\left\lfloor\frac{m_2-k}{n-t}\right\rfloor$	…	$\left\lceil\frac{m_2-k}{n-t}\right\rceil$

The following simple observation explains why split allocations are PO.

Observation 3. *For each $(t,k) \in \mathcal{T}$, the (t,k)-split-allocation is proper.*

2.5 Envy Direction

Our algorithm will encounter several split allocations. If at least one of the split allocations it encounters is EFX, the algorithm will be fine.

But if the (t,k)-split-allocation is not EFX for $(t,k) \in \mathcal{T}$, what can we say about the strong envy? We will show that our design of split allocations prevents strong envy in both directions simultaneously. We proceed to formal notions.

Definition 6 (Left-envious (LE) and right-envious (RE) allocations)**.** We say that an allocation X is *left-envious*, or *LE* (*right-envious*, or *RE*), if X is not EFX, and there exist agents i, j such that agent j strongly envies agent i for $i < j$ ($i > j$).

We will prove in Section 5 that no (t,k)-split-allocation can have envy in both directions.

Theorem 2. *For each $(t,k) \in \mathcal{T}$, the (t,k)-split-allocation cannot be LE and RE simultaneously.*

There are two natural "extremal" cases of split allocations. One case is the (n, m_2)-split-allocation, where all goods of the second type are given to agent n. This allocation maximizes the total utility provided by goods of the second type. The other case is when goods of the second type receive the leftmost positions possible for a split allocation. This case is the $(t, \lfloor m_2/(n-t+1) \rfloor)$-split-allocation, where $t \in [n]$ minimum possible such that $v_{t,2} \geq 1$ and $m_2 \geq n - t + 1$ (there can be no $(t', k') \in \mathcal{T}$ with $t' < t$). Following this logic, we define orderings of split allocations.

Definition 7 (Complete ordering of split allocations)**.** A complete linear ordering $\prec$ on the set $\mathcal{T}$ is defined by

$$(t_1, k_1) \prec (t_2, k_2)$$

if and only if $t_1 < t_2$ or $t_1 = t_2$ *and* $k_1 > k_2$.

The two split allocations described above are naturally maximal and minimal elements in $\mathcal{T}$ with respect to $\prec$. Our second result on envy directions uncovers that these two extremal allocations have specific envy directions (if they are not EFX).

Theorem 3. *Let (t_L, k_L) be the minimum element in $\mathcal{T}$ with respect to $\prec$. Let (t_R, k_R) be the maximum element in $\mathcal{T}$ with respect to $\prec$. Both of the following statements are true:*

(1) *The (t_L, k_L)-split-allocation is either EFX or LE,*
(2) *The (t_R, k_R)-split-allocation is either EFX or RE.*

The proof of the theorem above is also given in Section 5.

2.6 Reallocation

To get intuition behind the final concept, assume that none of the split allocations is EFX. Then Theorem 3 guarantees that the (t_L, k_L)-split-allocation and the (t_R, k_R)-split-allocation are left-envious and right-envious respectively. Since all split allocations are either LE or RE, there should be $(t,k) \in \mathcal{T}$ such that the (t,k)-split-allocation is LE, while for its immediate successor (w.r.t. $\prec$) $(t', k') \in \mathcal{T}$, the (t', k')-split-allocation is RE. That is, (t,k) and (t', k') form a point where the "envy direction changes". By the definition of $\prec$, either $t' = t$ and $k' = k - 1$, or $t' = t + 1$, $k = 1$, and $k' = \lfloor m_2/(n-t) \rfloor$. Thus, consecutive split allocations satisfy $t' - t \in \{0, 1\}$.

This can be, for example, $(t,k) \in \mathcal{T}$ for $t < n$ and $k \geq 2$, and $(t, k-1) \in \mathcal{T}$, such that the (t,k)-split-allocation is LE and the $(t, k-1)$-split-allocation is RE.

Table 4. Expressing (t,k)-reallocation as a series of prioritized equitable allocations.

(t,k)-reallocation as PEA sum		
Type (c)	Quantity (q)	Agent sequence $(a_1,\ldots,a_s)$
2	k	t
2	m_2-k	$t+1,\ldots,n$
1	$\lceil dv_{t,2}\rceil\cdot(t-1)$	$1,\ldots,t-1$
1	$\lceil dv_{t,2}\rceil-p$	t
1	$m_1-\lceil dv_{t,2}\rceil\cdot t+p$	$1,\ldots,\ell$ *or* $1,\ldots,t-1,t+1,\ldots,\ell,t$

Our final allocation construction slightly transforms such (t,k)-split-allocations by redistributing some goods of the first type from agents to the left of t to agents to the right of t. We now define this construction formally (see also Table 4).

Definition 8 ((t,k)-reallocation)**.** For $(t,k)\in\mathcal{T}$ such that $t<n$ and the (t,k)-split-allocation X satisfies

$$m_1\geq\lceil d\cdot v_{t,2}\rceil\cdot t-p,$$

where $p=\lceil kv_{t,2}\rceil$ and $d=x_{t+1,2}=\lfloor(m_2-k)/(n-t)\rfloor$, the (t,k)-*reallocation* is obtained by

(1) Start from X, but take away all items of the first type from each agent.
(2) Let $\ell\in[n]$ be maximum possible such that $x_{\ell,2}=d$.
(3) Give $\lceil d\cdot v_{t,2}\rceil$ items of the first type to each agent in $[t-1]$.
(4) Give $\lceil d\cdot v_{t,2}\rceil-p$ first-type items to agent t.
(5) Give the remaining $m_1-\lceil d\cdot v_{t,2}\rceil\cdot t+p$ items of the first type equitably prioritized to:
   - Agents $1,2,\ldots,\ell$, in case $\lceil dv_{t,2}\rceil-p>(d-k)\cdot v_{t,2}$;
   - Agents $1,2,\ldots,t-1,t+1,\ldots,\ell,t$, otherwise.

Note that the definition of (t,k)-reallocations forces additional constraints on (t,k). In Section 6 we will prove the following lemma, that guarantees that (t,k) satisfies these (and other) constraints, if it corresponds to the point of "envy direction change".

Lemma 1. *Let $(t,k),(t',k')\in\mathcal{T}$ be such that the (t,k)-split-allocation is left-envious, but (t',k')-split-allocation is right-envious. Moreover, there is no $(t'',k'')\in\mathcal{T}$ that satisfies $(t,k)\prec(t'',k'')\prec(t',k')$. Then $t<n$ and*

$$\lceil x_{t+1,2}\cdot v_{t,2}\rceil\cdot t-\lceil kv_{t,2}\rceil+1\leq m_1\leq\lceil x_{t+1,2}\cdot v_{t,2}\rceil\cdot t-\lceil(k-1)v_{t,2}\rceil.$$

The final of our results is the following theorem. If combined with Lemma 1, it grants an EFX+PO allocation.

Theorem 4. *Let X be the (t,k)-reallocation for some $(t,k)\in\mathcal{T}$. If $\lceil x_{t+1,2}\cdot v_{t,2}\rceil\cdot t-\lceil k\cdot v_{t,2}\rceil+1\leq m_1\leq\lceil x_{t+1,2}\cdot v_{t,2}\rceil\cdot t-\lceil(k-1)\cdot v_{t,2}\rceil$, then X is EFX+PO.*

The Pareto-optimality part follows by showing that a (t,k)-reallocation under the upper bound above is proper. This is established at the beginning of the proof of Theorem 4 in Section 6.

3 The Algorithm

We summarize the constructions and properties of the previous section into an algorithm that constructs the desired allocation. It proves, in particular, that EFX+PO allocations always exist for two types of goods, given that all agents' utilities are positive.

Algorithm 1: The algorithm constructing an EFX+PO allocation on preprocessed inputs.

$t_L \leftarrow$ smallest integer in $[n]$ with $(t_L, *) \in \mathcal{T}$;
$X_L \leftarrow$ the $(t_L, \lfloor \frac{m_2}{n-t_L+1} \rfloor)$-split-allocation;
if X_L *is EFX* **then return** X_L;
$t_R \leftarrow n$;
$X_R \leftarrow$ the (t_R, m_2)-split-allocation;
if X_R *is EFX* **then return** X_R;
`/* binary search to find` t `*/`
while $t_R - t_L > 1$ **do**
  $t_M \leftarrow \lfloor (t_L + t_R)/2 \rfloor$;
  $X_M \leftarrow$ the $(t_M, \lfloor \frac{m_2}{n-t_M+1} \rfloor)$-split-allocation;
  if X_M *is EFX* **then return** X_M;
  if X_M *is LE* **then** $t_L \leftarrow t_M$ **else** $t_R \leftarrow t_M$;
$t \leftarrow t_L$ and $k_L \leftarrow \lfloor \frac{m_2}{n-t+1} \rfloor$ and $k_R \leftarrow 1$;
if (t, k_R)*-split-allocation is RE* **then**
  `/* binary search to find` k `*/`
  while $k_L - k_R > 1$ **do**
    $k_M \leftarrow \lfloor (k_L + k_R)/2 \rfloor$;
    $X_M \leftarrow$ the (t, k_M)-split-allocation;
    if X_M *is EFX* **then return** X_M;
    if X_M *is LE* **then** $k_L \leftarrow k_M$ **else** $k_R \leftarrow k_M$;
  $k \leftarrow k_L$;
else
  $X_R \leftarrow$ the (t, k_R)-split-allocation;
  if X_R *is EFX* **then return** X_R;
  `/* LE changes to RE between` (t, k_R) `and` $(t+1, \lfloor m_2/(n-t) \rfloor)$ `*/`
  $k \leftarrow k_R$;
return *the* (t, k)*-reallocation.*

Note that in the theorem statement below, we assume that the input is given in its preprocessed form (see Section 2.1). The running time required to preprocess an input is at least the time required to sort the agents' utilities. Therefore, general-case inputs require additional $\Theta(n \log n)$ preprocessing time. This step is not included in Algorithm 1.

Theorem 5. *Given preprocessed input, an EFX+PO allocation can be found in* $O(\log n + \log m)$ *time.*

Proof. We present an algorithm that is based on envy directions of split allocations. Its pseudocode can be found in Algorithm 1, and we will refer to its lines throughout the proof.

We have to clarify first why the algorithm is able to work in logarithmic time, given that even just outputting an allocation ($2n$ integers) takes linear time. In the following claim, we explain why this is possible; moreover, we can even determine whether a split allocation is EFX in constant time.

Claim 1. *Given $(t,k) \in \mathcal{T}$, we can determine in $O(1)$ time whether the (t,k)-split-allocation is EFX, left-envious or right-envious.*

Proof of Claim 1. By definition of split allocations, there are $b \leq 5$ distinct bundles that agents receive in the (t,k)-split-allocation X. Additionally, there are b contiguous segments of agent numbers $[a_1, a_2 - 1], [a_2, a_3 - 1], \ldots, [a_b, n]$, where $a_1 = 1$ and $a_1 < a_2 < \ldots < a_b$. Agents within the same segment receive exactly the same bundle. Note that b and $a_1, a_2, \ldots, a_b$ are easily computable in $O(1)$ time from (t,k) following Definition 5, as well as the bundles themselves.

Moreover, if agent i strongly envies agent j, and agents $i-1$, i, $i+1$ receive exactly the same bundle, then either agent $i-1$ or agent $i+1$ necessarily strongly envies agent j.

To see the last paragraph, assume agent i strongly envies agent j. Equivalently, there is a choice of integers $z_1, z_2 \in \{0,1\}$ such that $z_1 + z_2 = 1$ and $z_c \leq x_{j,c}$, and

$$x_{i,1} + x_{i,2} \cdot v_{i,2} < (x_{j,1} - z_1) + (x_{j,2} - z_2) \cdot v_{i,2},$$

or, equivalently,

$$x_{i,1} - x_{j,1} + z_1 < (x_{j,2} - x_{i,2} - z_2) \cdot v_{i,2}. \tag{1}$$

Since $v_{i-1,2} \leq v_{i,2} \leq v_{i+1,2}$ and $x_{i-1,c} = x_{i,c} = x_{i+1,c}$ for each $c \in [2]$, we have that (1) holds if we replace i with either $i+1$ (if the right part is non-negative), or $i-1$ (if the right part is negative). Equivalently, either agent $i+1$ or agent $i-1$ strongly envies agent j.

Set $a_{b+1} = n+1$. It follows that there is no need to check whether an agent i with $a_s < i < a_{s+1} - 1$, for $1 \leq s \leq b$, strongly envies any other agent. The only agents we have to check are the agents with numbers in $\{a_s, a_{s+1} - 1 : 1 \leq s \leq b\}$. There are b distinct bundles in X, so for each agent we have to make $b-1$ comparisons (whether agent i strongly envies an agent with this specific bundle).

In total, one should make only a total of $2b(b-1)$ comparisons between agents and bundles. Each comparison is done in constant time. ⌟

We move on to the description of the algorithm itself. The algorithm first evaluates t_L from the statement of Theorem 3 (Line 1). Since

$$t_L = \max\{\min\{t \in [n] : v_{t,2} \geq 1\}, n - m_2 + 1\},$$

t_L is evaluated in $O(\log n)$ time via lower bound binary search over $(v_{1,2}, v_{2,2}, \ldots, v_{n,2})$.

The other integers from the statement of Theorem 3 are computable in $O(1)$ time, and the algorithm checks whether the (t_L, k_L)- or the (t_R, k_R)-split-allocation is EFX (Lines 2-6). If the algorithm does not return here, from Theorem 3 we know that the allocations are LE and RE respectively.

The algorithm then aims to find $(t,k) \in \mathcal{T}$ satisfying Lemma 1. This is done via two binary searches. In the first binary search, the algorithm finds t, and in the second one — the algorithm finds k (given that t is known). During these searches, the algorithm might encounter a split allocation that is EFX. If this happens, the algorithm returns such allocation as its final answer.

We move on to discussion of the first binary search. To perform this search, the algorithm treats t_L and t_R as binary search bounds and modify them correspondingly, until it reaches $t_R - t_L = 1$. It keeps the following invariant: the $(t_L, \lfloor m_2/(n - t_L + 1) \rfloor)$-split-allocation is LE, and the $(t_R, \lfloor m_2/(n - t_R + 1) \rfloor)$-split-allocation is RE. In a single iteration of the binary search, the algorithm takes t_M in the middle between t_L and t_R, checks whether the $(t_M, \lfloor m_2/(n - t_M + 1) \rfloor)$-split-allocation is EFX, LE, or RE, and returns the correct allocation, puts t_L equal to t_M, or puts t_R equal to t_M correspondingly (Lines 8-11).

After the binary search, the algorithm puts $t := t_L$. We know that the desired "envy direction change" point has the form (t, k) for some integer k between $\lfloor m_2/(n-t+1) \rfloor$ and 1 (because the successor of $(t, 1)$ w.r.t. to $\prec$ is RE). If the $(t, 1)$-split-allocation is right-envious, then the algorithm performs binary search over k starting with $k_L = \lfloor m_2/(n-t+1) \rfloor$ and $k_R = 1$ in exactly the same way as in the first binary search (Lines 14–18). The only difference from the first binary search is that $k_L > k_R$ (instead of $k_L < k_R$), as imposed by the definition of $\prec$.

If the $(t, 1)$-split-allocation is not right-envious, then it is either EFX (and forms the correct solution), or $k = 1$ is a viable choice for Lemma 1 (Lines 21–23).

In either of the two cases, the algorithm finds (t, k) viable for Lemma 1 (or encounters an EFX split allocation and returns before). The algorithm constructs the (t, k)-reallocation and returns it as a final solution (Line 24). This allocation is EFX+PO by Theorem 4.

Similarly to split allocations, the (t, k)-reallocation is found in $O(1)$ time, since there is at most a constant number of bundles given to contiguous segments of agents. □

4 Pareto Optimality of Proper Allocations

In this section we prove Theorem 1, stating that any proper allocation (Definition 3) is PO. We start with the following simple lemma that allows natural restrictions on a Pareto-dominating allocation.

Lemma 2. *Let $X = \{(x_{i,1}, x_{i,2})\}$ be an item allocation. If X is not PO, then there exists a Pareto-dominating item allocation $Y = \{(y_{i,1}, y_{i,2})\}$, such that for each $i \in [n], j \in [2]$ it holds $\Delta_{i,1} \cdot \Delta_{i,2} \le 0$, where $\Delta_{i,j} = y_{i,j} - x_{i,j}$.*

Proof. The proof is by contradiction. Let X be a given (non-Pareto-optimal) allocation and assume that no allocation Y satisfies the lemma statement. Then let Y be an allocation that Pareto-dominates X, with minimum possible

$$\sum_{i \in [n], j \in [2]} |\Delta_{i,j}|.$$

By our assumption, there exists $i^* \in [n]$ such that $\Delta_{i^*,1} \cdot \Delta_{i^*,2} > 0$. If both $\Delta_{i^*,1}, \Delta_{i^*,2} < 0$, then the utility of agent i^* in X is at least $\sum_{j \in [2]} |\Delta_{i^*,j}| \cdot v_{i^*,j} > 0$ greater than its utility in Y, and this is impossible.

Consequently, we have $\Delta_{i^*,1}, \Delta_{i^*,2} > 0$. Since $\sum_{i \in [n]} \Delta_{i,1} = 0$, there exists $k^* \in [n]$ such that $\Delta_{k^*,1} < 0$. Now construct another allocation Y' by transferring exactly one item of type 1 from agent i^* to agent k^* in Y.

Formally, we define $\Delta'_{i^*,1} = \Delta_{i^*,1} - 1 \ge 0$ and $\Delta'_{k^*,1} = \Delta_{k^*,1} + 1$. For each other $i \in [n], j \in [2]$, we put $\Delta'_{i,j} = \Delta_{i,j}$. Then $Y' = \{(y'_{i,1}, y'_{i,2})\}$ is given by $y'_{i,j} = x_{i,j} + \Delta'_{i,j}$ similarly to Y and Δ. Note that Y' is valid since $y'_{i,j} \ge 0$ for every $i \in [n], j \in [2]$, and $\sum_{i \in [n]} \Delta'_{i,j} = 0$ for each $j \in [2]$.

We claim that Y' Pareto-dominates X. To see this, note that the only agent that receives less utility in Y' than in Y, is agent i^*. Then we only have to verify that i^* still receives more utility in Y' than in X. Indeed, the difference between these utilities is at least

$$\sum_{j \in [2]} \Delta'_{i^*,j} \cdot v_{i^*,j} \ge \Delta'_{i^*,2} \cdot v_{i^*,2} = \Delta_{i^*,2} \cdot v_{i^*,2} > 0.$$

Therefore, Y' Pareto-dominates X, but $\sum |\Delta'_{i,j}| = \sum |\Delta_{i,j}| - 2$. This contradicts the initial choice of Y. □

4.1 Proof of Theorem 1

Using Lemma 2, we move on to proving that any *proper* allocation is PO.

Proof of Theorem 1. Let X be a proper allocation, where agent i receives $x_{i,1}, x_{i,2}$ goods of types 1 and 2 respectively, for each $i \in [n]$. Let $t \in [n]$ be the integer guaranteed by Definition 3: t is such that for each $i \in [t-1]$ $x_{i,2} = 0$, and, for each $i \in [t+1, n]$, it holds $x_{i,1} < v_{t,2}$.

Targeting towards a contradiction, assume that X is not PO, and there exists an allocation $Y = \{(y_{i,1}, y_{i,2})\}$ that Pareto-dominates X and satisfies Lemma 2. We use $\Delta_{i,j} = y_{i,j} - x_{i,j}$. We have $\Delta_{i,1} \cdot \Delta_{i,2} \leq 0$ for each $i \in [n]$.

For each $j \in [2]$, define $I_j = \{i : \Delta_{i,j} > 0\}$, that is, I_j is a set of agents that receive extra items of type j in Y (when compared to X). Before showing a contradiction, we prove two useful claims on properties of $\Delta_{i,j}$. Both claims together show that the ratio between the number of exchanged goods of two types is tied to $v_{t,2}$.

Claim 2. *It holds*

$$\sum_{i \in I_1} \Delta_{i,1} \geq v_{t,2} \cdot \sum_{i \in I_1} (-\Delta_{i,2}).$$

If the utility of some agent $i \in I_1$ is strictly greater in Y than in X, then the inequality is strict.

Proof of Claim 2. Consider an arbitrary agent $i \in I_1$. Its utility in Y is not less than its utility in X, hence $\Delta_{i,1} \cdot v_{i,1} + \Delta_{i,2} \cdot v_{i,2} \geq 0$, equivalently $\Delta_{i,1} + \Delta_{i,2} \cdot v_{i,2} \geq 0$. If its utility is strictly greater in Y, then this inequality is strict.

If $\Delta_{i,2} = 0$, then $\Delta_{i,1} \geq 0$, and $\Delta_{i,1} \geq -\Delta_{i,2} \cdot v_{t,2}$ holds. If $\Delta_{i,2} < 0$, then $x_{i,2} > 0$, and by definition of t we have $i \geq t$. Consequently, $v_{i,2} \geq v_{t,2}$. Then obtain

$$\Delta_{i,1} \geq -\Delta_{i,2} \cdot v_{i,2} \geq -\Delta_{i,2} \cdot v_{t,2}.$$

The first part of the claim follows by summing over all $i \in I_1$.

To see the second part of the claim, note that $\Delta_{i,1} > -\Delta_{i,2} \cdot v_{t,2}$ is strict for the choice of $i \in I_1$ with greater utility in Y. The resulting sum inequality also becomes strict. The proof is complete. ⌟

The second claim gives a symmetrical lower bound via different arguments.

Claim 3. *It holds*

$$\sum_{i \in I_2} \Delta_{i,2} \geq \frac{1}{v_{t,2}} \cdot \sum_{i \in I_2} (-\Delta_{i,1}).$$

If the utility of some agent $i \in I_2$ is strictly greater in Y than in X, then the inequality is strict.

Proof of Claim 3. Consider an agent $i \in I_2$. Similarly to the proof of the previous claim, it is enough to show that $\Delta_{i,2} \geq -\Delta_{i,1}/v_{t,2}$.

We consider two cases depending on whether $i \leq t$. If $i \leq t$, then $v_{i,2} \leq v_{t,2}$. Since the utility of agent i in Y is not less than in X, we have $\sum_{j \in [2]} \Delta_{i,j} \cdot v_{i,j} \geq 0$, hence

$$\Delta_{i,2} \cdot v_{t,2} \geq \Delta_{i,2} \cdot v_{i,2} \geq -\Delta_{i,1} \cdot v_{i,1} = -\Delta_{i,1}$$

as desired.

The remaining case is $i > t$. By definition of t, $x_{i,1} < v_{t,2}$. Since $\Delta_{i,2} > 0$, it holds $\Delta_{i,1} \leq 0$, and $0 \leq y_{i,1} \leq x_{i,1} < v_{t,2}$. Then $-\Delta_{i,1} = x_{i,1} - y_{i,1} < v_{t,2}$. Finally, we know that $\Delta_{i,2} \geq 1$, and $\Delta_{i,2} \geq -\Delta_{i,1}/v_{t,2}$ follows. This finishes the proof of the first part of the claim.

The second part is the same as in the proof of Claim 2. If for at least one $i \in I_2$ we have strictly greater utility, this choice of i gives strict $\Delta_{i,2} > -\Delta_{i,1}/v_{t,2}$, and the resulting sum is strict. ⌟

To combine Claim 2 and Claim 3, note that $\sum_{i \in [n]} \Delta_{i,j} = 0$ for each $j \in [2]$ by definition of $\Delta_{i,j}$. By definition of I_j and since Y Pareto-dominates X, for each $i \in [n] \setminus (I_1 \cup I_2)$ we have $\Delta_{i,1} = \Delta_{i,2} = 0$. Consequently, for each $j \in [2]$ we have

$$\sum_{i \in [n]} \Delta_{i,j} = \sum_{i \in I_1} \Delta_{i,j} + \sum_{i \in I_2} \Delta_{i,j},$$

and

$$-\sum_{i \in I_j} \Delta_{i,j} = \sum_{i \in I_{3-j}} \Delta_{i,j}.$$

Substituting this in the right parts of Claim 2 (for $j = 2$) and Claim 3 (for $j = 1$), we obtain

$$\sum_{i \in I_1} \Delta_{i,1} = v_{t,2} \cdot \sum_{i \in I_2} \Delta_{i,2},$$

and neither Claim 2 nor Claim 3 can give strict inequality. Thus, no agent $i \in I_1 \cup I_2$ can have greater utility in Y than in X.

But Y Pareto-dominates X, so there is some agent $i^* \in [n] \setminus (I_1 \cup I_2)$ whose utility is strictly greater. But this agent has both $\Delta_{i^*,1} \leq 0$ and $\Delta_{i^*,2} \leq 0$; that is, the agent did not receive any new item in Y compared with X, so the agent's utility in Y cannot exceed the utility in X. The obtained contradiction finishes the proof of the theorem. □

5 Envy Directions in Split Allocations

This section is devoted to properties of strong envy and envy directions in split allocations. Our main goal is to prove Theorem 2 and Theorem 3, that are fundamental to our algorithm.

5.1 Auxiliary Properties

Before proceeding, we formulate split allocations numerically for referencing throughout the proof.

Definition 9 (Numerical definition of split allocations)**.** For $(t, k) \in \mathcal{T}$, the (t, k)-split-allocation $\{(x_{i,1}, x_{i,2})\}$ is given by

$$x_{i,2} = \begin{cases} 0, \text{ if } i < t, \\ k, \text{ if } i = t, \\ q_2, \text{ if } t < i \leq n - r_2, \\ q_2 + 1, \text{ if } i > n - r_2, \end{cases}$$

where $m_2 - k = q_2(n - t) + r_2$ for $0 \leq r_2 < n - t$, and, for $p = \lceil k \cdot v_{t,2} \rceil$,

(a) in case of $m_1 < p(t - 1)$,

$$x_{i,1} = \begin{cases} 0, \text{ if } i \geq t, \\ q_1, \text{ if } i \leq t - 1 - r_1, \\ q_1 + 1, \text{ if } i \in [t - r_1, t - 1], \end{cases}$$

where $m_1 = q_1 \cdot (t - 1) + r_1$ for $0 \leq r_1 < (t - 1)$,

(b) in case of $m_1 \geq p(t - 1)$,

$$x_{i,1} = \begin{cases} 0, \text{ if } i > t, \\ q_1, \text{ if } i = t \text{ and } r_1 = 0, \\ q_1 + 1, \text{ if } i = t \text{ and } r_1 > 0, \\ p + q_1, \text{ if } i \leq \min\{t - r_1, t - 1\}, \\ p + q_1 + 1, \text{ if } i \in [t - r_1 + 1, t - 1], \end{cases}$$

where $m_1 - p(t - 1) = q_1 \cdot t + r_1$ for $0 \leq r_1 < t$.

We start with a series of lemmas. The first simple lemma demonstrates that there can be no strong envy between agents that go before t or between two agents that go after t in a (t, k)-split-allocation.

Lemma 3. *Let X be a (t, k)-split-allocation for $(t, k) \in \mathcal{T}$. If there is strong envy between agents $i < j$ in X for $i, j \in [n]$, then $i \leq t$ and $j \geq t$.*

Proof. The proof is by contradiction. Suppose that X is not EFX and there is strong envy between agent i and agent j with $i < j$ in X, but $i > t$ or $j < t$ holds.

Consider the case when $j < t$. By definition of split allocations, $|x_{i,1} - x_{j,1}| \leq 1$ for each $i, j \in [t-1]$, while $x_{i,2} = x_{j,2} = 0$. Hence, there cannot be strong envy between agent i and agent j since their bundles differ by at most one item.

Then it should be the case when $i > t$. Similarly, in the (t,k)-allocation X for any $i, j \in [t+1, n]$ we have $|x_{i,2} - x_{j,2}| \leq 1$ and $x_{i,1} = x_{j,1} = 0$. There cannot be strong envy between agent i and agent j if $i, j > t$. The proof is complete. □

The next lemma restricts any left-envious (LE) split allocation to the case (b) from Definition 9 of split allocations.

Lemma 4. *Let X be the (t,k)-split-allocation for $(t,k) \in \mathcal{T}$. If X is LE, then $m_1 \geq p \cdot (t-1)$, where $p = \lceil k \cdot v_{t,2} \rceil$.*

Proof. Targeting towards a contradiction, assume that $m_1 < p(t-1)$, and X is left-envious. That is, there is strong envy between agent i and agent j for $i < j$ in X, and agent j strongly envies agent i. By Lemma 3, $i \leq t$ and $j \geq t$.

There are three possible cases. The first case is when $i < t$ and $j = t$. By definition of split allocations, we have $x_{i,1} \leq \lfloor m_1/(t-1) \rfloor + 1$ and $x_{i,2} = 0$, hence $x_{i,1} \leq p$. Agent t received exactly k items of the second type, so $x_{t,2} = k$. Then $p = \lceil x_{t,2} \cdot v_{t,2} \rceil$, so utility of t in X is at least $x_{t,2} v_{t,2} > p - 1$. If we remove one arbitrary item (it's necessarily a first-type item) from the bundle of i, the utility of the resulting bundle (in terms of agent t) is exactly $(x_{i,1} - 1) \leq p - 1$. Hence, agent t cannot strongly envy agent $i < t$.

The next case is when $i < t$ and $j > t$. By definition of the (t,k)-split-allocation, agent j receives at least $\lfloor \frac{m_2 - k}{n-t} \rfloor$ items of the second type, so

$$x_{j,2} \geq \left\lfloor \frac{m_2 - k}{n-t} \right\rfloor \geq \left\lfloor \frac{m_2 - m_2/(n-t+1)}{n-t} \right\rfloor = \left\lfloor \frac{m_2}{n-t+1} \right\rfloor \geq k = x_{t,2},$$

where the upper-bound on k follows from $(t,k) \in \mathcal{T}$. That is, agent j receives at least as many items of the second type as agent t. Then j cannot strongly envy i, similarly to the previous case.

The remaining case is when $i = t$ and $j > t$. By definition of X, $x_{t,1} = 0$. On the other hand, $x_{j,2} \geq x_{t,2}$ as described in the previous case. Hence, agent j has as many items of each type as agent t has. Agent j cannot strongly envy agent t.

The list of the cases is exhausted. The obtained contradiction concludes the proof. □

The final lemma in the series explains that $m_1 \geq p(t-1)$ also rules out strong envy between any agent $i < t$ and agent t.

Lemma 5. *Let X be the (t,k)-split-allocation for $(t,k) \in \mathcal{T}$, and let $p = \lceil k \cdot v_{t,2} \rceil$. If $m_1 \geq p \cdot (t-1)$, then for each $i \in [t-1]$ there is no strong envy between agent i and agent t in X.*

Proof. We consider two cases of strong envy. In each of the cases, we come to a contradiction.

Case 1. Agent i strongly envies agent t.

The first case is when agent i strongly envies agent t. If $x_{t,1} = 0$, then the strong envy from i to t is equivalent to

$$x_{i,1} + (x_{i,2} + 1) \cdot v_{i,2} < x_{t,2} \cdot v_{i,2}. \tag{2}$$

From the definition of split allocations, we know that $x_{i,1} \geq p$. From definition of p, we know that $x_{t,2} \cdot v_{i,2} \leq x_{t,2} \cdot v_{t,2} \leq p$. Then from (2) we have $p < p$, which is a contradiction. Then it is necessary that $x_{t,1} > 0$.

Then the strong envy from i to t is equivalent to

$$x_{i,1} + x_{i,2} \cdot v_{i,2} + \min\{1, v_{i,2}\} < x_{t,1} + x_{t,2} \cdot v_{i,2}. \tag{3}$$

From definition of the (t,k)-split-allocation (Definition 9 case (b)), we have that $x_{i,1} \geq p + q_1$ and $x_{i,2} = 0$, while $x_{t,1} \leq q_1 + 1$. Using these with (3), obtain

$$(p + q_1) + 0 \cdot v_{i,2} + \min\{1, v_{i,2}\} < (q_1 + 1) + x_{t,2} \cdot v_{i,2},$$

and rewrite as

$$p + \min\{1, v_{i,2}\} < x_{t,2} \cdot v_{i,2} + 1. \tag{4}$$

Use definition of p and obtain

$$p + \min\{1, v_{i,2}\} < x_{t,2} \cdot v_{i,2} + 1 \leq x_{t,2} \cdot v_{t,2} + 1 \leq p + 1.$$

Consequently, $v_{i,2} \geq 1$ is not possible so we have $v_{i,2} < 1$ and by (4) we have

$$p + v_{i,2} < x_{t,2} \cdot v_{i,2} + 1,$$

so

$$p < v_{i,2} \cdot (x_{t,2} - 1) + 1 \leq (x_{t,2} - 1) + 1 = x_{t,2}.$$

From definition of p we obtain $x_{t,2} \cdot v_{t,2} < x_{t,2}$, and this is a contradiction with $v_{t,2} \geq 1$ that comes from $(t,k) \in \mathcal{T}$.

The case when agent i strongly envies agent t is not possible. We move on to the remaining case.

Case 2. Agent t strongly envies agent i.

In this case, the strong envy from agent t to agent i in X is equivalent to

$$(x_{t,1} + 1) + x_{t,2} \cdot v_{t,2} < x_{i,1}, \tag{5}$$

since agent i has no items of the second type. From definition of the (t,k)-split-allocation X, we have that $x_{i,1} - x_{t,1} \leq p$ (we can get $p - 1$ or p depending on i and r_1). Using this with (5), obtain

$$x_{t,2} \cdot v_{t,2} + 1 < p.$$

This contradicts the definition of p, as $p = \lceil x_{t,2} \cdot v_{t,2} \rceil \leq x_{t,2} \cdot v_{t,2} + 1$. The proof of the case and the lemma is complete. □

5.2 Proof of Theorem 2

We are ready to prove our main result on envy directions in split allocations, Theorem 2. For convenience of the reader, we restate it here first.

Theorem 2. *For each $(t,k) \in \mathcal{T}$, the (t,k)-split-allocation cannot be LE and RE simultaneously.*

Proof. Targeting towards a contradiction, assume that there exists $(t,k) \in \mathcal{T}$ such that the (t,k)-split-allocation $X = \{(x_{i,1}, x_{i,2})\}$ is LE and RE simultaneously.

By Lemma 4, we know that $m_1 \geq p \cdot (t-1)$ holds for $p = \lceil x_{t,2} \cdot v_{t,2} \rceil$. In particular, X comes from case (b) of Definition 9. Then, by Lemma 5 we have that there is no strong envy between any agent $i \in [t-1]$ and agent t in X. Consequently, if there is strong envy between agent i and agent j in X, where $i < j$, then $i \leq t$ and $j > t$ necessarily holds.

Under the initial assumption that X is LE and RE simultaneously, we prove two mutually contradictory claims. Recall that q_1 and r_1 from case (b) of Definition 9 are non-negative integers satisfying $m_1 - p(t-1) = q_1 t + r_1$, where $0 \leq r_1 < t$.Similarly, q_2 and r_2 satisfy $m_2 - k = q_2(n-t) + r_2$, where $q_2 \geq k \geq 1$ and $0 \leq r_2 < n - t$.

The first claim comes from X being right-envious.

Claim 4. *Both of the following statements are true:*

- *If* $r_1 > 0$, *then* $p + q_1 < q_2 \cdot v_{t,2}$.
- *If* $r_1 = 0$, *then* $v_{t,2} \cdot x_{t,2} + q_1 < q_2 \cdot v_{t,2}$.

Proof of Claim 4. Since X is right-envious, there is an agent $i \in [t]$ that strongly envies some agent $j \in [t+1, n]$. There are two cases depending on whether $i < t$ or $i = t$.

Then agent i has no items of the second type, and agent j has no items of the first type. The strong envy from agent i to agent j is equivalent to $x_{i,1} + 1 \cdot v_{i,2} < x_{j,2} \cdot v_{i,2}$. From Definition 9 we have that $x_{i,1} \geq p + q_1$ and $x_{j,2} \leq q_2 + 1$. Consequently, $p + q_1 + v_{i,2} < (q_2 + 1) \cdot v_{i,2}$. It follows $p + q_1 < q_2 \cdot v_{i,2} \leq q_2 \cdot v_{t,2}$, since $v_{i,2} \leq v_{t,2}$.

We have that if $i < t$, then $p + q_1 < q_2 \cdot v_{t,2}$. Since $p \geq k \cdot v_{t,2} = x_{t,2} \cdot v_{t,2}$, it also follows $v_{t,2} \cdot x_{t,2} + q_1 < q_2 \cdot v_{t,2}$. Therefore, both parts of the claim follow if $i < t$.

We now consider the case when agent t strongly envies agent $j > t$. The strong envy is equivalent to

$$x_{t,1} + (x_{t,2} + 1) \cdot v_{t,2} < x_{j,2} \cdot v_{t,2}.$$

Recall that $x_{j,2} \leq q_2 + 1$ and rewrite the above as

$$x_{t,1} + x_{t,2} \cdot v_{t,2} < q_2 \cdot v_{t,2}. \tag{6}$$

If $r_1 = 0$, then $x_{t,1} = q_1$, and (6) becomes

$$x_{t,2} \cdot v_{t,2} + q_1 < q_2 \cdot v_{t,2},$$

as required by the second part of the claim statement.

If $r_1 > 0$, then $x_{t,1} = q_1 + 1$. Then, by $p < x_{t,2} \cdot v_{t,2} + 1$ and (6) obtain

$$p + q_1 < x_{t,2} \cdot v_{t,2} + 1 + q_1 \leq x_{t,1} + x_{t,2} \cdot v_{t,2} < q_2 \cdot v_{t,2}.$$

This proves the first part of the claim statement. This concludes the proof of the claim. ⌟

The second claim comes from X being left-envious.

Claim 5. *Both of the following statements are true:*

- *If* $r_1 > 0$, *then* $p + q_1 > q_2 \cdot v_{t,2}$.
- *If* $r_1 = 0$, *then* $v_{t,2} \cdot x_{t,2} + q_1 > q_2 \cdot v_{t,2}$.

Proof of Claim 5. Since X is left-envious, there is an agent $j > t$ that strongly envies some agent $i \in [t]$. There are two cases depending on whether $i < t$ or $i = t$.

Case 1. Agent $i < t$**.** The strong envy from agent j to agent i is then equivalent to $1 + x_{j,2} \cdot v_{j,2} < x_{i,1}$. We have that $x_{j,2} \geq q_2$ from definition of X, and $v_{j,2} \geq v_{t,2}$ since $j > t$. Combining the inequalities, obtain

$$q_2 \cdot v_{t,2} < x_{i,1} - 1. \tag{7}$$

If $r_1 > 0$, then $x_{i,1} \leq p + q_1 + 1$. Then (7) gives

$$q_2 \cdot v_{t,2} < p + q_1 + 1 - 1,$$

which is essentially the first part of the claim. If $r_1 = 0$, then $x_{i,1} = p + q_1$. Combining this, (7) and $v_{t,2} \cdot x_{t,2} > p - 1$ gives

$$q_2 \cdot v_{t,2} < (p - 1) + q_1 < v_{t,2} \cdot x_{t,2} + q_1.$$

The second part is also proved for case $i < t$.

Case 2. Agent $i = t$**.** Strong envy from agent j to agent t implies

$$1 + x_{j,2} v_{j,2} < x_{t,1} + k v_{j,2}.$$

Since $x_{j,2} \geq q_2 \geq k$ and $v_{j,2} \geq v_{t,2}$,

$$(q_2 - k) v_{t,2} \leq (x_{j,2} - k) v_{t,2} \leq (x_{j,2} - k) v_{j,2} < x_{t,1} - 1.$$

Hence

$$q_2 v_{t,2} < x_{t,1} - 1 + k v_{t,2}. \tag{8}$$

If $r_1 > 0$, $x_{t,1} = q_1 + 1$. Combining this, (8) and $p \geq x_{t,2} \cdot v_{t,2}$ gives the first part of the claim. If $r_1 = 0$, then $x_{t,1} = q_1$. Combining (8) with $x_{t,1} = q_1$ gives

$$q_2 \cdot v_{t,2} < q_1 - 1 + x_{t,2} \cdot v_{t,2} < q_1 + x_{t,2} \cdot v_{t,2}.$$

The proof of the case $i = t$ and the whole claim is complete. ⌟

Clearly, Claim 4 and Claim 5 are mutually contradictory statements that are both true under the initial assumption. The obtained contradiction proves the theorem. □

5.3 Proof of Theorem 3

While Theorem 2 is fundamental to our split allocation approach, Theorem 3 is pivotal to the binary search over $\mathcal{T}$. We now proceed to prove Theorem 3.

Theorem 3. *Let (t_L, k_L) be the minimum element in $\mathcal{T}$ with respect to $\prec$. Let (t_R, k_R) be the maximum element in $\mathcal{T}$ with respect to $\prec$. Both of the following statements are true:*

(1) *The (t_L, k_L)-split-allocation is either EFX or LE,*
(2) *The (t_R, k_R)-split-allocation is either EFX or RE.*

Proof. By Theorem 2 it is enough to prove that the (t_L, k_L)-split-allocation is not right-envious (first part), and that the (t_R, k_R)-split-allocation is not left-envious (second part).

We start our proof with the second part of the theorem. Note that the maximal element in $\mathcal{T}$ with respect to $\prec$ is (n, m_2) (by definition of $\mathcal{T}$). Denote by X the (n, m_2)-split-allocation. Targeting towards a contradiction, assume that X is left-envious. Then there are $i, j \in [n]$ with $i < j$ such that agent j strongly envies agent i in X. On the one hand, by Lemma 3, we have that $i \leq t$ and $j \geq t$. In our case $t = n$, consequently $j = n$ and $i < n$. On the other hand, by Lemma 4 and Lemma 5 combined, there can be no strong envy between agent $i < n$ and $j = n$. This contradiction proves the second part of the theorem.

We proceed to prove the first part of the theorem. For convenience, we denote the smallest element (t_L, k_L) of $\mathcal{T}$ by just (t, k). By definition of $\mathcal{T}$, $t = t_L$ is the smallest integer in $[n]$ such that

- $v_{t,2} \geq 1$, and
- $m_2 \geq n - t + 1$.

Then, t is such that $m_2 = n - t + 1$ or $t - 1 \leq n_1$ (since $n_1 < n$ and $v_{n_1+1,2} \geq 1$ by definition of n_1).

Then $k = k_L = \lfloor m_2/(n - t + 1) \rfloor$. We denote the (t, k)-split-allocation by X. To prove the first part of the theorem, we assume that X is right-envious, aiming to obtain a contradiction.

By our assumption, there exist $i, j \in [n]$ such that $i < j$ and agent i strongly envies agent j in X. By Lemma 3, we have $i \leq t$ and $j \geq t$.

If $m_2 = n - t + 1$, then $k = 1$ and every agent $j > t$ receives exactly one second-type good and no first-type goods, so no agent can strongly envy such an agent. If $x_{t,1} = 0$, the same holds for agent t. Otherwise, case (b) of Definition 9 applies, and Lemma 5 rules out strong envy towards t from any agent $i < t$. Together with Lemma 3, this contradicts the assumption that X is right envious. Therefore $m_2 > n - t + 1$, so $t - 1 \leq n_1$ and $v_{i,2} \leq 1$ for every $i < t$.

We consider several cases depending on whether $i = t$ or $j = t$, and in each case we come to a contradiction.

Case 1. Agent $i < t$ strongly envies agent $j = t$. In this case, by Lemma 5, we have that $m_1 < p(t - 1)$. Then by definition of split allocations, $x_{t,1} = 0$. Then the strong envy from agent i to agent t is equivalent to

$$x_{i,1} < (x_{t,2} - 1) \cdot v_{i,2}.$$

Consequently, by $v_{i,2} \leq 1$, we have $x_{i,1} < x_{t,2} - 1 = k - 1$. From definition of split allocations we know that $x_{s,1} \leq x_{i,1} + 1$ for each $s \in [t-1]$, and $m_1 = \sum_{s \in [t-1]} x_{s,1}$. Then

$$(k-1)(t-1) \geq m_1,$$

so $k \geq 2$. This means that $m_2 > n - t + 1$, hence $t - 1 \leq n_1$ holds. Finally obtain

$$\frac{m_2}{n_2} = \frac{m_2}{n - n_1} \geq \frac{m_2}{n - t + 1} \geq k > \frac{m_1}{t-1} \geq \frac{m_1}{n_1}.$$

This contradicts the initial assumption $m_1/n_1 \geq m_2/n_2$. The case $j = t$ is ruled out.

In the remaining two cases, we have $j > t$. The following claim is important for both cases.

Claim 6. *For each $j > t$, $x_{j,2} - 1 \leq k$.*

Proof of Claim 6. By definition of split allocations, we know that $x_{j,2} \leq \lceil (m_2 - k)/(n - t) \rceil$.

Targeting towards a contradiction, suppose

$$k < \left\lceil \frac{m_2 - k}{n - t} \right\rceil - 1. \tag{9}$$

If $m_2 - k$ is divisible by $n - t$, then

$$k < \frac{m_2 - k}{n - t} - 1,$$

and

$$(n-t) \cdot k < (m_2 - k) - (n - t),$$

so

$$(n - t + 1) \cdot k + (n - t) < m_2.$$

This contradicts the definition of k.

Then $m_2 - k$ has to be not divisible by $n - t$. Let $r > 0$ be the remainder of division of $m_2 - k$ by $n - t$. Then (9) becomes

$$k < \left\lfloor \frac{m_2 - k}{n - t} \right\rfloor = \frac{m_2 - k - r}{n - t}.$$

Multiplying both sides by $(n - t)$ yields

$$(n-t) \cdot k < m_2 - k - r,$$

so

$$(n - t + 1) \cdot k < m_2 - r.$$

Let r' be the remainder of division of m_2 by $(n - t + 1)$, then $(n - t + 1) \cdot k + r' = m_2$. Consequently,

$$m_2 - r' < m_2 - r,$$

and $r' > r$. By definition of r' and r we have

$$((n - t + 1) \cdot k + r') - k = (n - t) \cdot q_2 + r,$$

so

$$r' - r = (n - t) \cdot (q_2 - k).$$

All parts are integers, so $r' - r$ is divisible by $n - t$. Consequently, $r' - r \geq n - t$. Since $r > 0$, it has to be $r' \geq n - t + 1$. This contradicts the definition of r'. The proof of the claim is complete. ⌟

We move on to the remaining two cases of strong envy between i and j.

Case 2. Agent $i < t$ strongly envies agent $j > t$. In this case, we have

$$x_{i,1} < (x_{j,2} - 1) \cdot v_{i,2}.$$

Then $x_{i,1} < x_{j,2} - 1 \le k$, since $v_{i,2} \le 1$ by $i < t \le n_1 + 1$. Thus, X must satisfy case (a) of Definition 9: case (b) would imply $x_{i,1} \ge p = \lceil v_{t,2} \cdot k \rceil \ge k$, contradicting $x_{i,1} < k$. By definition, all items of the first type are distributed between agents in $[t-1]$, so

$$\sum_{s=1}^{t-1} x_{s,1} = m_1.$$

For each $s \in [t-1]$, $x_{s,1} \le x_{i,1} + 1 \le k$, while $x_{i,1} < k$. Consequently, $\sum_{s\in[t-1]} x_{s,1} < k(t-1)$ and $k(t-1) > m_1$. Recall that it's either $t - 1 \le n_1$ or $m_2 = n - t + 1$.

Subcase 2.1 . $t - 1 \le n_1$**.** Then, using $m_1/n_1 \ge m_2/n_2$ and $n_1 + n_2 = n$, obtain

$$k > \frac{m_1}{t-1} \ge \frac{m_1}{n_1} \ge \frac{m_2}{n_2} = \frac{m_2}{n - n_1} \ge \frac{m_2}{n - t + 1}.$$

This contradicts the definition of k.

Subcase 2.2. $t - 1 > n_1$**.** Then $m_2 = n - t + 1$. So $k = 1$ and $m_1 < k(t-1) = t - 1$. Consequently,

$$m_1 + m_2 < (t-1) + (n - t + 1) = n.$$

This contradicts the initial setup.

The analysis of the case $t \notin \{i, j\}$ is finished.

Case 3. Agent $i = t$ strongly envies agent $j > t$. The strong envy is then expressed as

$$x_{t,1} + x_{t,2} \cdot v_{t,2} < (x_{j,2} - 1) \cdot v_{t,2},$$

since agent j receives no item of the first type in X. Consequently, $x_{t,2} < x_{j,2} - 1 \le k$. This contradicts the definition of X, since $x_{t,2} = k$. The case $i = t$ is also ruled out.

The possible cases are exhausted, in each of the cases a contradiction is obtained. The proof of the theorem is complete. □

6 Reallocation

In this section we study reallocations at a transition between consecutive split allocations, from a left-envious allocation to a right-envious one. The bounds at this transition are what make the resulting reallocation EFX and Pareto optimal.

6.1 Proof of Lemma 1

We first prove Lemma 1. Its lower bound ensures that at least one first-type good is distributed in the final equitable step; its upper bound limits that distribution and yields Pareto optimality.

Lemma 1. *Let $(t,k), (t',k') \in \mathcal{T}$ be such that the (t,k)-split-allocation is left-envious, but (t',k')-split-allocation is right-envious. Moreover, there is no $(t'',k'') \in \mathcal{T}$ that satisfies $(t,k) \prec (t'',k'') \prec (t',k')$. Then $t < n$ and*

$$\lceil x_{t+1,2} \cdot v_{t,2} \rceil \cdot t - \lceil k v_{t,2} \rceil + 1 \le m_1 \le \lceil x_{t+1,2} \cdot v_{t,2} \rceil \cdot t - \lceil (k-1) v_{t,2} \rceil.$$

Proof. Let us denote by Y the (t',k')-split-allocation. We will use $y_{i,1}$ and $y_{i,2}$ to denote the number of items of each type allocated to agent i in Y.

First, $t < n$. Indeed, if $t = n$, then $(t,k) = (n, m_2)$. Since X is left-envious, Lemma 4 gives $m_1 \ge \lceil m_2 v_{n,2} \rceil (n-1)$, while Lemmas 3 and 5 exclude left-directed strong envy in this case, a contradiction.

We begin the proof with the right inequality. Consider two cases: $t = t'$ and $t \ne t'$.

Case 1. $t = t'$**.** By definition of (t,k)-split-allocations, we then have $k' = k-1$. Assume the contrary:

$$m_1 \geq \lceil x_{t+1,2}\, v_{t,2} \rceil\, t - \lceil (k-1)\, v_{t,2} \rceil + 1.$$

By Lemma 3, strong envy can only occur between agents $i \leq t$ and $j \geq t$. Also, by the definition of a (t,k)-split-allocation, $x_{t+1,2} \geq x_{t,2} = k$, so

$$\lceil x_{t+1,2} v_{t,2} \rceil \geq \lceil k v_{t,2} \rceil \geq \lceil k' v_{t,2} \rceil.$$

Therefore,

$$m_1 \geq \lceil k'\, v_{t,2} \rceil\, (t-1) + 1 = p\,(t-1) + 1,$$

where $p = \lceil k'\, v_{t,2} \rceil$.

From Lemma 5, it follows that there is no strong envy between agents $i \leq t$ and $j \leq t$. However, by assumption, the (t,k')-allocation is right-envious, i.e., there exists an agent $i \leq t$ who strongly envies an agent $j > t$. Denote:

$$m_1 - p\,(t-1) = q_1\, t + r_1, \qquad 0 \leq r_1 < t.$$

Then the number of items of the first type for agents is:

$$y_{i,1} = \begin{cases} q_1, & \text{if } i = t \text{ and } r_1 = 0, \\ q_1 + 1, & \text{if } i = t \text{ and } r_1 > 0, \\ p + q_1, & \text{if } i \leq \min\{t - r_1, t-1\}, \\ p + q_1 + 1, & \text{if } i \in [t - r_1 + 1, t-1]\,. \end{cases}$$

Thus, for the agent t we get:

$$y_{t,1} = \left\lfloor \frac{m_1 - p\,(t-1)}{t} \right\rfloor = \left\lfloor \frac{m_1 - \lceil k'\, v_{t,2} \rceil\, (t-1)}{t} \right\rfloor.$$

Using the contrary assumption and $k' = k-1$, we obtain the following lower bound:

$$\left\lfloor \frac{\lceil x_{t+1,2}\, v_{t,2} \rceil\, t - \lceil (k-1)\, v_{t,2} \rceil + 1 - \lceil (k-1)\, v_{t,2} \rceil\, (t-1)}{t} \right\rfloor$$
$$= \left\lfloor \frac{\lceil x_{t+1,2}\, v_{t,2} \rceil\, t - \lceil (k-1)\, v_{t,2} \rceil\, t + 1}{t} \right\rfloor \geq \lceil x_{t+1,2}\, v_{t,2} \rceil - \lceil (k-1)\, v_{t,2} \rceil + 1.$$

For any agent $i < t$ we have

$$y_{i,1} \geq p + \left\lfloor \frac{m_1 - p\,(t-1)}{t} \right\rfloor \geq p + (y_{t,1} - 1) \geq p + (\lceil x_{t+1,2}\, v_{t,2} \rceil - \lceil (k-1)\, v_{t,2} \rceil) = \lceil x_{t+1,2}\, v_{t,2} \rceil\,.$$

For agents $j > t$, we have $y_{j,1} = 0$ and $y_{j,2} \leq x_{t+1,2} + 1$. To see the latter, write

$$m_2 - k = x_{t+1,2}(n-t) + r, \qquad 0 \leq r < n - t.$$

Since $k' = k-1$, the allocation Y distributes

$$m_2 - k' = x_{t+1,2}(n-t) + (r+1)$$

goods of the second type equitably among the $n-t$ agents to the right of t. Because $1 \leq r+1 \leq n-t$, each such agent receives at most $x_{t+1,2} + 1$ goods.

Subcase 1.1. Agent t **strongly envies some agent** $j > t$**.** Then

$$y_{t,1} + y_{t,2}\, v_{t,2} < (y_{j,2} - 1)\, v_{t,2} \leq x_{t+1,2}\, v_{t,2}.$$

On the other hand:

$$y_{t,1} + y_{t,2}v_{t,2} \geq \lceil x_{t+1,2}v_{t,2} \rceil - \lceil (k-1)v_{t,2} \rceil + 1 + (k-1)v_{t,2} > x_{t+1,2}v_{t,2},$$

because $\lceil (k-1)v_{t,2} \rceil < (k-1)v_{t,2} + 1$. Contradiction.

Subcase 1.2. An agent $i < t$ strongly envies some agent $j > t$. Then

$$y_{i,1} < (y_{j,2} - 1)\, v_{i,2} \leq x_{t+1,2}\, v_{t,2}.$$

We have $y_{i,1} \geq \lceil x_{t+1,2}\, v_{t,2} \rceil$. Then:

$$\lceil x_{t+1,2}v_{t,2} \rceil \leq y_{i,1} < x_{t+1,2}v_{t,2} \leq \lceil x_{t+1,2}v_{t,2} \rceil,$$

which is impossible. Contradiction.

Case 2. $t \neq t'$.

Subcase 2.1. $t' = n$. In this case, $k = 1$, $k' = m_2$, and $t = n - 1$. It follows that $x_{t+1,2} = m_2 - 1$. We need to show that

$$m_1 < \lceil (m_2 - 1)\, v_{t,2} \rceil\, (n-1).$$

Suppose the contrary, that is,

$$m_1 \geq \lceil (m_2 - 1)\, v_{t,2} \rceil\, (n-1).$$

Now consider the allocation Y, and write $p' = \lceil m_2 v_{n,2} \rceil$. If $y_{n,1} > 0$, then $m_1 > p'(n-1)$. In that case, Lemmas 3 and 5 exclude all strong envy in Y, contradicting the assumption that Y is right-envious. Thus $y_{n,1} = 0$, so all first-type goods are distributed equitably among agents $1, \ldots, n-1$. Our contrary assumption therefore implies that, for any $i < n$,

$$y_{i,1} \geq \lceil (m_2 - 1)\, v_{t,2} \rceil.$$

But then agent i cannot strongly envy agent n, because otherwise

$$\lceil (m_2 - 1)\, v_{t,2} \rceil \leq y_{i,1} < (y_{n,2} - 1)\, v_{i,2} \leq (m_2 - 1)\, v_{t,2},$$

which is a contradiction. Therefore, this case is resolved.

Subcase 2.2. $t' < n$. In this case, by the definition of the (t,k)-ordering, $k = 1$, $k' = \lfloor m_2/(n-t) \rfloor$, and $t' = t + 1$. Let us again consider a possible strong-envy situation. By assumption, there exist agents $i < j$ such that agent i strongly envies agent j. By Lemma 3, we have $i \leq t'$ and $j \geq t'$.

Now, observe that $i \neq t'$. Indeed, for any agent $j > t'$ we have

$$y_{j,2} \leq \left\lceil \frac{m_2}{n-t} \right\rceil \leq \left\lfloor \frac{m_2}{n-t} \right\rfloor + 1 = y_{t',2} + 1,$$

Therefore, agent t' cannot strongly envy another agent, and the strong envy must come from some agent $i < t'$.

Let $s = n - t$ and write

$$m_2 - 1 = qs + r, \qquad 0 \leq r < s.$$

Since $k = 1$, the construction of X gives

$$q = x_{t+1,2}.$$

Moreover,

$$k' = \left\lfloor \frac{m_2}{s} \right\rfloor = \begin{cases} q, & r \leq s-2, \\ q+1, & r = s-1. \end{cases}$$

If $r \leq s - 2$, then

$$m_2 - k' = q(s-1) + (r+1);$$

if $r = s - 1$, then

$$m_2 - k' = (q + 1)(s - 1).$$

In either case, equitably allocating these goods among the $s - 1$ agents to the right of t' gives each of them at most

$$q + 1 = x_{t+1,2} + 1$$

goods. Therefore,

$$\max_{i>t'} y_{i,2} \leq x_{t+1,2} + 1.$$

Put $A = \lceil x_{t+1,2} v_{t,2} \rceil$ and $p' = \lceil k' v_{t',2} \rceil$. Suppose, for a contradiction, that $m_1 \geq At + 1$. Since $k' \geq x_{t+1,2}$ and $v_{t',2} \geq v_{t,2}$, we have $p' \geq A$.

If $m_1 < p't$, all first-type goods in Y are distributed equitably among the t agents to the left of t', so each receives at least

$$y_{i,1} \geq \left\lfloor \frac{m_1}{t} \right\rfloor \geq A.$$

If $m_1 \geq p't$, each such agent receives at least $p' \geq A$. Thus $y_{i,1} \geq A$ for every $i < t'$.

Such an agent i cannot strongly envy any $j > t'$, since removing one second-type good from Y_j leaves value at most

$$(y_{j,2} - 1)v_{i,2} \leq x_{t+1,2} v_{i,2} \leq x_{t+1,2} v_{t,2} \leq A.$$

Agent i cannot strongly envy t' either. If $y_{t',1} = 0$, removing a second-type good from $Y_{t'}$ leaves at most

$$(k' - 1)v_{i,2} \leq x_{t+1,2} v_{i,2} \leq A,$$

because $k' \leq x_{t+1,2} + 1$. If $y_{t',1} > 0$, then $m_1 \geq p't$, and Lemma 5 excludes strong envy between i and t'. The preceding argument also excludes strong envy from t' to agents to its right. Thus Y cannot be right-envious, a contradiction.

Therefore, the right inequality is established.

Continue the proof with the left inequality. Assume the opposite:

$$m_1 \leq \lceil x_{t+1,2}\, v_{t,2} \rceil\, t - \lceil k\, v_{t,2} \rceil.$$

Since X is left-envious, Lemmas 4 and 5 imply that agents $i \leq t$ and $j \leq t$ do not strongly envy each other. Therefore, by Lemma 3, there must exist an agent $j > t$ who strongly envies some agent $i \leq t$. Consider two possible cases.

Case 1. Agent $j > t$ strongly envies an agent $i < t$. In this case we have

$$x_{t+1,2} v_{t,2} \leq x_{t+1,2} v_{j,2} \leq x_{j,2} v_{j,2} < x_{i,1} - 1.$$

It follows that

$$x_{i,1} > x_{t+1,2}\, v_{t,2} + 1.$$

Since for all agents $i_1 < t$, the number of first-type items differs by at most one, we have

$$x_{i_1,1} \geq \lceil x_{t+1,2}\, v_{t,2} \rceil.$$

Here we use the ceiling on the right-hand side, since $x_{i_1,1}$ is an integer. In the final equitable step, agent t is prioritized after every agent $i < t$, so $x_{t,1} \geq x_{i,1} - \lceil k v_{t,2} \rceil$. In particular, for the agent t we have

$$x_{t,1} > x_{t+1,2}\, v_{t,2} + 1 - \lceil k\, v_{t,2} \rceil.$$

Summing over all such agents, we obtain

$$m_1 \geq \lceil x_{t+1,2}\, v_{t,2} \rceil (t - 2) + (\lceil x_{t+1,2}\, v_{t,2} \rceil + 1) + (\lceil x_{t+1,2}\, v_{t,2} \rceil + 1 - \lceil k\, v_{t,2} \rceil) = \lceil x_{t+1,2}\, v_{t,2} \rceil t - \lceil k\, v_{t,2} \rceil + 2,$$

which contradicts our assumption.

Case 2. Agent $j > t$ strongly envies agent t.

Since $v_{j,2} \geq v_{t,2} \geq 1$, every good has value at least 1 to agent j. Strong envy therefore implies

$$x_{j,2}v_{j,2} < x_{t,1} - 1 + kv_{j,2}.$$

As $x_{j,2} \geq x_{t+1,2} \geq k$ and $v_{j,2} \geq v_{t,2}$, we obtain

$$x_{t,1} > (x_{j,2} - k)v_{j,2} + 1 \geq (x_{t+1,2} - k)v_{t,2} + 1.$$

Let $p = \lceil kv_{t,2} \rceil$. Since X is left-envious, Lemma 4 gives $m_1 \geq p(t-1)$. In the final equitable step, t is prioritized after every $i < t$, and each such agent receives at most one fewer additional good than t. Hence

$$x_{i,1} \geq x_{t,1} + p - 1 > (x_{t+1,2} - k)v_{t,2} + p \geq x_{t+1,2}v_{t,2}.$$

Since the item counts are integers, for every $i < t$ we have

$$x_{i,1} \geq \lceil x_{t+1,2}v_{t,2} \rceil.$$

Moreover,

$$x_{t,1} > x_{t+1,2}v_{t,2} + 1 - p,$$

so

$$x_{t,1} \geq \lceil x_{t+1,2}v_{t,2} \rceil + 1 - p.$$

Summing these bounds gives

$$\begin{aligned} m_1 &\geq \lceil x_{t+1,2}v_{t,2} \rceil (t-1) + \lceil x_{t+1,2}v_{t,2} \rceil + 1 - p \\ &= \lceil x_{t+1,2}v_{t,2} \rceil t - p + 1, \end{aligned}$$

which contradicts our assumption.

Both possible cases lead to a contradiction, so the assumption was wrong. The left inequality is proved. Thus, we have established both inequalities, and therefore the proof of the lemma is complete. □

6.2 Proof of Theorem 4

We are ready to restate and prove the main result on feasibility of reallocations. Combined with Lemma 1, it guarantees that the (t,k)-reallocation returned in the final line of Algorithm 1 is a feasible solution.

Theorem 4. *Let X be the (t,k)-reallocation for some $(t,k) \in \mathcal{T}$. If $\lceil x_{t+1,2} \cdot v_{t,2} \rceil \cdot t - \lceil k \cdot v_{t,2} \rceil + 1 \leq m_1 \leq \lceil x_{t+1,2} \cdot v_{t,2} \rceil \cdot t - \lceil (k-1) \cdot v_{t,2} \rceil$, then X is EFX+PO.*

Proof. Let $d = x_{t+1,2}$, $p = \lceil kv_{t,2} \rceil$, and let ℓ be the largest index such that $x_{\ell,2} = d$. Admissibility gives $d \geq k \geq 1$ and $v_{t,2} \geq 1$.

We use five claims to establish Pareto optimality and rule out the possible cases of strong envy. Their proofs require technical case analyses, which we defer to Section 6.3. Readers interested in the details can find each claim restated together with its complete proof there.

We begin with Pareto optimality, for which only the upper bound on m_1 is needed.

Claim 7. *Let X be a (t,k)-reallocation satisfying*

$$m_1 \leq \lceil dv_{t,2} \rceil t - \lceil (k-1)v_{t,2} \rceil, \qquad d = x_{t+1,2}.$$

Then X is proper and hence Pareto optimal.

The upper bound in the theorem ensures that Claim 7 applies to X. Hence X is proper and Pareto optimal. It remains to establish EFX.

The next claim excludes strong envy between agents lying strictly on the same side of the split point.

Claim 8. *If there is strong envy between two agents i and j, then $(i-t)(j-t) \le 0$.*

Thus, strong envy cannot arise entirely on either side of the split point. We next rule out pairs lying strictly on opposite sides.

Claim 9. *There can be no strong envy between agents $i < t$ and $j > t$.*

Consequently, any remaining strong envy must involve agent t. We first consider its interactions with the agents to the right that receive d goods of the second type.

Claim 10. *There can be no strong envy between agent t and agent i with $t < i \le \ell$.*

The remaining agents to the right receive $d+1$ goods of the second type and no first-type goods. The next claim covers these pairs in both directions.

Claim 11. *There can be no strong envy between agent t and agent $i > \ell$.*

Together, these claims leave only pairs consisting of agent t and an agent $i < t$. We verify these directly. Let r_i and r_t denote the numbers of first-type goods received by these agents in the final equitable step. In either priority order, t is after i, so

$$0 \le r_t - r_i \le 1, \qquad x_{i,1} = \lceil dv_{t,2} \rceil + r_i, \qquad x_{t,1} = \lceil dv_{t,2} \rceil - p + r_t.$$

Agent i receives only first-type goods, and

$$x_{t,1} + kv_{t,2} - (x_{i,1} - 1) = kv_{t,2} - p + (r_t - r_i) + 1 > 0.$$

Thus, t cannot strongly envy i.

For the reverse direction, if $v_{i,2} \ge 1$, removing any good from X_t leaves a bundle worth at most $x_{t,1} + kv_{i,2} - 1$ to i. Since $v_{i,2} \le v_{t,2}$,

$$x_{i,1} - (x_{t,1} + kv_{i,2} - 1) = p - kv_{i,2} - (r_t - r_i) + 1 \ge 0.$$

If $v_{i,2} < 1$, a least-valued good in X_t is of the second type, since $k \ge 1$. Removing such a good leaves a bundle worth $x_{t,1} + (k-1)v_{i,2}$ to i, and

$$\begin{aligned} x_{i,1} - (x_{t,1} + (k-1)v_{i,2}) &= p - (k-1)v_{i,2} - (r_t - r_i) \\ &\ge k - (k-1)v_{i,2} - 1 \\ &= (k-1)(1 - v_{i,2}) \ge 0. \end{aligned}$$

Therefore, i cannot strongly envy t either.

All pairs of agents have now been covered. Hence X is EFX and Pareto optimal. □

6.3 Proofs of Technical Claims

We now prove the five claims used in Theorem 4, restating each one before its proof. We use $d = x_{t+1,2}$, $p = \lceil kv_{t,2} \rceil$, and let ℓ be the largest index such that $x_{t,2} = d$. Admissibility gives $d \ge k \ge 1$, $v_{t,2} \ge 1$, and $\ell \ge t+1$.

We start with Pareto optimality. For this first claim, the upper bound on m_1 suffices; the lower bound will be used later in the EFX analysis.

Claim 7. *Let X be a (t,k)-reallocation satisfying*

$$m_1 \le \lceil dv_{t,2} \rceil t - \lceil (k-1)v_{t,2} \rceil, \qquad d = x_{t+1,2}.$$

Then X is proper and hence Pareto optimal.

Proof of Claim 7. Set $d = x_{t+1,2}$ and $p = \lceil kv_{t,2} \rceil$, and let ℓ be the largest index such that $x_{t,2} = d$. Since $(t,k) \in \mathcal{T}$ and $t < n$, we have $v_{t,2} \geq 1$, $d \geq k \geq 1$, and $\ell \geq t+1$. Write

$$R = m_1 - \lceil dv_{t,2} \rceil t + p.$$

By the definition of reallocation and the upper bound in the claim,

$$0 \leq R \leq \lceil kv_{t,2} \rceil - \lceil (k-1)v_{t,2} \rceil.$$

By Theorem 1, any proper allocation is Pareto optimal. We will show that X is *proper*. By the definition of a *proper* allocation, it is enough to verify that

$$\max\{x_{i,1} : i > t\} < v_{t,2}. \tag{10}$$

Note that $x_{i,2} = 0$ for all $i < t$. If $R = 0$, every agent $i > t$ receives no first-type goods, and the desired inequality follows immediately. Hence assume $R \geq 1$. We consider two cases depending on whether $v_{t,2}$ is an integer or not.

Case 1. $v_{t,2}$ **is not an integer.**

Assume the opposite, and let there exist an agent $i > t$ such that

$$x_{i,1} \geq v_{t,2}.$$

For $t < i \leq \ell$, first-type goods are received only in the final equitable step; agents $i > \ell$ receive none. Since $\ell \geq t+1$, for every $i > t$ we have

$$x_{i,1} \leq \left\lceil \frac{m_1 - \lceil dv_{t,2} \rceil t + \lceil kv_{t,2} \rceil}{1+t} \right\rceil.$$

Now, assuming $x_{i,1} \geq v_{t,2}$, we get

$$\left\lceil \frac{m_1 - \lceil dv_{t,2} \rceil t + \lceil kv_{t,2} \rceil}{1+t} \right\rceil \geq v_{t,2}. \tag{11}$$

By the upper bound in the claim, we have

$$m_1 \leq \lceil dv_{t,2} \rceil t - \lceil (k-1)v_{t,2} \rceil.$$

Substituting this into inequality (11), we obtain

$$\left\lceil \frac{[\lceil dv_{t,2} \rceil t - \lceil (k-1)v_{t,2} \rceil] - \lceil dv_{t,2} \rceil t + \lceil kv_{t,2} \rceil}{1+t} \right\rceil \geq v_{t,2},$$

which simplifies to

$$\left\lceil \frac{\lceil kv_{t,2} \rceil - \lceil (k-1)v_{t,2} \rceil}{1+t} \right\rceil \geq v_{t,2}.$$

For the numerator, subadditivity of the ceiling gives

$$\lceil kv_{t,2} \rceil - \lceil (k-1)v_{t,2} \rceil \leq \lceil v_{t,2} \rceil = \lfloor v_{t,2} + 1 \rfloor.$$

Then

$$\left\lceil \frac{\lfloor v_{t,2} + 1 \rfloor}{1+t} \right\rceil \geq v_{t,2}.$$

Since $t \geq 1$, we have

$$\left\lceil \frac{\lfloor v_{t,2} + 1 \rfloor}{2} \right\rceil \geq v_{t,2}.$$

Then

$$\left\lceil \frac{\lfloor v_{t,2} \rfloor + 1}{2} \right\rceil \geq v_{t,2}.$$

Since $v_{t,2} \notin \mathbb{N}$, this implies

$$\left\lceil \frac{\lfloor v_{t,2} \rfloor + 1}{2} \right\rceil > \lfloor v_{t,2} \rfloor,$$

so

$$\frac{\lfloor v_{t,2} \rfloor + 1}{2} > \lfloor v_{t,2} \rfloor,$$

which in turn implies

$$\frac{\lfloor v_{t,2} \rfloor}{2} < \frac{1}{2}.$$

But this is impossible, since $v_{t,2} \geq 1$, hence $\lfloor v_{t,2} \rfloor \geq 1$, and then $\frac{\lfloor v_{t,2} \rfloor}{2} \geq \frac{1}{2}$ — a contradiction.

Case 2. $v_{t,2}$ **is an integer.**

Assume for a contradiction that $x_{i,1} \geq v_{t,2}$ for some $i > t$. In this case we have

$$\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil = (d-k)v_{t,2}, \tag{12}$$

since d, k are both integers. Therefore,

$$x_{i,1} \leq \left\lceil \frac{m_1 - dv_{t,2}t + kv_{t,2} - 1}{1+t} \right\rceil. \tag{13}$$

To prove this, note that, by (12), the agents are prioritized in the order $1, 2, \ldots, t-1, t+1, \ldots, \ell, t$. Since t is the last in this order, every agent $t < i \leq \ell$ receives at most $\lceil (R-1)/\ell \rceil$ first-type goods. As $R \geq 1$ and $\ell \geq t+1$, the displayed bound on $x_{i,1}$ follows; it also holds for $i > \ell$, where $x_{i,1} = 0$.

Substitute the upper bound on m_1 from the claim statement combined with (12), that is, $m_1 \leq dv_{t,2}t - (k-1)v_{t,2}$, in the right part of (13) and obtain

$$v_{t,2} \leq x_{i,1} \leq \left\lceil \frac{dv_{t,2}t - (k-1)v_{t,2} - dv_{t,2}t + kv_{t,2} - 1}{1+t} \right\rceil = \left\lceil \frac{v_{t,2} - 1}{1+t} \right\rceil.$$

And since $t \geq 1$, it follows that

$$\left\lceil \frac{v_{t,2} - 1}{2} \right\rceil \geq v_{t,2}.$$

However, since $v_{t,2}$ is a positive integer,

$$\left\lceil \frac{v_{t,2} - 1}{2} \right\rceil \leq v_{t,2} - 1 < v_{t,2},$$

which is a contradiction.

Therefore, X is a *proper* allocation and, by Theorem 1, it is Pareto-optimal. Thus, the claim is proved. ⌟

Having established Pareto optimality, we turn to EFX. We retain the bound $x_{i,1} < v_{t,2}$ for $i > t$ from (10), which will control comparisons with agents to the right of t. From this point on, X satisfies both bounds in Theorem 4.

The final equitable step distributes

$$R = m_1 - \lceil dv_{t,2} \rceil t + p, \qquad 1 \leq R \leq \lceil kv_{t,2} \rceil - \lceil (k-1)v_{t,2} \rceil \leq \lceil v_{t,2} \rceil.$$

Thus, each participating agent receives at most

$$\left\lceil \frac{R}{\ell} \right\rceil \leq \left\lceil \frac{\lceil v_{t,2} \rceil}{2} \right\rceil \leq \max\{1, \lceil v_{t,2} \rceil - 1\} \leq v_{t,2}$$

first-type goods in this step. We also use the stronger bound $x_{i,1} < v_{t,2}$ for $i > t$, established in (10).

When an envious agent a has $v_{a,2} \geq 1$, every good has value at least 1 to that agent. Consequently, strong envy towards b implies

$$x_{a,1} + x_{a,2}v_{a,2} < x_{b,1} + x_{b,2}v_{a,2} - 1.$$

We use this as a necessary condition, including when X_b contains no first-type goods.

We first consider pairs whose agents lie strictly on the same side of t. The proof distinguishes agents to the right according to whether they receive d or $d+1$ second-type goods.

Claim 8. *If there is strong envy between two agents i and j, then $(i-t)(j-t) \leq 0$.*

Proof of Claim 8. Let us proceed by contradiction. Suppose that $(i-t)(j-t) > 0$.

Case 1. $i < t$, $j < t$.

These agents receive only items of the first type. If $x_{j,1} \geq x_{i,1}$, j cannot envy i. If $x_{i,1} > x_{j,1}$, the difference is exactly 1, so removing the single excess type 1 good eliminates the envy. Hence strong envy is impossible.

Case 2. $i > t$, $j > t$.

Let us consider the possible cases based on the positions of the agents relative to agent ℓ.

Subcase 2.1. $i > \ell$, $j > \ell$.

By construction of allocation X, all agents in this group receive identical sets of items. Therefore, strong envy is impossible.

Subcase 2.2. $t < i \leq \ell$, $t < j \leq \ell$.

Suppose that agent i strongly envies agent j. Then

$$x_{i,1} + dv_{i,2} < x_{j,1} + dv_{i,2} - 1.$$

Here $x_{i,2} = x_{j,2} = d$ and $v_{i,2} \geq v_{t,2} \geq 1$. After cancellation we obtain $x_{i,1} < x_{j,1} - 1$, contradicting the fact that the first-type quantities differ by at most 1.

Subcase 2.3. Agent $j > \ell$ strongly envies agent $t < i \leq \ell$.

Then, by the definition of strong envy,

$$x_{j,2}v_{j,2} < (x_{i,1} - 1) + x_{i,2}v_{j,2}.$$

Substituting $x_{j,2} = d+1$ and $x_{i,2} = d$, we get

$$(d+1)v_{j,2} < (x_{i,1} - 1) + dv_{j,2}.$$

Rearranging,

$$(d+1)v_{j,2} - dv_{j,2} < x_{i,1} - 1.$$

So,

$$v_{j,2} < x_{i,1} - 1.$$

By condition (10), we have

$$x_{i,1} < v_{t,2}.$$

But

$$v_{t,2} \leq v_{j,2}.$$

So, we have

$$v_{j,2} < x_{i,1} - 1 < x_{i,1} < v_{j,2}.$$

This is a contradiction.

Subcase 2.4. Agent $t < i \leq \ell$ strongly envies agent $j > \ell$.

Then

$$x_{i,2}v_{i,2} + x_{i,1} < (x_{j,2} - 1)v_{i,2}.$$

Substituting $x_{j,2} = d + 1$ and $x_{i,2} = d$, we get

$$dv_{i,2} + x_{i,1} < ((d + 1) - 1)v_{i,2},$$

hence

$$dv_{i,2} + x_{i,1} < dv_{i,2}.$$

Simplifying,

$$x_{i,1} < 0,$$

a contradiction.

All cases have been considered and lead to contradictions. Therefore, no strong envy can occur between i and j, and the claim is proved. ⌟

The preceding claim excludes strong envy within either side of the split point. It remains to compare agents on opposite sides and pairs involving t. We next handle the former, considering both possible directions of envy.

Claim 9. *There can be no strong envy between agents $i < t$ and $j > t$.*

Proof of Claim 9. Consider all possible cases of strong envy between an agent $i < t$ and an agent $j > t$.

Case 1. Agent $j > \ell$ strongly envies agent $i < t$.

Agent j receives only items of the second type, and agent i receives only items of the first type. If agent j strongly envied agent i, then

$$x_{j,2}v_{j,2} < x_{i,1} - 1.$$

However, by construction $x_{i,1} \le \lceil dv_{t,2} \rceil + v_{t,2}$ and $x_{j,2} = d + 1$. Therefore,

$$(d + 1)v_{j,2} < \lceil dv_{t,2} \rceil + v_{t,2} - 1 \le (d + 1)v_{t,2}.$$

And since $v_{j,2} \ge v_{t,2}$, this is impossible. Contradiction.

Case 2. Agent $\ell \ge j > t$ strongly envies agent $i < t$.

Agent j receives $x_{j,2} = d$ items of the second type and $x_{j,1} \ge 0$ items of the first type; agent i receives only first-type items. If agent j strongly envied agent i, then

$$dv_{j,2} + x_{j,1} < x_{i,1} - 1.$$

Every agent $j > t$ participating in the final equitable step is prioritized after every agent $i < t$. Consequently, $x_{i,1} \le \lceil dv_{t,2} \rceil + x_{j,1}$. Hence

$$dv_{j,2} + x_{j,1} < \lceil dv_{t,2} \rceil + x_{j,1} - 1,$$

so

$$dv_{j,2} < \lceil dv_{t,2} \rceil - 1.$$

But $v_{j,2} \ge v_{t,2}$, which implies $dv_{j,2} \ge dv_{t,2}$, whereas $\lceil dv_{t,2} \rceil - 1 < dv_{t,2}$ — again a contradiction.

Case 3. Agent $i < t$ strongly envies agent $\ell \ge j > t$.

Agent i receives only first-type items; agent j receives d items of the second type and $x_{j,1}$ items of the first type (with $x_{j,1} \le x_{i,1} - \lceil dv_{t,2} \rceil + 1$ by construction). Two subcases are possible.

Subcase 3.1. $1 \le v_{i,2}$**.**

Then strong envy implies

$$x_{i,1} < dv_{i,2} + x_{j,1} - 1.$$

But $x_{j,1} \le x_{i,1} - \lceil dv_{t,2} \rceil + 1$, hence

$$x_{i,1} < dv_{i,2} + x_{i,1} - \lceil dv_{t,2} \rceil,$$

so

$$0 < dv_{i,2} - \lceil dv_{t,2} \rceil,$$

which is impossible, since $v_{i,2} \le v_{t,2}$.

Subcase 3.2. $1 > v_{i,2}$**.**

Then strong envy implies

$$x_{i,1} < (d-1)v_{i,2} + x_{j,1}.$$

And $x_{j,1} \le x_{i,1} - \lceil dv_{t,2} \rceil + 1$, therefore

$$x_{i,1} < (d-1)v_{i,2} + x_{i,1} - \lceil dv_{t,2} \rceil + 1,$$

which simplifies to

$$0 < (d-1)v_{i,2} - \lceil dv_{t,2} \rceil + 1.$$

However, since $d \ge 1$, $v_{i,2} < 1$, and $v_{t,2} \ge 1$,

$$(d-1)v_{i,2} - \lceil dv_{t,2} \rceil + 1 \le d - \lceil dv_{t,2} \rceil \le 0,$$

a contradiction.

Case 4. Agent $i < t$ **strongly envies agent** $j > \ell$**.**

Agent j receives only goods of the second type: $x_{j,2} = d + 1$, $x_{j,1} = 0$. If agent i strongly envied agent j, then

$$x_{i,1} < (d + 1 - 1)v_{i,2} = dv_{i,2}.$$

But by construction $x_{i,1} \ge \lceil dv_{t,2} \rceil$, and $v_{i,2} \le v_{t,2}$. Hence

$$\lceil dv_{t,2} \rceil \le x_{i,1} < dv_{i,2},$$

which is impossible, since $dv_{i,2} \le dv_{t,2} \le \lceil dv_{t,2} \rceil$.

Thus, in all cases strong envy is impossible, and the claim is proved. ⌟

The first two EFX claims show that any remaining strong envy must involve agent t. We first compare t with the agents $t < i \le \ell$, each of whom receives d second-type goods. Both directions of envy are considered for each priority order.

Claim 10. *There can be no strong envy between agent* t *and agent* i *with* $t < i \le \ell$*.*

Proof of Claim 10. Let us consider two cases, depending on whether agent t strongly envies agent i or agent i strongly envies agent t.

Case 1. Agent i **strongly envies agent** t**.**

Consider two subcases, depending on the relation between $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil$ and $(d-k)v_{t,2}$.

Subcase 1.1. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > (d-k)v_{t,2}$**.**

In this case, by the construction of the allocation,

$$x_{i,1} \ge x_{t,1} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil),$$

because the agents are prioritized in the following order:

$$1, 2, \ldots, t-1, t, t+1, \ldots, \ell.$$

If strong envy were possible, then

$$x_{i,1} + dv_{i,2} = x_{i,1} + x_{i,2}v_{i,2} < x_{t,1} + x_{t,2}v_{i,2} - 1 = x_{t,1} - 1 + kv_{i,2},$$

which implies

$$x_{t,1} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) + dv_{i,2} < x_{t,1} - 1 + kv_{i,2}.$$

Subtract $x_{t,1}$ from both parts and obtain

$$\begin{aligned} dv_{i,2} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) &< kv_{i,2} - 1, \\ (d-k)v_{i,2} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) &< -1, \\ (d-k)v_{i,2} &< (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) - 1, \\ (d-k)v_{i,2} &< (dv_{t,2} + 1 - kv_{t,2}) - 1. \end{aligned}$$

This is a contradiction, since $d \geq k$ and $v_{i,2} \geq v_{t,2}$.

Subcase 1.2. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil \leq (d-k)v_{t,2}$**.**

Here, similarly,

$$x_{i,1} \geq x_{t,1} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) - 1.$$

If strong envy were possible,

$$x_{i,1} + dv_{i,2} < x_{t,1} - 1 + kv_{i,2},$$

then

$$x_{t,1} - (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) - 1 + dv_{i,2} < x_{t,1} - 1 + kv_{i,2},$$

which simplifies to

$$(d-k)v_{i,2} < \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil.$$

Since $d \geq k$ and $v_{i,2} \geq v_{t,2}$, the last inequality implies

$$(d-k)v_{t,2} \leq (d-k)v_{i,2} < \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil,$$

contradicting the assumption of this subcase.

Case 2. Agent t **strongly envies agent** i**.**

Consider two subcases depending on the result of comparing $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil$ and $(d-k)v_{t,2}$.

Subcase 2.1. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > (d-k)v_{t,2}$**.**

In this case, by construction,

$$x_{t,1} \geq x_{i,1} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) - 1.$$

If strong envy were possible,

$$x_{i,1} + dv_{t,2} - 1 > x_{t,1} + kv_{t,2},$$

then substituting the bound for $x_{t,1}$ gives

$$x_{i,1} + dv_{t,2} - 1 > x_{i,1} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) - 1 + kv_{t,2},$$

which simplifies to

$$\begin{aligned} dv_{t,2} &> (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) + kv_{t,2}, \\ dv_{t,2} - kv_{t,2} &> \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil, \\ (d-k)v_{t,2} &> \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil. \end{aligned}$$

But by assumption of this subcase,

$$\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > (d-k)v_{t,2}.$$

This is a contradiction.

Subcase 2.2. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil \leq (d-k)v_{t,2}$**.**

In this case, by construction,

$$x_{t,1} \geq x_{i,1} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil).$$

This follows because the agents are prioritized in the order

$$1, 2, \ldots, t-1, t+1, \ldots, \ell, t.$$

If strong envy were possible,

$$x_{i,1} + dv_{t,2} - 1 > x_{t,1} + kv_{t,2},$$

then substituting the bound for $x_{t,1}$ gives

$$x_{i,1} + dv_{t,2} - 1 > x_{i,1} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) + kv_{t,2},$$

which simplifies to

$$dv_{t,2} - 1 > (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) + kv_{t,2},$$
$$(d-k)v_{t,2} - 1 > \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil.$$

But the ceiling inequalities give

$$\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > dv_{t,2} - (kv_{t,2} + 1) = (d-k)v_{t,2} - 1,$$

which is a contradiction.

The claim is proved. ⌟

We have now excluded strong envy between t and every agent $t < i \le \ell$. To complete the right-side analysis, we consider the remaining agents $i > \ell$, who receive $d+1$ second-type goods and no first-type goods.

Claim 11. *There can be no strong envy between agent t and agent $i > \ell$.*

Proof of Claim 11. Let us consider two cases, depending on whether agent t strongly envies agent i or agent i strongly envies agent t.

Case 1. Agent t strongly envies agent i.

If strong envy exists, then

$$x_{t,1} + x_{t,2}v_{t,2} < (x_{i,2} - 1)v_{t,2} = dv_{t,2}.$$

We distinguish the two possible priority orders.

If $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > (d-k)v_{t,2}$, then already the initial first-type allocation to t gives

$$x_{t,1} + kv_{t,2} \ge \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil + kv_{t,2} > dv_{t,2},$$

a contradiction.

Otherwise, t is last in the final equitable step. The lower bound in Theorem 4 ensures that

$$R = m_1 - \lceil dv_{t,2} \rceil t + \lceil kv_{t,2} \rceil \ge 1.$$

Therefore t receives at least one good in that step, so

$$x_{t,1} \ge \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil + 1.$$

Substituting this into the strong-envy inequality gives

$$\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil + 1 + kv_{t,2} < dv_{t,2}.$$

Since $\lceil dv_{t,2} \rceil \ge dv_{t,2}$, this implies

$$-\lceil kv_{t,2} \rceil + 1 + kv_{t,2} < 0,$$

contradicting $\lceil kv_{t,2} \rceil < kv_{t,2} + 1$. Thus, agent t cannot strongly envy an agent $i > \ell$.

Case 2. Agent $i > \ell$ strongly envies agent t.

Consider two subcases, depending on the relation between $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil$ and $(d-k)v_{t,2}$.

Subcase 2.1. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil > (d-k)v_{t,2}$.

In this case, by construction,

$$x_{t,1} \le x_{j,1} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) < v_{t,2} + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil),$$

where j is any agent with $t < j \le \ell$. Such an agent exists because $\ell \ge t+1$, and $x_{j,1} < v_{t,2}$ by (10).

If strong envy were possible, then

$$(d+1)v_{i,2} < x_{t,1} - 1 + kv_{i,2}.$$

Subtracting $kv_{i,2}$ and using the bound on $x_{t,1}$, we obtain

$$(d+1-k)v_{i,2} < x_{t,1} - 1 < v_{t,2} + \lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil - 1.$$

By the ceiling inequalities,

$$\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil < (d-k)v_{t,2} + 1.$$

Consequently,

$$(d+1-k)v_{i,2} < (d+1-k)v_{t,2}.$$

But $d \ge k$ and $v_{i,2} \ge v_{t,2}$, so

$$(d+1-k)v_{i,2} \ge (d+1-k)v_{t,2},$$

a contradiction.

Subcase 2.2. $\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil \le (d-k)v_{t,2}$**.**

Then by our construction,

$$x_{t,1} \le x_{j,1} + 1 + (\lceil dv_{t,2} \rceil - \lceil kv_{t,2} \rceil) < v_{t,2} + 1 + (d-k)v_{t,2},$$

where j is an agent with $t < j \le \ell$, and $x_{j,1} < v_{t,2}$ by (10).

If strong envy were possible, then

$$(d+1)v_{i,2} < x_{t,1} - 1 + kv_{i,2}.$$

Subtracting $kv_{i,2}$ gives

$$(d+1-k)v_{i,2} < x_{t,1} - 1 < (d+1-k)v_{t,2} \le (d+1-k)v_{i,2},$$

where the last inequality follows from $d \ge k$ and $v_{i,2} \ge v_{t,2}$. This is impossible.

Thus, all cases have been considered, and the claim is proved. ⌟

Together, the last two claims exclude strong envy between t and every agent to its right. Combined with the preceding claims, they leave only pairs consisting of t and an agent $i < t$; these are verified directly in the proof of Theorem 4. This completes the proofs of all five claims.

7 Conclusion

In this work, we established that an EFX+PO allocation always exists for two types of goods under positive additive valuations. We also presented an algorithm that computes such an allocation in $O(n \log n + \log m)$ time for general input, including preprocessing, and in $O(\log n + \log m)$ time once the input has been preprocessed. Finally, we showed that EFX and fractional Pareto optimality are not always compatible, even for two types of goods with positive valuations.

Several natural questions remain open. For example, one can ask whether the main result can be extended to more types of goods or to instances in which the utility matrix has bounded rank.

Open Question 1. *Under positive additive valuations, do EFX and EFX+PO allocations always exist in either of the following settings?*

(a) *the utility matrix has rank 2;*
(b) *there are three types of goods.*

We remark that the existence of EFX for submodular or subadditive positive valuations was recently disproved by (Akrami, Mayorov, et al. 2026; Mackenzie and Suzuki 2026). The question of whether EFX is always guaranteed for general positive additive valuations remains widely open.

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